

CESÀRO MEANS OF FOURIER TRANSFORMS AND MULTIPLIERS ON $L^1(\mathbf{R})$

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ABSTRACT. We prove that the Cesàro mean σ of a multiplier λ on $L^1(\mathbf{R})$ is also a multiplier on $L^1(\mathbf{R})$. In the particular cases when (i) λ is odd, we prove that σ is the Fourier transform of an odd function in the Hardy space $H^1(\mathbf{R})$, and (ii) λ is even, we give a necessary and sufficient condition in order that σ be a Fourier transform of an even function in $L^1(\mathbf{R})$. As a corollary, we obtain a nontrivial condition for λ in order to be a multiplier on $L^1(\mathbf{R})$; namely,

$$\int_0^\infty \left| \frac{1}{t} \int_0^t \{\lambda(\xi) - \lambda(-\xi)\} d\xi \right| \frac{dt}{t} < \infty.$$

We also prove Hardy type inequalities for multipliers and Hilbert transforms.

1. PRELIMINARY RESULTS

Let f be a function defined on the real line $\mathbf{R} := (-\infty, \infty)$. We recall that if f is Lebesgue integrable over \mathbf{R} , in sign: $f \in L^1(\mathbf{R})$, then the Fourier transform of f is defined by

$$\hat{f}(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-itx} dx, \quad t \in \mathbf{R},$$

while the Hilbert transform of f is defined as an improper integral by

$$\tilde{f}(x) := -\frac{1}{\pi} \lim_{\delta \downarrow 0} \int_{\delta}^{\infty} \frac{f(x+t) - f(x-t)}{t} dt.$$

Hille and Tamarkin [5] proved that $\tilde{f}(x)$ exists for almost all x in \mathbf{R} . They also proved that if $\hat{f} \in L^1(\mathbf{R})$, then we necessarily have

$$(1.1) \quad \int_0^\infty \frac{|\hat{f}(t)|}{t} < \infty.$$

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We note that (1.1) is the extension of Hardy's inequality (see, e.g., [7, Volume 1, p. 286]) from Fourier series to Fourier transforms. To honor Hardy, by $H^1(\mathbf{R})$ we denote the space of functions f in $L^1(\mathbf{R})$ whose Hilbert transform \hat{f} is also in $L^1(\mathbf{R})$.

We say that a function λ , measurable and bounded on \mathbf{R} , is a multiplier on $L^1(\mathbf{R})$ if for every $f \in L^1(\mathbf{R})$ there exists a function $f_1 \in L^1(\mathbf{R})$ such that

$$\lambda(t)\hat{f}(t) = \hat{f}_1(t), \quad t \in \mathbf{R}.$$

As is known (see, e.g., [1, p. 269]), a necessary and sufficient condition for a function λ to be a multiplier on $L^1(\mathbf{R})$ is that λ is the Fourier-Stieltjes transform of a function μ of bounded variation on \mathbf{R} ; i.e.,

$$(1.2) \quad \lambda(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} d\mu(x), \quad t \in \mathbf{R}.$$

We remind the reader that the integral on the right-hand side of (1.2) is a Lebesgue-Stieltjes integral (see, e.g., [6, Chapter 1]), while $\mu(x)$ is a complex-valued function of bounded variation, i.e., whose total variation over \mathbf{R} is finite (see, e.g., [6, Chapter 6]). Then the one-sided limits $\mu(x-0)$ and $\mu(x+0)$ exist at each $x \in \mathbf{R}$, and the limits

$$\mu(\infty) := \lim_{x \rightarrow \infty} \mu(x) \quad \text{and} \quad \mu(-\infty) := \lim_{x \rightarrow -\infty} \mu(x)$$

also exist and are finite. For the sake of definiteness, we always assume that μ is continuous on the left. (Without loss of generality, we may also assume that $\mu(-\infty) = 0$.)

2. MAIN RESULTS: CESÀRO MEANS OF MULTIPLIERS ON $L^1(\mathbf{R})$

We distinguish between the cases of odd and even multipliers.

Case (i). Let λ be an odd multiplier on $L^1(\mathbf{R})$. Then there exists a function μ of bounded variation on \mathbf{R} such that (1.2) is satisfied. An integration by substitution gives

$$(2.1) \quad \lambda(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} d\mu(-x).$$

The oddness of λ implies that μ is also odd: $\mu(-x) = -\mu(x)$, $x \in \mathbf{R}$. In particular, $\mu(0) = 0$.

We define the Cesàro mean of λ by

$$(2.2) \quad \sigma(t) := \frac{1}{t} \int_0^t \lambda(\xi) d\xi, \quad t \in \mathbf{R}, \quad t \neq 0.$$

Clearly, $\sigma(t)$ is also an odd function. We will prove the following

Theorem 1. *The Cesàro mean of an odd multiplier λ on $L^1(\mathbf{R})$ is the Fourier transform of an odd function in $H^1(\mathbf{R})$. In particular, it is also a multiplier on $L^1(\mathbf{R})$.*

Combining Theorem 1 with (1.1) yields the following

Corollary 1. *If λ is an odd multiplier on $L^1(\mathbf{R})$, then*

$$\int_0^\infty \left| \frac{1}{t} \int_0^t \lambda(\xi) d\xi \right| \frac{dt}{t} < \infty.$$

If, in addition, $\lambda(t) \geq 0$ for $t \geq 0$, then $\int_0^\infty \frac{\lambda(\xi)}{\xi} d\xi < \infty$.

The next corollary is due to the fact that if λ is a multiplier on $L^1(\mathbf{R})$, then $\lambda(t) - \lambda(-t)$ is an odd multiplier on $L^1(\mathbf{R})$.

Corollary 2. *If λ is a multiplier on $L^1(\mathbf{R})$, then*

$$\int_0^\infty \left| \frac{1}{t} \int_0^t \{\lambda(\xi) - \lambda(-\xi)\} d\xi \right| \frac{dt}{t} < \infty.$$

Case (ii). Now let λ be an even multiplier on $L^1(\mathbf{R})$. Then there exists a function μ of bounded variation on \mathbf{R} such that (1.2) is satisfied. An integration by substitution gives (2.1). The evenness of λ implies that μ is also even: $\mu(-x) = \mu(x)$, $x \in \mathbf{R}$.

We consider the Cesàro mean σ of λ defined by (2.2). This time σ is an even function. We will prove the following

Theorem 2. *The Cesàro mean σ of an even multiplier λ on $L^1(\mathbf{R})$ is also a multiplier on $L^1(\mathbf{R})$. Furthermore, σ is a Fourier transform if and only if $\mu(x)$ is continuous at $x = 0$, where μ is the function of bounded variation associated with λ according to (1.2).*

Since any multiplier can be written as the sum of an even and an odd multiplier, from Theorems 1 and 2 we conclude the following

Corollary 3. *The Cesàro mean of a multiplier on $L^1(\mathbf{R})$ is also a multiplier on $L^1(\mathbf{R})$.*

3. PROOFS OF THEOREMS 1 AND 2

We begin with an auxiliary result on Lebesgue-Stieltjes integrals.

Lemma. *If a function ν is continuous and μ is nondecreasing on the interval $[0, \infty)$ with $\mu(\infty) < \infty$, then*

$$(3.1) \quad \lim_{\delta \downarrow 0} \int_\delta^\infty \nu(t) d\mu(t) = \int_0^\infty \nu(t) d\mu(t) - \nu(0)\{\mu(+0) - \mu(0)\}.$$

Proof. Consider an approximating sum for the integral on the left-hand side of (3.1):

$$(3.2) \quad \nu(\delta)\{\mu(\delta_1) - \mu(\delta)\} + \nu(\delta_1)\{\mu(\delta_2) - \mu(\delta_1)\} + \dots$$

where $0 < \delta < \delta_1 < \delta_2 < \dots$ is a partition of the interval $[\delta, \infty)$. If we add the extra term $\nu(0)\{\mu(\delta) - \mu(0)\}$ to (3.2), then we obtain an approximating sum for the integral on the right-hand side of (3.1). Now letting δ tend to 0 and at the same time making the partition finer and finer, we conclude that

$$\lim_{\delta \downarrow 0} \left\{ \nu(0)\{\mu(\delta) - \mu(0)\} + \int_\delta^\infty \nu(t) d\mu(t) \right\} = \int_0^\infty \nu(t) d\mu(t).$$

This is equivalent to (3.1) to be proved.

Proof of Theorem 1. First we prove that σ is the Fourier transform of an odd function in $L^1(\mathbb{R})$. Indeed, the function in question is defined by

$$(3.3) \quad h(x) := \begin{cases} \int_x^\infty \frac{d\mu(\xi)}{\xi} & \text{if } x > 0, \\ -h(-x) & \text{if } x < 0, \end{cases}$$

and let $h(0) = 0$. Clearly, $h(x)$ is odd.

We show that $h \in L^1(\mathbb{R})$. By Fubini's theorem,

$$(3.4) \quad \begin{aligned} \int_0^\infty |h(x)| dx &\leq \int_0^\infty dx \int_x^\infty \frac{d|\mu|(\xi)}{\xi} \\ &= \int_0^\infty \frac{d|\mu|(\xi)}{\xi} \int_0^\xi dx = \int_0^\infty d|\mu|(\xi) < \infty. \end{aligned}$$

Here and in the sequel, by $|\mu|(x)$ we denote the total variation of μ over $(-\infty, x]$.

Second we prove that σ is the Fourier transform of h . By Fubini's theorem and the lemma,

$$(3.5) \quad \begin{aligned} \hat{h}(t) &= -i\sqrt{\frac{2}{\pi}} \int_0^\infty h(x) \sin tx dx = -i\sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \int_x^\infty \frac{d\mu(\xi)}{\xi} \right\} \sin tx dx \\ &= -i\sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \int_0^\xi \frac{\sin tx}{\xi} dx \right\} d\mu(\xi) = -i\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1 - \cos t\xi}{t\xi} d\mu(\xi). \end{aligned}$$

On the other hand,

$$(3.6) \quad \begin{aligned} \sigma(t) &= -i\sqrt{\frac{2}{\pi}} \frac{1}{t} \int_0^t \left\{ \int_0^\infty \sin \xi x d\mu(x) \right\} d\xi \\ &= -i\sqrt{\frac{2}{\pi}} \frac{1}{t} \int_0^\infty \left\{ \int_0^t \sin x\xi d\xi \right\} d\mu(x) \\ &= -i\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1 - \cos tx}{tx} d\mu(x). \end{aligned}$$

Comparing (3.5) and (3.6) shows that $\sigma(t) = \hat{h}(t)$, $t \in \mathbb{R}$.

Third we prove that \tilde{h} , the Hilbert transform of h , also belongs to $L^1(\mathbb{R})$. Without loss of generality, we may assume that μ is nondecreasing on \mathbb{R} . By definition,

$$(3.7) \quad \begin{aligned} \pi\tilde{h}(x) &= \lim_{\delta \downarrow 0} \int_\delta^\infty \frac{h(x-t) - h(x+t)}{t} dt \\ &= \lim_{\delta \downarrow 0} \int_\delta^{x/2} + \int_{x/2}^\infty =: H_1(x) + H_2(x), \quad \text{say.} \end{aligned}$$

Clearly, $H_1(x) \geq 0$ and $H_2(x) \geq 0$ for $x > 0$.

By (3.3)

$$(3.8) \quad H_1(x) = \lim_{\delta \downarrow 0} \int_0^{x/2} \frac{dt}{t} \int_{x-t}^{x+t} \frac{d\mu(\xi)}{\xi}.$$

Let $\delta > 0$ be fixed. By Fubini's theorem,

$$\begin{aligned} \int_{2\delta}^{\infty} dx \int_{\delta}^{x/2} \frac{dt}{t} \int_{x-t}^{x+t} \frac{d\mu(\xi)}{\xi} &= \int_{\delta}^{\infty} \frac{dt}{t} \int_{2t}^{\infty} dx \int_{x-t}^{x+t} \frac{d\mu(\xi)}{\xi} \\ &= \int_{\delta}^{\infty} \frac{dt}{t} \int_t^{\infty} \frac{d\mu(\xi)}{\xi} \int_{\max(\xi-t, 2t)}^{\xi+t} dx \\ &\leq 2 \int_{\delta}^{\infty} dt \int_t^{\infty} \frac{d\mu(\xi)}{\xi} = 2 \int_{\delta}^{\infty} \frac{d\mu(\xi)}{\xi} \int_{\delta}^{\xi} dt. \end{aligned}$$

By Fatou's lemma, $H_1 \in L^1(\mathbf{R})$ and

$$(3.9) \quad \int_0^{\infty} H_1(x) dx \leq 2\{\mu(\infty) - \mu(+0)\}.$$

Now we consider H_2 , which we decompose as

$$(3.10) \quad H_2(x) = - \int_{x/2}^{\infty} \frac{h(x+t)}{t} dt + \int_{x/2}^{\infty} \frac{h(x-t)}{t} dt =: \beta_1(x) + \beta_2(x), \quad \text{say.}$$

By (3.4) it is not hard to check that

$$(3.11) \quad \begin{aligned} \int_0^{\infty} |\beta_1(x)| dx &= \int_0^{\infty} dx \int_{3x/2}^{\infty} \frac{h(u)}{u-x} du \\ &= \int_0^{\infty} h(u) du \int_0^{2u/3} \frac{dx}{u-x} = \int_0^{\infty} h(u) du \int_{u/3}^u \frac{dt}{t} < \infty. \end{aligned}$$

By simple substitutions and manipulations,

$$(3.12) \quad \begin{aligned} \beta_2(x) &= \left\{ \int_{x/2}^x + \int_x^{\infty} \right\} \frac{h(x-t)}{t} dt \\ &= \int_{x/2}^x \frac{dt}{t} \int_{x-t}^{\infty} \frac{d\mu(\xi)}{\xi} - \int_x^{\infty} \frac{dt}{t} \int_{t-x}^{\infty} \frac{d\mu(\xi)}{\xi} \\ &= \int_0^{\infty} \frac{d\mu(\xi)}{\xi} \int_{\max(x-\xi, x/2)}^x \frac{dt}{t} - \int_0^{\infty} \frac{d\mu(\xi)}{\xi} \int_x^{x+\xi} \frac{dt}{t} \\ &= \int_0^{x/2} \ln \left(\frac{x}{x-\xi} \right) \frac{d\mu(\xi)}{\xi} + (\ln 2) \int_{x/2}^{\infty} \frac{d\mu(\xi)}{\xi} - \int_0^{\infty} \ln \left(\frac{x+\xi}{x} \right) \frac{d\mu(\xi)}{\xi} \\ &= \int_0^{x/2} \ln \left(\frac{x^2}{x^2-\xi^2} \right) \frac{d\mu(\xi)}{\xi} + (\ln 2) \int_{x/2}^{\infty} \frac{d\mu(\xi)}{\xi} - \int_{x/2}^{\infty} \ln \left(\frac{x+\xi}{x} \right) \frac{d\mu(\xi)}{\xi} \\ &=: \beta_3(x) + \beta_4(x) + \beta_5(x), \quad \text{say.} \end{aligned}$$

We begin with β_3 . Making use of Fubini's theorem, the substitution $x = \xi t$, and the inequality

$$(3.13) \quad \ln(1+u) \leq u, \quad u \geq 0,$$

we find

$$\begin{aligned}
 \int_0^\infty |\beta_3(x)| dx &= \int_0^\infty \frac{d\mu(\xi)}{\xi} \int_{2\xi}^\infty \ln\left(\frac{x^2}{x^2 - \xi^2}\right) dx \\
 (3.14) \qquad &= \int_0^\infty d\mu(\xi) \int_2^\infty \ln\left(\frac{t^2}{t^2 - 1}\right) dt \\
 &\leq \int_0^\infty d\mu(\xi) \int_2^\infty \frac{1}{t^2 - 1} dt < \infty.
 \end{aligned}$$

Since

$$(3.15) \qquad \beta_4(x) = (\ln 2)h(x/2), \quad x > 0,$$

and h is an odd function, by (3.4) we see that $\beta_4 \in L^1(\mathbf{R})$. Finally, applying Fubini's theorem and the substitutions $x = \xi t$ and $u = \frac{1}{t}$, in turn we get

$$\begin{aligned}
 \int_0^\infty |\beta_5(x)| dx &= \int_0^\infty \frac{d\mu(\xi)}{\xi} \int_0^{2\xi} \ln\left(\frac{x + \xi}{x}\right) dx \\
 (3.16) \qquad &= \int_0^\infty d\mu(\xi) \int_0^2 \ln\left(\frac{t+1}{t}\right) dt \\
 &= \int_0^\infty d\mu(\xi) \int_{1/2}^\infty \frac{\ln(1+u)}{u^2} du < \infty.
 \end{aligned}$$

Combining (3.7), (3.9)–(3.12), (3.14)–(3.16), and the fact that \tilde{h} is an even function (since h is odd), we conclude that $\tilde{h} \in L^1(\mathbf{R})$. This completes the proof of Theorem 1.

Proof of Theorem 2. First we prove that σ is the Fourier transform of an even function, up to an additive constant. In particular, it follows that σ is a multiplier on $L^1(\mathbf{R})$. In fact, the function in question is defined by

$$(3.17) \qquad g(x) := \begin{cases} \int_x^\infty \frac{d\mu(\xi)}{\xi} & \text{if } x > 0, \\ g(-x) & \text{if } x < 0 \end{cases}$$

(cf. (3.3)). Clearly, g is even. Similarly to (3.4), we deduce that g is in $L^1(\mathbf{R})$.

Second we consider \hat{g} , the Fourier transform of g ,

$$(3.18) \qquad \hat{g}(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos tx \, dx.$$

Again fix $\delta > 0$. By (3.17) and Fubini's theorem,

$$\begin{aligned}
 \int_\delta^\infty g(x) \cos tx \, dx &= \int_\delta^\infty \cos tx \, dx \int_x^\infty \frac{d\mu(\xi)}{\xi} = \int_\delta^\infty \frac{d\mu(\xi)}{\xi} \int_\delta^\xi \cos tx \, dx \\
 (3.19) \qquad &= \int_\delta^\infty \frac{\sin t\xi}{t\xi} d\mu(\xi) - \frac{\sin t\delta}{t} \int_\delta^\infty \frac{d\mu(\xi)}{\xi}.
 \end{aligned}$$

Without loss of generality, we may assume that μ is nondecreasing on $(0, \infty)$. Then for $0 < \delta < \delta_1$ we may write that

$$\begin{aligned}
 0 &\leq \frac{\sin t\delta}{t} \int_\delta^\infty \frac{d\mu(\xi)}{\xi} \leq \delta \left\{ \int_\delta^{\delta_1} + \int_{\delta_1}^\infty \right\} \frac{d\mu(\xi)}{\xi} \\
 &\leq \{\mu(\delta_1) - \mu(\delta)\} + o(1) \rightarrow 0 \quad \text{as } \delta \rightarrow 0.
 \end{aligned}$$

Thus, from (3.18), (3.19), and the lemma it follows that

$$\begin{aligned}
 \hat{g}(t) &= \sqrt{\frac{2}{\pi}} \lim_{\delta \downarrow 0} \int_{\delta}^{\infty} \frac{\sin t\xi}{t\xi} d\mu(\xi) \\
 (3.20) \qquad &= \sqrt{\frac{2}{\pi}} \left\{ \int_0^{\infty} \frac{\sin t\xi}{t\xi} d\mu(\xi) - \{\mu(+0) - \mu(0)\} \right\}.
 \end{aligned}$$

On the other hand, the Cesàro mean σ of g defined in (2.2) can be rewritten as

$$\begin{aligned}
 \sigma(t) &= \sqrt{\frac{2}{\pi}} \frac{1}{t} \int_0^t d\xi \int_0^{\infty} \cos \xi x d\mu(x) \\
 (3.21) \qquad &= \sqrt{\frac{2}{\pi}} \frac{1}{t} \int_0^{\infty} d\mu(x) \int_0^t \cos x\xi d\xi = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin tx}{tx} d\mu(x).
 \end{aligned}$$

Combining (3.20) and (3.21) gives

$$(3.22) \qquad \sigma(t) = \hat{g}(t) + \sqrt{\frac{2}{\pi}} \{\mu(+0) - \mu(0)\},$$

whence the necessity and sufficiency part in Theorem 2 is obvious. This completes the proof of Theorem 2.

4. RELATED RESULTS

(i) We point out that in the case of an even multiplier λ on $L^1(\mathbf{R})$ we cannot state that the function g defined in (3.17) belongs to $H^1(\mathbf{R})$. Nevertheless, we are able to deduce a Hardy type inequality for the Hilbert transform \tilde{g} . More exactly, the following is true.

Theorem 3. *If λ is an even multiplier on $L^1(\mathbf{R})$, μ is the function of bounded variation associated with λ according to (1.2), σ is the Cesàro mean of λ defined by (2.2), and g is defined by (3.17), then*

$$(4.1) \qquad \int_0^{\infty} \left| \pi \tilde{g}(x) - \frac{2}{x} \int_0^{x/2} d\mu(\xi) \right| dx \leq C \int_0^{\infty} d|\mu|(\xi),$$

where C is an absolute constant.

The next corollary is an immediate consequence of Theorem 3.

Corollary 4. *If λ is an even multiplier on $L^1(\mathbf{R})$ and μ is the function of bounded variation associated with λ according to (1.2), then the function g defined by (3.17) belongs to $H^1(\mathbf{R})$ if and only if*

$$(4.2) \qquad \int_0^{\infty} \left| \frac{1}{x} \int_0^x d\mu(\xi) \right| dx < \infty.$$

Proof of Theorem 3. Similarly to the case of an odd multiplier on $L^1(\mathbf{R})$ (cf. (3.7)–(3.12)), we can represent the Hilbert transform \tilde{g} by

$$(4.3) \qquad \pi \tilde{g}(x) = G_1(x) + \beta_1(x) + \gamma_3(x) + \beta_4(x) + \beta_5(x),$$

where

$$G_1(x) := \lim_{\delta \downarrow 0} \int_{\delta}^{x/2} \frac{dt}{t} \int_{x-t}^{x+t} \frac{d\mu(\xi)}{\xi},$$

$$\beta_1 := - \int_{x/2}^{\infty} \frac{g(x+t)}{t} dt, \quad \gamma_3(x) := \int_0^{x/2} \ln \left(\frac{x+\xi}{x-\xi} \right) \frac{d\mu(\xi)}{\xi}$$

(there is an essential difference between β_3 and γ_3 , due to the evenness of g , in contrast to the oddness of h),

$$\beta_4(x) := (\ln 2) \int_{x/2}^{\infty} \frac{d\mu(\xi)}{\xi}, \quad \beta_5(x) := \int_{x/2}^{\infty} \ln \left(\frac{x+\xi}{x} \right) \frac{d\mu(\xi)}{\xi}.$$

Repeating analogous reasonings as in the proof of Theorem 1, we conclude that the functions G_1 , β_1 , β_4 , and β_5 are in $L^1(\mathbf{R})$. Thus, from (4.3) it follows that

$$(4.4) \quad \int_0^{\infty} |\pi \tilde{g}(x) - \gamma_3(x)| dx \leq C \int_0^{\infty} d|\mu|(\xi).$$

On the basis of (3.13) and the reversed inequality $\ln(1+u) \geq u - u^2/2$, $u \geq 0$, we see that the inequality

$$0 \leq \frac{2}{x-\xi} - \frac{1}{\xi} \ln \left(\frac{x+\xi}{x-\xi} \right) \leq \frac{4\xi}{(x-\xi)^2}$$

holds true for $0 \leq \xi \leq \frac{x}{2}$. Consequently, by setting

$$\gamma_4(x) := \int_0^{x/2} \frac{d\mu(\xi)}{x-\xi},$$

from (4.4) it follows that

$$(4.5) \quad \int_0^{\infty} |\pi \tilde{g}(x) - 2\gamma_4(x)| dx \leq \int_0^{\infty} d|\mu|(\xi) \int_{2\xi}^{\infty} \left| \frac{1}{\xi} \ln \left(\frac{x+\xi}{x-\xi} \right) - \frac{2}{x-\xi} \right| dx$$

$$\leq \int_0^{\infty} d|\mu|(\xi) \int_{2\xi}^{\infty} \frac{4\xi}{(x-\xi)^2} dx = 4 \int_0^{\infty} d|\mu|(\xi).$$

Analogously, by Fubini's theorem,

$$(4.6) \quad \int_0^{\infty} \left| \gamma_4(x) - \frac{1}{x} \int_0^{x/2} d\mu(\xi) \right| dx = \int_0^{\infty} \left| \int_0^{x/2} \frac{\xi}{x(x-\xi)} d\mu(\xi) \right| dx$$

$$\leq \int_0^{\infty} d|\mu|(\xi) \int_{2\xi}^{\infty} \frac{\xi}{x(x-\xi)} dx = \int_2^{\infty} \frac{dt}{t(t-1)} \int_0^{\infty} d|\mu|(\xi).$$

Combining (4.5) and (4.6) results in (4.1). This completes the proof of Theorem 3.

(ii) Adopting the relevant parts in the proofs of Theorems 1 and 2 with appropriate modifications, we can achieve an analogous result on the Cesàro mean σ of the Fourier transform \hat{f} defined by

$$(4.7) \quad \sigma(t) := \frac{1}{t} \int_0^t \hat{f}(\xi) d(\xi), \quad t \in \mathbf{R}, t \neq 0$$

(cf. definition (2.2)), where $f \in L^1(\mathbf{R})$ is a given function.

Corollary 5. *If $f \in L^1(\mathbf{R})$ and σ is defined by (4.7), then σ is also the Fourier transform of a function in $L^1(\mathbf{R})$.*

(iii) On closing, we note that Hardy [4], Goes [3], and Georgakis [2] achieved similar results on the arithmetic means of the Fourier-Stieltjes coefficients of periodic functions μ of bounded variation on the torus $[-\pi, \pi]$.

NOTE ADDED IN PROOF

After submitting this paper, it came to our knowledge that Georgakis [8] has proved our Theorems 1–2 and Corollaries 3–4 in a more general setting. However, our estimates in the proofs are different from those in his proofs.

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