PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 2, October 1994

H^p ESTIMATES FOR BI-INVARIANT OPERATORS ON COMPACT LIE GROUPS

DASHAN FAN AND ZENGFU XU

(Communicated by J. Marshall Ash)

ABSTRACT. The main purpose of this paper is to extend to compact Lie groups a Hörmander multiplier theorem concerning translation invariant operators on Hardy spaces H^p, $0 < p \le 1$.

1. INTRODUCTION

Suppose that G is a semisimple compact Lie group and $\widehat{G}$ is a maximal collection of inequivalent irreducible unitary representations of G. Given a bounded multisequence $\{m(\lambda)\}_{\lambda\in\widehat{G}}$, $m(\lambda) \in \mathbb{C}$, define the operator T on the space of finite linear combinations of entry functions on G by writing

$$(T\hat{f})(\lambda) = m(\lambda)\hat{f}(\lambda), \qquad \lambda \in \widehat{G}. \tag{1.1}$$

T commutes with left and right translation. If $f \in H^p(G)$, then the Hilbert-Schmidt norm of $\hat{f}$ satisfies

$$|||\hat{f}(\lambda)||| \le C\|f\|_{H^p} d_\lambda^{n+2}|\lambda + \beta|^{n(1/p-1)} \tag{1.2}$$

(see [X]). It follows that if $m \in L^\infty(\widehat{G})$, then $f \to Tf$ is well defined from $H^p(G)$ to the Schwartz distribution space $\mathscr{S}'(G)$. We say m is a Fourier multiplier of $H^p(G)$ if this mapping takes H^p continuously into H^p.

Let $\dim G = n$, $l = \operatorname{rank} G$. Then $\widehat{G}$ can be identified with a subset of a lattice in $\mathbb{R}^l$. Hence we can speak of the partial difference operator δ^J acting on multisequences $\{m(\lambda)\}_{\lambda\in\widehat{G}}$ with $J = (j_1, \dots, j_l)$ a multi-index of order $|J| = j_1 + \cdots + j_l$.

The following theorem is the main result of this paper.

Theorem 1.3. *Let s be the smallest even integer such that $s > \frac{n}{p} - \frac{n}{2}$, where $0 < p \le 1$. Suppose that $m \in L^\infty$ and that for all J with $|J| \le s$ and all*

Received by the editors June 23, 1992 and, in revised form, January 4, 1993.
1991 *Mathematics Subject Classification.* Primary 43A15, 43A77; Secondary 22E30.
Key words and phrases. Hörmander multiplier theorem, H^p spaces, compact Lie group.

$R > 0$

$$\sum_{R \leq |\lambda| \leq 2R} |\delta^J m(\lambda)|^2 \leq C R^{l-2|J|}. \tag{1.4}$$

Then m is a Fourier multiplier of $H^p(G)$.

Similar theorems in other spaces can be found in [BS, CW, W].

2. Notation and definitions

Let G be a connected, simply connected, compact semisimple Lie group of dimension n. Let $\mathscr{g}$ be the Lie algebra of G and $\mathscr{t}$ the Lie algebra of a fixed maximal torus $\mathbb{T}$ in G of dimension l. Let A be a system of positive roots for $(\mathscr{g}, \mathscr{t})$ so that $\mathrm{Card}(A) = \frac{n-l}{2}$, and let $\beta = \frac{1}{2}\sum_{\alpha \in A} \alpha$.

Let $|\cdot|$ be the norm on $\mathscr{g}$ induced by the negative of the Killing form B on $\mathscr{g}^{\mathbb{C}}$, the complexification of $\mathscr{g}$. Then $|\cdot|$ induces a bi-invariant metric d on G. Furthermore, since $B|_{\mathscr{t}^{\mathbb{C}} \times \mathscr{t}^{\mathbb{C}}}$ is nondegenerate, given $\lambda \in \hom_{\mathbb{C}}(\mathscr{t}^{\mathbb{C}}, \mathbb{C})$, there is a unique H_λ in $\mathscr{t}^{\mathbb{C}}$ such that $\lambda(H) = B(H, H_\lambda)$ for each $H \in \mathscr{t}^{\mathbb{C}}$. We let $\langle\ ,\ \rangle$ and $\|\ \|$ denote the inner product and norm transferred from $\mathscr{t}$ to $\mathscr{t}^*$ (the dual of $\mathscr{t}$) by means of this canonical isomorphism and let ξ be this natural map of $\mathscr{t}$ onto $\mathscr{t}^*$.

Let $\mathbb{N} = \{H \in \mathscr{t},\ \exp H = e\}$, e being the identity in G. The weight lattice P is defined by $P = \{\lambda \in \mathscr{t},\ \langle \lambda, x\rangle \in 2\pi\mathbb{Z}, \text{ any } x \in \mathbb{N}\}$ with dominant weights defined by $\Lambda = \{\lambda \in P,\ \langle \lambda, \alpha\rangle \geq 0, \text{ any } \alpha \in A\}$. We can identify $\widehat{G}$ with this Λ because Λ provides a full set of parameters for the equivalence classes of unitary irreducible representations of G. For $\lambda \in \Lambda$ the representation U_λ has dimension

$$d_\lambda = \prod_{\alpha \in A} \frac{\langle \lambda + \beta, \alpha\rangle}{\langle \beta, \alpha\rangle}, \tag{2.1}$$

and its associated character is

$$\chi_\lambda(x) = \frac{\sum_{w \in W} \varepsilon(w) e^{i\langle w(\lambda+\beta), x\rangle}}{\sum_{w \in W} \varepsilon(w) e^{i\langle w\beta, x\rangle}}, \tag{2.2}$$

where $x \in \mathscr{t}$, W is the Weyl group, and $\varepsilon(w)$ is the signature of $w \in W$. Let $X_1, \dots, X_n$ be an orthonormal basis of $\mathscr{g}$. Form the Casimir operator

$$\Delta = \sum_{i=1}^{n} X_1^2. \tag{2.3}$$

This is an elliptic bi-invariant operator which is independent of the choice of orthonormal basis of $\mathscr{g}$. The solution of the Poisson equation on $G \times \mathbb{R}^+$

$$\Delta\phi(g, t) = \frac{d^2\phi}{dt^2}(g, t), \qquad \phi(g, 0) = f(g), \tag{2.4}$$

for $f \in L(G)$ is given by

$$\phi(g, t) = (P_t * f)(g), \tag{2.5}$$

where

$$P_t(x) = \sum_{\lambda \in \widehat{G}} e^{-t(\|\lambda+\beta\|^2 - \|\beta\|^2)^{1/2}} d_\lambda \chi_\lambda(x) \tag{2.6}$$

is the Poisson kernel.

Definition. For $0 < p < \infty$ the Hardy space $H^p(G)$ is the collection of $f \in \mathscr{S}'$ such that

$$\|f\|_{H^p(G)} = \left\| \sup_{t>0} |P_t * f| \right\|_{L^p(G)} < \infty . \tag{2.7}$$

Since the Hardy-Littlewood maximal function is a bounded operator from $L^p(G)$ to $L^p(G)$ if $1 < p < \infty$ and since the Hardy-Littlewood maximal function majorizes the function $\sup_{t>0} |P_t * f|$, we see from [BF] that $H^p(G) = L^p(G)$ if $1 < p < \infty$. In [W] Weiss proved the same multiplier theorem for the case of $1 < p < \infty$ by a method of Littlewood-Paley theory. Following his method, we easily obtain the boundedness result for the case $0 < p \leq 1$ if m satisfies the Hörmander condition $s > \frac{n}{p}$. But obviously this result is not as precise as the condition $s > \frac{n}{p} - \frac{n}{2}$ in Theorem 1.3. To get this precise valve s, we need to introduce the atomic characterization of $H^p(G)$ space. The following definition of atomic H^p on G was given in [C] (see [CW1, FS, L, CL] for the equivalent definitions on other spaces).

An exceptional atom is an L^∞ function bounded by 1. In order to define a regular atom, we consider a faithful unitary representation Π of G such that $\Pi(G) \cong U(L, \mathbb{C})$. Then G can be identified as a submanifold in a real vector space $\mathbb{E}$ underlying $\operatorname{End}(\mathbb{C}^L)$. Now a regular (p, ∞, N) atom for $0 < p \leq 1$ is a function $a(x)$ supported in some ball $B(y, \rho)$ such that

$$\|a\|_\infty \leq \rho^{-n/p} , \tag{2.8}$$

$$\int_G a(x) P(\Pi(x))\, dx = 0 , \tag{2.9}$$

where P is any polynomial on $\mathbb{E}$ of degree less than or equal to $N = [n(\frac{1}{p} - 1)]$. The space $H^p_a(G)$, $0 < p \leq 1$, is the space of all $f \in \mathscr{S}'(G)$ having the form

$$f = \sum c_k a_k \quad \text{with} \sum |c_k|^p < \infty , \tag{2.10}$$

where each $a(x)$ is either a regular atom or an exceptional atom. The "norm" $\|f\|_{H^p_a}$ is the infimum of all expressions $(\sum |c_k|^p)^{1/p}$ for which we have such a representation of f. Various characterizations of Hardy spaces on compact Lie groups were studied in [BF]. In particular, we have $\|f\|_{H^p} \cong \|f\|_{H^p_a}$; hence, $H^p(G) = H^p_a(G)$. For this reason in the sequel we only need to prove Theorem 1.3 for the atomic H^p spaces.

Recall that difference operators δ^j are defined on sequences by

$$\delta^1(a_n) = a_{n+1} - a_n , \qquad \delta^{j+1}(a_n) = \delta^1(\delta^j(a_n)) . \tag{2.11}$$

Given an l-tuple $J = (j_1, \ldots, j_l)$ of nonnegative integers, the partial difference operator δ^J is defined analogously on multisequences $\{m(\lambda)\}_{\lambda \in \widehat{G}}$.

Finally, in the remaining portion of this paper, the letter C will denote (possibly different) constants that are independent of the essential variables in the argument; this independence will be clear from the context.

3. Some lemmas

Take radial functions η, $\psi \in C^\infty(\mathfrak{t})$ with

(3.1)
$$0 \le \eta \le 1, \qquad \eta(H) = 1 \quad \text{for } \tfrac{1}{2} \le |H| \le 2, \qquad \operatorname{supp}\eta \subset \{\tfrac{1}{4} \le |H| \le 4\};$$

(3.2)
$$0 \le \psi \le 1, \qquad \operatorname{supp}\psi \subset \{\tfrac{1}{2} \le |H| \le 2\}, \qquad \sum_{j=-\infty}^{\infty} \psi(2^{-j}H) = 1.$$

Then $\psi\eta = \psi$, and for any atom $a(x)$

$$Ta(x) = \sum_{j=-\infty}^{\infty} f_j * b_j(x) = \sum_{j=0}^{\infty} f_j * b_j(x), \tag{3.3}$$

where

$$f_j(x) = \sum_{\lambda\in\widehat{G}} d_\lambda m(\lambda)\eta\left(\frac{|\lambda+\beta|}{2^j}\right)\chi_\lambda(x),$$

$$b_j(x) = \sum_{\lambda\in\widehat{G}} d_\lambda \psi\left(\frac{|\lambda+\beta|)}{2^j}\right) Tr(\hat{a}(\lambda)U_\lambda(x)).$$

Thus $f_j = b_j = 0$ if $j < 0$. For these f_j's and $b_j's$ we have the following lemmas.

Lemma 3.4. *Suppose $m(\lambda)$ satisfies* (1.4) *with s being the smallest even integer such that $s > \frac{n}{p} - \frac{n}{2}$. Then*

$$\|f_j\|_1 \le C, \tag{3.5}$$

$$\int_G |f_j(x)|^2[1 + (2^j|x|)^2]^s\,dx \le C2^{jn}, \tag{3.6}$$

where $|x| = d(x, e)$ is the distance between x and e.

Proof. First we introduce a function $\gamma(x)$ which was defined in [W]. Let $|W|$ be the order of the Weyl group. For $x \in G$, x conjugate to a point $\exp(\xi^{-1}\tau)$, $\tau \in \mathfrak{t}^*$, we set $\gamma(x) = (\sum_{w\in W} e^{i\langle w(\beta),\tau\rangle}) - |W|$. Then $\gamma(x) \sim |x|^2$ (see [W, Lemma 9]). Now by Hölder's inequality

$$\|f_j\|_1 \le \left\{\int_G |f_j(x)|^2[1 + (2^j|x|)^2]^s\,dx\right\}^{1/2} \left\{\int_G [1 + (2^j|x|^2]^{-s}\,dx\right\}^{1/2} = I_1 \times I_2.$$

We estimate I_1 and I_2 separately:

$$\begin{aligned}(I_1)^2 &= \sum_{k=0}^{s} C_k \int_G (2^j|x|)^{2k}|f_j(x)|^2\,dx \\ &\le \sum_{k=0}^{s} C_k(2^j)^{2k} \int_G \gamma(x)^k|f_j(x)|^2\,dx = \sum_{k=0}^{s} C_k(2^j)^{2k}\|\gamma^{k/2}f_j\|_2^2.\end{aligned}$$

Using [C, Corollary (20) and Lemma 10] (together with the corollaries following that lemma), we have

$$\begin{aligned}(I_1)^2 &\le C\sum_{k=0}^{2} C_k(2^j)^{2k}\sum_{\lambda\in\widehat{G}}\left[\delta^{k/2}\left(d_\lambda m(\lambda)\eta\left(\frac{|\lambda+\beta|}{2^j}\right)\right)\right]^2\\ &\le C\sum_{k=0}^{s} C_k 2^{2kj}\sum_{2^{j-1}<|\lambda+\beta|<2^{j+1}}\left[\delta'(m(\lambda))\delta''(|\lambda|^{(n-l)/2})\delta'''\left(\eta\left(\frac{|\lambda+\beta|}{2^j}\right)\right)\right]^2\\ &\le C\sum_{k=0}^{s} C_k(2^j)^{2k}[(2^j)^{(n-l)/2-q''-q'''}]^2\sum_{2^{j-1}\le|\lambda+\beta|<2^{j+1}}(\delta'(m(\lambda)))^2\\ &\le C\sum_{k=0}^{s} C_k(2^j)^{2k+n-l-2q''-2q'''+l-2q'}\le C2^{jn},\end{aligned}$$

where δ', δ'', δ''' are partial difference operators of order q', q'', q''' and $q'+q''+q'''=k$. This proves (3.6), but it is easy to check that $(I_2)^2\le C2^{-jn}$. So Lemma 3.4 is proved.

Lemma 3.7. *Let $a(x)$ be a regular (p,∞,N) atom with* $\operatorname{supp} a\subset B(e,\rho)$ *and $r\in\mathbb{N}^+$. Suppose that $i_0\in\mathbb{N}$ is such that 2^{i_0} is sufficiently large. Then for $j\ge i_0$*

$$|b_j(x)|\le\begin{cases} C\rho^{1+n+N-n/p}2^{j(N+n+1-r)}(d(x,e)^{-r}+\Delta^{(j)}(x)^{-1}) \\ \qquad\qquad \textit{if } d(x,e)>2^{-j}+\rho \textit{ and } 2^j\rho\le 1,\\ C\rho^{-n/p+n+N+1}2^{j(N+n+1)} \\ \qquad\qquad \textit{if } d(x,e)<2^{-j+2}-\rho \textit{ and } 2^j\rho\le 1,\\ C\rho^{-n/p+n}2^{j(n-r)}[d(x,e)^{-r}+M(\Delta^{(j)-1})(x)] \\ \qquad\qquad \textit{whenever } 2^j\rho>1 \textit{ and } d(x,e)>2\rho,\end{cases}$$

where

$$\Delta^{(j)}(x)=\Delta^{(j)}(\exp H)=\prod_{\alpha\in A}\sup(2^{-j},\sin\alpha(H)/2) \tag{3.8}$$

and $M(\Delta^{(j)-1})$ is the Hardy-Littlewood maximal function of $(\Delta^{(j)})^{-1}$.

Proof. Observe that $b_j=a*\Psi_j$, where

$$\Psi_j(x)=\sum_{\lambda\in\widehat{G}} d_\lambda\psi\left(\frac{|\lambda+\beta|}{2^j}\right)\chi_\lambda(x). \tag{3.9}$$

A similar argument to the proof of (6.2) in [C] gives the lemma.

Lemma 3.10. *Let $a(x)$ be either an exceptional atom or a (p,∞,N) atom for $0<p\le 1$. If $(Tf)^\wedge(\lambda)=m(\lambda)\hat{f}(\lambda)$ with $m(\lambda)$ satisfying (1.4), then $\|Ta\|_p\le C$, where C is a constant independent of $a(x)$ and $N=[n(\frac{1}{p}-1)]$.*

Proof. We know from [W] that this T is a bounded operator in $L^2(G)$. Thus, the lemma follows for any exceptional atom $a(x)$. For a regular (p,∞,N) atom $a(x)$ without loss of generality we can assume that $\operatorname{supp} a\subset B(e,\rho)$. Take a positive integer $j_0=j_0(\rho)$ such that $2^{j_0}\rho\le\varepsilon_0/4<2^{j_0+1}\rho$. Here ε_0 is a positive number lying in the interval $(0,1)$ such that $\exp^{-1}\cdot L_{x^{-1}}$ is an analytic diffeomorphism of $B(x, \varepsilon_0)$ onto $B(0, \varepsilon_0)$, a ball centered at the origin of $\mathscr{g}$. (L_x is the left translation by x.)

Let

$$\begin{aligned} U_0 &= \{x \in G : d(x, e) \leq 2\rho\}, \\ U_k &= \{x \in G : 2^k\rho \leq d(x, e) < 2^{k+1}\rho\} \quad \text{for } k = 1, 2, \dots, j_0, \\ U &= \{x \in G : d(x, e) > 2^{j_0-2}\rho\}. \end{aligned}$$

By Hölder's inequality we have

$$\|Ta\|_p \leq C\left\{\sum_{k=0}^{j_0(\rho)}\left(\int_{U_k}|Ta(x)|\,dx\right)^p(2^k\rho)^{n(1-p)} + \left(\int_U |Ta(x)|\,dx\right)^p\right\}^{1/p}.$$

Fix an $i_0 \in \mathbb{N}$ so that 2^{i_0} is as large as in Lemma 3.7. Then we need only show

$$\sigma = \sum_{k=i_0}^{j_0(\rho)}\left(\int_{U_k}|Ta(x)|\,dx\right)^p(2^k\rho)^{n(1-p)} \leq C, \tag{3.11}$$

$$\int_U |Ta(x)|\,dx \leq C \tag{3.12}$$

with C independent of a. We will only give a proof of (3.11) and (3.12) can be proved in a similar way. By (3.3) we have $\sigma \leq \sum_{i=1}^3 \sigma_i$, where

$$\sigma_1 = \sum_{k=i_0}^{j_0(\rho)}\sum_{j=0}^{i_0-1}\left(\int_{U_k}|f_j * b_j(x)|\,dx\right)^p(2^k\rho)^{n(1-p)}, \tag{3.13}$$

$$\sigma_2 = \sum_{j=i_0}^{j_0(\rho)}\sum_{k=i_0}^{j_0(\rho)}\left(\int_{U_k}|f_j * b_j(x)|\,dx\right)^p(2^k\rho)^{n(1-p)}, \tag{3.14}$$

$$\sigma_3 = \sum_{j=j_0+1}^{\infty}\sum_{k=i_0}^{j_0(\rho)}\left(\int_{U_k}|f_j * b_j(x)|\,dx\right)^p(2^k\rho)^{n(1-p)}. \tag{3.15}$$

We can assume that $\rho \leq 1$. Observe that $\|X^J\Psi_j\|_\infty \leq C$ for $j \leq i_0$. As a result we have $\|b_j\|_\infty \leq C$, and then by Lemma 3.4, $\sigma_1 \leq C$.

For σ_2 we observe that $i_0 \leq j \leq j_0(\rho)$. Hence $2^j\rho < 1$. Let $\widetilde{U}_k$ be the union of U_{k-1}, U_k, and U_{k+1}. Then for $x \in U_k$

$$\begin{aligned} |f_j * b_j(x)| \leq{}& \sum_{l=0}^{k-2}\int_{U_l}|f_j(y)b_j(y^{-1}x)|\,dy + \int_{\widetilde{U}_k}|f_j(y)b_j(y^{-1}x)|\,dy \\ &+ \sum_{l=k+2}^{j_0(\rho)}\int_{U_l}|f_j(y)b_j(y^{-1}x)|\,dy \int_U |f_j(y)b_j(y^{-1}x)|\,dy \\ ={}& \sum_{i=1}^4 I_i(x). \end{aligned}$$

$$\int_{U_k} I_1(x)\,dx \leq \sum_{l=0}^{k-2}\int_{U_l}|f_j(y)|\left(\int_{U_k}|b_j(y^{-1}x)|\,dx\right)dy. \tag{3.16}$$

Notice that $y \in U_l$ $(l \le k-2)$ and $x \in U_k$ imply that $d(y^{-1}x, e) > \rho 2^{k-1}$. We estimate (3.16) in the following two cases.

Case (i). $2^{k-1}\rho > 2^{-j+1}$. In this case by Lemma 3.7 we have

$$\int_{U_k} |b_j(y^{-1}x)|\,dx \\ \le C(2^j\rho)^{n+N+1-n/p}(2^j)^{n/p-r}\int_{U_k}(d(y^{-1}x, e)^{-r} + \Delta^{(j)}(y^{-1}x)^{-1})\,dx.$$

By the Weyl integral formula

$$\int_G |\Delta^{(j)}(x)|^{-2}\,dx \le \int_Q |D(H)|^2\Delta^{(j)}(\exp H)^{-2}\,dH \le 1,$$

where $D(H) = \prod_{\alpha\in A}\sin\alpha(H)/2$ is the Weyl denominator and Q is the fundamental domain of $\mathbb{T}$ containing 0. Hence, owing to $2^k\rho \le 1$,

$$\int_{U_k}|b_j(y^{-1}x)|\,dx \le C(2^j\rho)^{n+N+1-n/p}(2^j)^{n/p-r}\{(2^k\rho)^{n-r} + (2^k\rho)^{n/2}\} \\ \le C(2^j\rho)^{-n/p+n+N+1}2^{j(n/p-r)}(2^k\rho)^{-r+n}.$$

Put $k_0 = |\ln(2^j\rho)|/\ln 2$. Then $2^{k_0} \cong (2^j\rho)^{-1}$ and

$$\sum_{\substack{k=i_0\\ 2^{k-1}\rho>2^{1-j}}}^{j_0(\rho)} (2^k\rho)^{n-rp} \le C\sum_{k=k_0}^{j_0(\rho)}(2^k\rho)^{n-rp} \le C2^{jp(r-n/p)}.$$

Thus by Lemma 3.4,

$$\sum_{j=i_0}^{j_0(\rho)}\sum_{\substack{k=i_0\\ 2^{k-1}\rho>2^{1-j}}}^{j_0(\rho)} (2^k\rho)^{n(1-p)}\left(\int_{U_k} I_1(x)\,dx\right)^p \le C\sum_{j=i_0}^{j_0(\rho)}(2^j\rho)^{p(1+n+N-n/p)} \le C.$$

Case (ii). $2^{k-1}\rho \le 2^{-j+1}$. In this case we write

$$\int_{U_k}|b_j(y^{-1}x)|\,dx = \int_{U_k\cap\{d(y^{-1}x, e)\le 2^{-j+1}\}} + \int_{U_k\cap\{d(y^{-1}x, e)>2^{-j+1}\}} = J_1 + J_2.$$

By Lemma 3.7,

$$J_1 \le C\rho^{N+n+1-n/p}2^{j(N+n+1)}(2^{-j})^n = C(2^j\rho)^{n+N+1-n/p}(2^j)^{-n+n/p}$$

and J_2 is bounded by

$$C\rho^{n+N+1-n/p}(2^j)^{N+n+1-r}\int_{U_k\cap\{d(y^{-1}x, e)>2^{-j+1}\}}(d(y^{-1}x, e)^{-r} + \Delta^{(j)}(y^{-1}x)^{-1})\,dx \\ \le C\rho^{n+N+1-n/p}(2^j)^{n+N+1-r}\{(2^{-j})^{-r}(2^k\rho)^n + (2^k\rho)^{n/2}\} \\ \le C\rho^{n+N+1-n/p}(2^j)^{n+N+1-r}\{(2^{-j})^{-r+n}(2^{-j})^{n/2}\} \\ \le C(2^j\rho)^{N+n+1-n/p}(2^j)^{-n+n/p}.$$

If $p < 1$, then

$$\sum_{\substack{k=i\\ 2^{k-1}\rho\le 2^{1-j}}}^{j_0(\rho)} (2^k\rho)^{n(1-p)} \le \sum_{k=i_0}^{k_0}(2^{n(1-p)})^k\rho^{n(1-p)} \le C2^{jp(n-n/p)}.$$

If $p = 1$, then

$$\sum_{\substack{k=i \\ 2^{k-1}\rho \le 2^{-j+1}}}^{j_0(\rho)} (2^k\rho)^{n(1-p)} \le \sum_{k=i_0}^{k_0} (2^k\rho)^{-1/2}\sqrt{2}^{-j} \le C\rho^{-1/2}\sqrt{2}^{-j}.$$

Therefore, for $0 < p \le 1$ we have, by Lemma 3.4 and the choice of $j_0(\rho)$,

$$\sum_{j=i_0}^{j_0(\rho)} \sum_{\substack{k=i_0 \\ 2^{k-1}\rho \le 2^{1-j}}}^{j_0(\rho)} (2^k\rho)^{n(1-p)} \left(\int_{U_k} I_1(x)\,dx\right)^p \le C$$

and

$$\sum_{j=i_0}^{j_0(\rho)} \sum_{k=i_0}^{j_0(\rho)} (2^k\rho)^{n(1-p)} \left(\int_{U_k} I_1(x)\,dx\right)^p \le C. \tag{3.17}$$

Similarly we have for $0 < p \le 1$

$$\sum_{j=i_0}^{j_0(\rho)} \sum_{k=i_0}^{j_0(\rho)} (2^k\rho)^{n(1-p)} \left(\int_{U_k} I_3(x)\,dx\right)^p \le C. \tag{3.18}$$

Also notice

$$\int_{U_k} I_2(x)\,dx \le \int_{\widetilde{U}_k} |f_j(y)| \left(\int_{U_k} |b_j(y^{-1}x)|\,dx\right) dy.$$

By Lemma 3.7 and the Weyl integral formula we have for $x \in U_k$, $y \in \widetilde{U}_k$

$$\begin{aligned} \int_{U_k} |b_j(y^{-1}x)|\,dx &= \int_{U_k \cap \{d(y,x) < 2^{-j+1}\}} + \int_{U_k \cap \{d(y,x) \ge 2^{-j+1}\}} \\ &\le C\rho^{1+n+N-n/p}(2^j)^{n+N+1}(2^{-j})^n + C\rho^{n+N+1-n/p}(2^j)^{n+N+1-r} \\ &\quad \times \int_{U_k \cap \{d(y,x) \ge 2^{-j+1}\}} (d(y,x)^{-r} + \Delta^{(j)}(y^{-1}x))\,dx \\ &\le C(2^j\rho)^{n+N+1-n/p}\{(2^j)^{-n+n/p} + (2^j)^{-r+n/p}(2^{j(r-n)} + (2^k\rho)^{n/2})\} \\ &\le C(2^j\rho)^{n+1+N-n/p}2^{j(-n+n/p)}. \end{aligned} \tag{3.19}$$

By Lemma 3.4 we have

$$\int_{\widetilde{U}_k} |f_j(y)|\,dy \le C2^{j(-s+n/2)}(2^k\rho)^{-s+n/2}. \tag{3.20}$$

Thus for $2^k\rho > 2^{-j}$

$$\begin{aligned} &\sum_{j=i_0}^{j_0(\rho)} \sum_{\substack{k=i_0 \\ 2^k\rho > 2^{-j}}}^{j_0(\rho)} \left(\int_{U_k} I_2(x)\,dx\right)^p (2^k\rho)^{n(1-p)} \\ &\quad \le C\sum_{j=i_0}^{j_0(\rho)} (2^j\rho)^{(n+N+1-n/p)p}2^{jp(n/p-s-n/2)} \sum_{k=k_0}^{j_0(\rho)} (2^k\rho)^{p(n/p-s-n/2)} \le C. \end{aligned}$$

For $2^k\rho \le 2^{-j}$ the same estimate can be obtained by using the fact that $\|f_j\|_1 \le C$ instead of (3.20). Therefore,

$$\sum_{j=i_0}^{j_0(\rho)} \sum_{k=i_0}^{j_0(\rho)} (2^k\rho)^{n(1-p)} \left(\int_{U_k} I_2(x)\,dx \right)^p \le C. \tag{3.21}$$

As in the proof of Lemma 3.4, we can show that

$$\int_U |f_j(x)|\,dx \le C 2^{j(-s+n/2)}$$

and then obtain

$$\sum_{j=i_0}^{j_0(\rho)} \sum_{k=i_0}^{j_0(\rho)} (2^k\rho)^{n(1-p)} \left(\int_{U_k} I_4(x)\,dx \right)^p \le C. \tag{3.22}$$

The estimates (3.17), (3.18), (3.21), and (3.22) gives $\sigma_2 \le C$. Similarly we can prove $\sigma_3 \le C$. Estimate (3.11) is therefore proved.

4. Proof of the main theorem

Now we are ready to prove Theorem 1.3. By the atomic decomposition of Hardy spaces it is enough to prove that

$$\|Ta\|_{H^p} \le C \tag{4.1}$$

uniformly for each regular (p, ∞, N) atom $a(x)$.

In order to prove (4.1) we will introduce the generalized Riesz transforms on compact Lie groups which were studied in [BX]. For an integer $L \ge 0$ and a multi-index $J = (j_1, \dots, j_L) \in \{0, 1, \dots, n\}^L$ let $R_J(f)$ denote the generalized transform $R_J(f) = R_{j_1} \cdots R_{j_L} f$, where $R_j(f)$ is the jth Riesz transform of f if $j \ne 0$ and $R_0 f = f$. We can prove (see [BX] for the proof) that for $p > (n-1)/(n-1+L)$ and all $f \in L^2 \cap H^p$

$$\sum_J \|R_J(f)\|_p \cong \|f\|_{H^p}, \qquad \|R_J(f)\|_{H^p} \le C\|f\|_{H^p}. \tag{4.2}$$

The proof of Theorem 1.3 is now easy to obtain from (4.2) and Lemma 3.10. For any $p \in (0, 1]$ take L large enough so that $p > (n-1)/(n-1+L)$. For any (p, ∞, N) atom $a(x)$ we have the atomic decomposition of $R_J(a)$, $R_J(a) = \sum \lambda_i b_i$, where the b_i's are atoms and $\sum |\lambda_j|^p \cong R_J(a)|_{H^p}^p \le C$. Now notice that both T and R_J are convolution operators, so by (4.2) and Lemma 3.10 we have

$$\|Ta\|_{H^p}^p \le C \sum_J \|T(R_J(a))\|_p^p \le C \sum_i |\lambda_i|^p \|Tb_i\|_p^p \le C.$$

Thus Theorem 1.3 is proved. We now easily obtain an application as follows.

Theorem 4.3. *Suppose T is the bi-invariant operator associated to the multiplier $m(\lambda) = \|\lambda + \beta\|^{ia}$ for some $a \in \mathbb{R}$. Then T is bounded in H^p spaces for any $0 < p \le 1$.*

5. A theorem on SU(2)

The operators studied above are all bi-invariant (central operators). But it is also possible to study some kind of non-bi-invariant operators including the well-known Riesz transforms (see [S] for the definitions). For the sake of simplicity we will study this case on the simplest compact Lie group $SU(2)$, but the idea should work on general compact Lie groups. Recall X_i $(i = 1, 2, 3)$ is an orthonormal basis on the Lie algebra of $SU(2)$. For any bi-invariant operator T associated with a multiplier $m(\lambda)$ we write $Tf = f * K$ and define a new operator $\mathscr{T}_j f = f * X_j K$. Then we have the following theorem.

Theorem 5.1. *Let s be the smallest even integer such that $s > \frac{3}{p} - \frac{3}{2}$. Suppose that $m(\lambda) \in L^\infty$ and that for all integer J with $0 \le J \le s$ and all $R > 0$*

$$\sum_{R \le |\lambda| \le 2R} |\delta^J m(\lambda)|^2 \le C R^{-2J-1}. \tag{5.2}$$

Then $\mathscr{T}_j$ is a linear bounded operator on $H^p(SU(2))$ $(0 < p \le 1)$.

Proof. On $SU(2)$ the kernel of this operator $\mathscr{T}_j$ is

$$\mathscr{K}(x) = \sum_{\lambda \ge 1} m(\lambda) \lambda X_j \chi_\lambda(x). \tag{5.3}$$

By [M] we know

$$X_j \chi_\lambda(x) = \{(\lambda + 1)\chi_{\lambda-1}(x) - (\lambda - 1)\chi_{\lambda+1}(x)\} E(x), \tag{5.4}$$

where $E(x) = X_j \chi_3(x)\{3 - \chi_3(x)\}^{-1} \cong |x|^{-1}$. Thus a simple computation shows

$$\begin{aligned} \mathscr{K}(x) = & \sum_{\lambda \ge 2} (m(\lambda + 1) - m(\lambda))(\lambda^2 + 2)\chi_\lambda(x) E(x) \\ & + \sum_{\lambda \ge 2} (m(\lambda) - m(\lambda - 1))(\lambda^2 + 2)\chi_\lambda(x) E(x) \\ & + \sum_{\lambda \ge 2} 3\lambda (m(\lambda + 1) - m(\lambda - 1))\chi_\lambda(x) E(x) + O(|x|^{-1}). \end{aligned} \tag{5.4}$$

Now the proof of the theorem is easily obtained by mimicking the proofs of Lemmas 3.4 and 3.10 and (4.1). We leave these details to the reader.

Acknowledgment

We thank the referee for his most helpful suggestions.

References

[BF] B. Blank and D. Fan, *H^p spaces on compact Lie groups* (submitted).

[BS] A. Baernstein and E. T. Sawyer, *Embedding and multiplier theorems for $H^p(\mathbf{R}^n)$*, Mem. Amer. Math. Soc., No. 318, Amer. Math. Soc., Providence, RI, 1985.

[BX] W. R. Bloom and Zengfu Xu, *Approximation of H^p functions by Bochner-Riesz means on compact Lie groups*, Math. Z. (to appear).

[C] J. L. Clerc, *Bochner-Riesz means of H^p functions $(0 < p < 1)$ on compact Lie groups*, Lecture Notes in Math., vol. 1243, Springer, New York, 1987, pp. 86–107.

[CT] A. P. Calderon and A. Torchinsky, *Parabolic maximal functions associated with a distribution* II, Adv. Math. **24** (1977), 101–171.

[Co] L. Colzani *Hardy space on sphere*, Ph.D. Thesis, Washington Univ., St. Louis, 1982.

[CW] R. Coifman and G. Weiss, *Theoremes sur les multiplicateurs de Fourier sur* $SU(2)$ *et* Σ_2, C. R. Acad. Sci. Paris Sér I. Math. **271** (1970).

[CW1] ——, *Extension of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.

[FS] C. Fefferman and E. Stein, *H^p space of several variable*, Acta Math. **129** (1972), 137–193.

[L] R. H. Latter, *A characterization of $H^p(\mathbb{R}^n)$ in term of atoms*, Studia Math. **62** (1978), 93–101.

[M] R. Mayer, *Fourier series on differentiable functionson* SU(2), Duke Math. J. **34** (1967), 549–554.

[S] E. M. Stein, *Topics in harmonic analysis*, Ann. of Math. Studies, vol. 63, Princeton Univ. Press, Princeton, NJ, 1970.

[W] N. J. Weiss, *L^p estimate for bi-invariant operators on compact Lie groups*, Amer. J. Math. **94** (1972), 103–118.

[X] Zengfu Xu, *The generalized Abel means of H^p functions on compact Lie groups*, Chinese Ann. Math. Ser. A **13** (1992), 101–110.

Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201

E-mail address: `fan@csd4.csd.uwm.edu`

Mathematics-MPS, Murdoch University, Murdoch, Western Australia, 6150, Australia

E-mail address: `zengfu@csuvax1.murdoch.edu.au`