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ABSOLUTE SUMMABILITY FACTORS OF TYPE (X, Y)

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ABSTRACT. In Theorem 1 a set of sufficient conditions is investigated for the summability factor of type $\theta_n \in (|\overline{N}, q|_k, |\overline{N}, p|_k)$, $k \geq 1$. In the special case when $k = 1$ three different sets of necessary and sufficient conditions are established in Theorem 2 for the factor θ_n. Various known and some new results are also deduced as special cases.

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Let $\{p_n\}$ be a sequence of positive real constants such that $P_n \to \infty$ as $n \to \infty$, where $P_n = p_0 + \cdots + p_n \neq 0$, $P_{-1} = p_{-1} = 0$. A series $\sum a_n$ with the sequence of partial sums $\{s_n\}$ is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$, if [1]

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |T_n - T_{n-1}|^k < \infty, \tag{1.1}$$

where $T_n = P_n^{-1} \sum_{\nu=0}^{n} p_\nu s_\nu$.

If $\sum a_n \theta_n$ is summable by the method Y whenever $\sum a_n$ is summable by the method X, then we say that the factor θ_n is of type (X, Y) and write

$$\theta_n \in (X, Y). \tag{1.2}$$

Such a type of factor has been investigated in detail by Bosanquet and Das [6].

Recently, Bor [2] has proved the following theorem.

Theorem A. *Let $\{p_n\}$ be a sequence of positive real constants such that, as $n \to \infty$.*

$$\text{(i) } np_n = O(P_n), \qquad \text{(ii) } P_n = O(np_n). \tag{1.3}$$

If $\sum a_n$ is summable $|C, 1|_k$, then it is also summable $|\overline{N}, p_n|_k$, $k \geq 1$.

He also obtained the converse of Theorem A under condition (1.3) [3]. He proved

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Theorem B. *Let* $\{p_n\}$ *be such that* (1.3) *is satisfied. If* $\sum a_n$ *is summable* $|\overline{N}, p_n|_k$, *then it is also summable* $|C, 1|_k$.

Very recently Bor and Thorpe [4] improved the above results by replacing $|C, 1|_k$ by more general summability, viz. $|\overline{N}, q_n|_k$ ($\{q_n\}$ is defined in the same way as $\{p_n\}$, and we write $Q_n = q_0 + \cdots + q_n$). They proved the following theorem.

Theorem C. *If* $\sum a_n$ *is summable* $|\overline{N}, q_n|_k$, *then it is also summable* $|\overline{N}, p_n|_k$ $(k \geq 1)$ *provided* $\{p_n\}$ *and* $\{q_n\}$ *are positive sequences such that*

$$\text{(1.4)} \qquad \text{(i) } p_n/P_n = O(q_n/Q_n), \qquad \text{(ii) } q_n/Q_n = O(p_n/P_n).$$

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In view of (1.2) we can say that when (1.3) and (1.4)]are satisfied, then

$$1 \in (|C, 1|_k, |\overline{N}, p_n|_k) \quad \text{(Theorem A)},$$
$$1 \in (|\overline{N}, p_n|_k, |C, 1|_k) \quad \text{(Theorem B)},$$

and

$$1 \in (|\overline{N}, q_n|_k, |\overline{N}, p_n|_k) \quad \text{(Theorem C)}.$$

Suppose the sequences $\{p_n\}$ and $\{q_n\}$ are such that (1.4)(ii) is satisfied. Now the following question naturally arises: what would be the order conditions on a factor, say θ_n, such that

$$\theta_n \in (|\overline{N}, q_n|_k, |\overline{N}, p_n|_k), \qquad k \geq 1.$$

In an attempt to answer this question we establish the following theorem.

Theorem 1. *Let the sequences* $\{p_n\}$ *and* $\{q_n\}$ *be such that* $p_n > 0$, $q_n > 0$, $P_n \to \infty$, $Q_n \to \infty$,[1] *and*

$$\text{(2.1)} \qquad P_n/p_n = O[Q_n/q_n].$$

Then in order that

$$\text{(2.2)} \qquad \theta_n \in (|\overline{N}, q_n|_k, |\overline{N}, p_n|_k), \qquad k \geq 1,$$

it is sufficient that

$$\text{(2.3)} \qquad \theta_n = O[q_n P_n / p_n Q_n]$$

and

$$\text{(2.4)} \qquad P_n Q_{n-1} \Delta\theta_n + (q_n P_n - p_n Q_n)\theta_n = O[q_n P_n],$$

where $\Delta\theta_n = \theta_n - \theta_{n+1}$.

The question now again arises: is it possible to obtain order conditions on a factor θ_n which are necessary and sufficient for (2.2) to hold? We will later see that the answer to this question is not in the affirmative. However, for $k = 1$ we can obtain a set of conditions which are necessary and sufficient for (2.2) to hold.

Some special cases are also of sufficient interest and worth including in the theorem. The referee suggested we include in the theorem as part (c) the special case in which $p_n Q_n = O(q_n P_n)$. The proof of this part was also supplied by the referee. We state the following theorem in three parts.

[1]We note that this implies that $(\overline{N}, p)$ and $(\overline{N}, q)$ are absolutely regular.

Theorem 2. (a) *Let the sequences* $\{p_n\}$ *and* $\{q_n\}$ *be such that* $p_n > 0$, $q_n > 0$, $P_n \to \infty$, *and* $Q_n \to \infty$. *Then in order that*

$$\theta_n \in (|\overline{N}, q_n|, |\overline{N}, p_n|) \tag{2.5}$$

it is necessary and sufficient that conditions (2.3) *and* (2.4) *are satisfied.*

(b) *Let the sequences* $\{p_n\}$ *and* $\{q_n\}$ *be such that condition* (2.1) *is satisfied. Then in order that* (2.5) *holds it is necessary and sufficient that*

$$\theta_n = O(1), \tag{2.6}$$

(2.3) *holds, and*

$$\Delta\theta_n = O[q_n/Q_{n-1}]. \tag{2.7}$$

(c) *Let the sequences* $\{p_n\}$ *and* $\{q_n\}$ *be such that*

$$p_n Q_n = O(q_n P_n). \tag{2.8}$$

Then in order that (2.5) *holds it is necessary and sufficient that conditions* (2.6) *and* (2.7) *are satisfied.*

In order to obtain necessary and sufficient conditions we need the following lemmas.

Lemma 1 [7]. *In order that the series-to-series transformation*

$$d_n = \sum_{\nu=0}^{\infty} \alpha_{n\nu} a_\nu \tag{2.9}$$

be absolutely conservative (*i.e., absolute convergence of* $\sum a_\nu$ *implies the absolute convergence of* $\sum d_n$) *it is necessary and sufficient that*

$$\sum_{n=0}^{\infty} |\alpha_{n\nu}| = O(1). \tag{2.10}$$

In order that (2.9) *be absolutely regular* (*i.e., absolute convergence of* $\sum a_\nu$ *implies* $\sum d_n$ *converges absolutely and that* $\sum_{n=0}^{\infty} d_n = \sum_{\nu=0}^{\infty} a_\nu$) *it is necessary and sufficient that* (2.10) *holds and further that for all* ν

$$\sum_{n=0}^{\infty} \alpha_{n\nu} = 1. \tag{2.11}$$

Lemma 2. *Let* A *be an absolutely regular series-to-series transformation and* l *be the space of absolutely convergent series. Then in order that*

$$\theta_n \in (l, |A|) \tag{2.12}$$

it is necessary and sufficient that (2.6) *be satisfied.*

Proof. Let A be given by (2.9). Then the A transform of $\sum a_\nu \theta_\nu$ is

$$\sum_{\nu=0}^{\infty} \alpha_{n\nu} \theta_\nu a_\nu. \tag{2.13}$$

Thus (2.12) is equivalent to the assumption that (2.13) is absolutely conservative. By Lemma 1 a necessary and sufficient condition for this is

$$|\theta_\nu| \sum_{n=0}^{\infty} |\alpha_{n\nu}| = O(1). \tag{2.14}$$

But we are given that A is absolutely regular. Hence (2.12) holds, and further

$$\sum_{n=0}^{\infty} |\alpha_{n\nu}| \geq \left| \sum_{n=0}^{\infty} \alpha_{n\nu} \right| = 1 .$$

Hence (2.14) is equivalent to (2.6), and the lemma follows.

Lemma 3. *Let A and B be two absolutely regular series-to-series transformations. In order that*

$$\theta_n \in (|A|, |B|) \tag{2.15}$$

it is necessary that (2.6) *holds.*

Proof. Suppose (2.6) does not hold. In view of Lemma 2 there is a series $\sum a_\nu$ belonging to l such that $\sum a_\nu \theta_\nu$ does not belong to $|B|$. But since A is absolutely regular, $\sum a_\nu$ belongs to $|A|$. Thus (2.15) is false, which proves the lemma.

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Now we prove Theorem 2.

Proof of Theorem 2. (a) Let $t_n = Q_n^{-1} \sum_{\nu=0}^{n} q_\nu s_\nu = Q_n^{-1} \sum_{\nu=0}^{n} (Q_n - Q_{\nu-1}) a_\nu$. Write $t_n - t_{n-1} = b_n$ (write $t_{-1} = 0$) so that $t_n = b_0 + \cdots + b_n$. Thus we can say that the $(\overline{N}, q_n)$ transform of $\sum a_\nu$ is $\sum b_n$.

In a similar way suppose that in series form the $(\overline{N}, p_n)$ transform of $\sum q_\nu \theta_\nu$ is $\sum c_n$. Now we have for $n \geq 1$

$$b_n = q_n / Q_n Q_{n-1} \sum_{\nu=1}^{n} Q_{\nu-1} a_\nu , \tag{3.1}$$

where

$$a_n = (Q_n / q_n) b_n - (Q_{n-2} / q_{n-1}) b_{n-1} , \tag{3.2}$$

analogous to (3.1). Replacing a_ν by $a_\nu \theta_\nu$ and interchanging p, P with q, Q, we have for $n \geq 1$

$$c_n = p_n / P_n P_{n-1} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu \theta_\nu . \tag{3.3}$$

Substituting (3.2) in (3.3), we get

$$\begin{aligned} c_n &= p_n / P_n P_{n-1} \sum_{\nu=1}^{n} P_{\nu-1} \theta_\nu ((Q_\nu / q_\nu) b_\nu - (Q_{\nu-2} / q_{\nu-1}) b_{\nu-1}) \\ &= p_n / P_n P_{n-1} \sum_{\nu=1}^{n-1} b_\nu / q_\nu (P_{\nu-1} Q_\nu \theta_\nu - P_\nu Q_{\nu-1} \theta_{\nu+1}) \\ &\quad + (p_n Q_n / q_n P_n) \theta_n b_n \\ &= p_n / P_n P_{n-1} \sum_{\nu=1}^{n-1} b_\nu / q_\nu [P_\nu Q_{\nu-1} \Delta\theta_\nu + (q_\nu P_\nu - p_\nu Q_\nu) \theta_\nu] \\ &\quad + (p_n Q_n / q_n P_n) \theta_n b_n . \end{aligned} \tag{3.4}$$

It follows from the definitions that (2.5) holds if and only if (3.4) is absolutely conservative. Necessary and sufficient conditions for this are given by Lemma 1. We note, however, that in the case considered we have $\alpha_{n\nu} = 0$ for $\nu > n$ so that (2.10) takes the form

$$\sum_{n=\nu}^{\infty} |\alpha_{n\nu}| = O(1). \tag{3.5}$$

We note that (3.5) is equivalent to the assumption that

$$\alpha_{nn} = O(1) \tag{3.6}$$

and

$$\sum_{n=\nu+1}^{\infty} |\alpha_{n\nu}| = O(1) \tag{3.7}$$

are both satisfied. It is convenient to split up (3.5) in this way, because $\alpha_{n\nu}$ takes different forms in the cases $n = \nu$, $n > \nu$.

Referring to (3.4) we see that (3.6) becomes

$$\theta_n = O[q_n P_n / p_n Q_n].$$

Hence (2.3) is satisfied. Also, (3.7) becomes

$$\frac{1}{q_\nu} |P_\nu Q_{\nu-1} \Delta\theta_\nu + (q_\nu P_\nu - p_\nu Q_\nu)\theta_\nu| \sum_{n=\nu+1}^{\infty} p_n / P_n P_{n-1} = O(1),$$

which is precisely (2.4).

(b) From the proof of (a) we see that (2.3) is satisfied; that is,

$$\theta_n = O[q_n P_n / p_n Q_n] = O(1) \quad \text{(in view of (2.1))}.$$

Hence (2.6) is satisfied.

Following the proof of (a) we note that on the assumption that (2.3) holds (2.4) is equivalent to

$$P_n Q_{n-1} \Delta\theta_n + q_n P_n \theta_n = O(q_n P_n). \tag{3.8}$$

But it follows from Lemma 3 that (2.6) is necessary. We may therefore suppose that (2.6) is satisfied, and consequently (3.8) is equivalent to

$$P_n Q_{n-1} \Delta\theta_n = O(q_n P_n);$$

that is, (2.7) holds.

Thus we have proved the necessity parts of the theorem. The sufficiency parts of the theorem follow from the proof of Theorem 1, with $k = 1$.

(c) It may be noted by Bosanquet's theorem [5] (cf. Corollary 2) that (2.8) is equivalent to the assertion that $|\overline{N}, q|$ implies $|\overline{N}, p|$. Thus (c) shows that the necessary and sufficient conditions for $\theta_n \in (|\overline{N}, q|, |\overline{N}, p|)$ when $|\overline{N}, q|$ implies $|\overline{N}, p|$ are the same as those when $|\overline{N}, q| = |\overline{N}, p|$.

First we consider sufficiency, so suppose the conditions are satisfied. Taking $p_n = q_n$ in (2.8) it follows from (b) that

$$\theta_n \in (|\overline{N}, q|, |\overline{N}, p|).$$

Thus if $\sum a_n$ is summable $|\overline{N}, q|$, then $\sum a_n\theta_n$ is summable $|\overline{N}, p|$ and hence, by Bosanquet's theorem (since (2.8) holds), summable $|\overline{N}, p|$. Thus, $\theta_n \in (|\overline{N}, q|, |\overline{N}, p|)$, as required.

Necessity. Now suppose that $\theta_n \in (|\overline{N}, q|, |\overline{N}, p|)$. Since $(\overline{N}, q)$ is absolutely regular, it follows that $\theta_n \in (l, |\overline{N}, p|)$, so by Lemma 2 $\theta_n = O(1)$.

Also by part (a) it follows that

$$(q_nP_n - p_nQ_n)\theta_n = O(q_nP_n),$$

so (2.4) reduces to $\Delta\theta_n = O[q_n/Q_{n-1}]$; that is, (2.7) is satisfied. Hence the result.

Remark 1. The corresponding result with index $k > 1$ appears to be much harder, as there is now no known result analogous to Lemma 1 which we can approach. Hence it is not possible to obtain necessary conditions for (2.2).

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Proof of Theorem 1. In order to prove the sufficiency of Theorem 1 we have to prove that $\sum a_n\theta_n$ is summable $|\overline{N}, p_n|_k$ whenever $\sum a_n$ is summable $|\overline{N}, q_n|_k$.

The $|\overline{N}, p_n|$ means of $\sum a_n\theta_n$ is given by (3.4). Now using (2.4) in (3.4) we have

$$\begin{aligned} c_n &= O\left\{p_n/P_nP_{n-1}\sum_{\nu=1}^{n-1}|b_\nu|P_\nu\right\} + (p_nQ_n/q_nP_n)\theta_nb_n \\ &= O\left\{p_n/P_nP_{n-1}\sum_{\nu=1}^{n-1}|b\nu|P_\nu\right\} + O(|b_n|) \quad \text{(by (2.3))} \\ &= O(c_n^{(1)}) + O(c_n^{(2)}), \quad \text{say}. \end{aligned}$$

Now, in view of (1.1) we have

$$\begin{aligned} \sum_{k=1}^{\infty}(P_n/p_n)^{k-1}|c_n^{(2)}| &= O\left\{\sum_{n=1}^{\infty}(Q_n/q_n)^{k-1}|b_n|^k\right\} \quad \text{(by (2.1))} \\ &< \infty \end{aligned}$$

by the assumption that $\sum a_n$ is summable $|\overline{N}, q_n|_k$.

Also, by Hölder's inequality we get

$$\begin{aligned} \left\{\sum_{\nu=1}^{n-1}|b_\nu|P_\nu\right\}^k &\le \sum_{\nu=1}^{n-1}(|b_\nu|^kP_\nu^k/p_\nu^{k-1})\left(\sum_{\nu=1}^{n-1}p_\nu\right)^{k-1} \\ &\le P_{n-1}^{k-1}\sum_{\nu=1}^{n-1}|b_\nu|^kP_\nu^k/p_\nu^{k-1} \end{aligned}$$

so that

$$\begin{aligned}\sum_{n=1}^{\infty}(P_n/p_n)^{k-1}|c_n^{(1)}|^k &\le \sum_{n=1}^{\infty} p_n/P_nP_{n-1}\sum_{\nu=1}^{n-1}|b_\nu|^k p_\nu^k/p_\nu^{k-1}\\ &= \sum_{\nu=1}^{\infty}|b_\nu|^k P_\nu^k/p_\nu^{k-1}\sum_{n=\nu+1}^{\infty} p_n/P_nP_{n-1}\\ &= \sum_{\nu=1}^{\infty}(P_\nu/p_\nu)^{k-1}|b_\nu|^k\\ &= O\left[\sum_{\nu=1}^{\infty}(Q_\nu/q_\nu)^{k-1}|b_\nu|^k\right] \quad \text{(by (2.1))}\\ &< \infty,\end{aligned}$$

which completes the proof.

5. Some important corollaries and special cases

On taking $\theta_n = 1$ the following corollaries follow from Theorems 1 and 2(a) respectively.

Corollary 1. *Let the sequences $\{p_n\}$ and $\{q_n\}$ be such that $p_n > 0$, $q_n > 0$, $P_n \to \infty$, and $Q_n \to \infty$ as $n \to \infty$, and (2.1) is satisfied. Then in order that $|\overline{N}, q_n|_k$ implies $|\overline{N}, p_n|_k$ it is sufficient that $Q_n|q_n = O[P_n/p_n]$.*

Corollary 2. *Let the sequences $\{p_n\}$ and $\{q_n\}$ be such that $p_n > 0$, $q_n > 0$, $P_n \to \infty$, and $Q_n \to \infty$ as $n \to \infty$. Then the necessary and sufficient condition that $|\overline{N}, q_n|$ implies $|\overline{N}, p_n|$ is that $Q_n|q_n = O[P_n/p_n]$.*

Remark 2. The result of Corollary 2 was first pointed out by Bosanquet [5] in the review of the paper by Sunouchi [9].

In Theorem 1, if we take:

(i) $q_n = 1$ and $\theta_n = 1$, we get Theorem A;
(ii) $p_n = 1$ and $\theta_n = 1$, we get Theorem B;
(iii) $\theta_n = 1$, we get Theorem C.

Taking $q_n = 1$ in Theorem 1 and using the argument of Theorem 2(b) we find that in this case (2.4) reduces to $\Delta\theta_n = O(1/n)$. Thus we get the following theorem.

Theorem 3. *If the sequence $\{p_n\}$ is such that*

$$P_n = O(np_n), \tag{5.1}$$

then in order that

$$\theta_n \in (|C, 1|_k, |\overline{N}, p_n|_k), \qquad k \ge 1,$$

it is sufficient that

$$\theta_n = O[P_n/np_n] \tag{5.2}$$

and

$$\Delta\theta_n = O[1/n]. \tag{5.3}$$

It is interesting to note that Theorem A follows from Theorem 3 with $\theta_n = 1$.

Theorem 4. *Let the sequence* $\{p_n\}$ *be such that* (5.1) *is satisfied and*

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{5.4}$$

Then $P_n/np_n \in (|C, 1|_k, |\overline{N}, p_n|_k)$.

Proof. Taking $\theta_n = P_n/np_n$ in Theorem 3, we find that (5.2) is satisfied and $\Delta(P_n/np_n) = O(1/n)$, in view of [8, Lemma 2]. Hence Theorem 4 follows from Theorem 3.

Theorem 5. *Let the sequences* $\{p_n\}$ *and* $\{q_n\}$ *be as defined in Theorem* 1, *and suppose that* (2.1) *is satisfied. If*

$$P_n \Delta p_n = O(p_n p_{n+1}), \qquad Q_n \Delta q_n = O(q_n q_{n+1}) \tag{5.5}$$

and

$$q_{n+1} Q_{n-1} = O(q_n Q_n), \tag{5.6}$$

then

$$q_n P_n / p_n Q_n \in (|\overline{N}, q_n|_k, |\overline{N}, p_n|_k), \qquad k \geq 1.$$

Proof. In Theorem 1 if we take $\theta_n = q_n P_n / p_n Q_n$, then (2.3) is satisfied, using the argument of Theorem 2(b). We observe that (2.4) is equivalent to (2.7). Hence, in order to prove Theorem 5 we must show that $\Delta(q_n P_n / p_n Q_n) = O[q_n / Q_{n-1}]$. We have

$$\begin{aligned}
&\Delta(q_n P_n / p_n Q_n) \\
&\quad = (q_{n+1}/Q_{n+1})\Delta(P_n/p_n) + (P_n/p_n)\Delta(q_n/Q_n) \\
&\quad = q_{n+1}/Q_{n+1}[P_n\{1/p_n - 1/p_{n+1}\}] \\
&\qquad + P_n/p_n[\Delta q_n/Q_{n+1} + q_n \Delta(1/Q_n)] \\
&\quad = q_{n+1}/Q_{n+1}[-P_n \Delta p_n / p_n p_{n+1} - 1] \\
&\qquad + O(Q_n/q_n)[\Delta q_n/Q_{n+1} + q_n q_{n+1}/Q_n Q_{n+1}] \quad \text{(in view of (2.1))} \\
&\quad = O[q_{n+1}/Q_{n+1}] \quad \text{(in view of (5.5))} \\
&\quad = O[q_{n+1}/Q_n] = O[q_n/Q_{n-1}] \quad \text{(in view of (5.6))}.
\end{aligned}$$

Hence, (2.7) is satisfied, and the theorem follows.

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