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FREE ACTIONS OF ZERO-DIMENSIONAL COMPACT GROUPS ON MENGER MANIFOLDS

KATSURO SAKAI

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ABSTRACT. It is proved that every separable zero-dimensional compact group acts freely on any Menger manifold M. In case M is compact, this result was proved by Dranishnikov. Here is provided an alternative short proof.

A manifold modeled on the n-dimensional universal Menger compactum μ^n is called a *μ^n-manifold.* In [Dr], Dranishnikov established the following theorem in case M is compact. Here we generalize his theorem in case M is noncompact and provide an alternative short proof.

Theorem. *Every separable zero-dimensional compact group*[1] G *acts freely on any μ^n-manifold M.*

For a finite group G and $n \in \mathbb{N}$, we define the n-dimensional $(n-1)$-connected simplicial complex L_G^n as the join of $n+1$ copies of G and a free G-action on $|L_G^n|$ as follows:[2]

$$g(t_1 g_1 \oplus \cdots \oplus t_{n+1} g_{n+1}) = t_1 g g_1 \oplus \cdots \oplus t_{n+1} g g_{n+1}.$$

Note that each $g \in G$ induces the simplicial isomorphism of L_G^n onto itself.

Proof of Theorem. By a well-known theorem of Pontryagin [Po, §46, C], G is the inverse limit of an inverse sequence of finite groups G_i, $i \in \mathbb{N}$; whence G is a subgroup of $\prod_{i \in \mathbb{N}} G_i$. We may assume that each G_i is nontrivial.

By [Be, the analogue of 5.1.3 (p. 103)], we have a locally finite simplicial complex K_0 with $\dim K_0 \leqslant n$ and a proper map $f\colon |K_0| \to M$ which induces isomorphisms of homotopy groups of dim $< n$ and of homotopy groups of ends of dim $< n$. For each $i \in \mathbb{N}$, we have a nondegenerate n-dimensional $(n-1)$-connected finite simplicial complex $L_i = L_{G_i}^n$. We inductively define the simplicial complex K_i as the n-skeleton of $K_{i-1} \times L_i$, where $K \times L$ is the

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[1]Topological groups are assumed to be T_1. Then separable compact groups are metrizable.

[2] $|L_G^n|$ is the quotient space of the product space $G^{n+1} \times \Delta^n$, where

$$\Delta^n = \{(t_1, \cdots, t_{n+1}) \in \mathbf{I}^{n+1} \mid t_1 + \cdots + t_{n+1} = 1\},$$

and the equivalence class of $(g_1, \cdots, g_{n+1}; t_1, \cdots, t_{n+1})$ is denoted by $t_1 g_1 \oplus \cdots \oplus t_{n+1} g_{n+1}$.

simplicial complex defined by the barycentric subdivision of the cell complex $\{\sigma \times \tau \mid \sigma \in K, \tau \in L\}$. Let $p_i \colon |K_i| \to |K_{i-1}|$ be the restriction of the projection. Thus we have an inverse sequence

$$|K_0| \xleftarrow{p_1} |K_1| \xleftarrow{p_2} |K_2| \xleftarrow{p_3} \cdots .$$

By using the simplicial approximation theorem and the homotopy extension technique, it is easy to show that if a map $f\colon \mathbf{S}^k \to |K_i|$ $(k < n)$ extends to a map $f'\colon \mathbf{B}^{k+1} \to |K_{i-1} \times L_i|$, then f extends to a map $f''\colon \mathbf{B}^{k+1} \to |K_i|$ such that each $f'(x)$ and $f''(x)$ are contained in the same simplex in $K_{i-1} \times L_i$. By modifying [GHW, Theorem 1],[3] we can show similarly as [GHW, Theorem 2] that the inverse limit X is a μ^n-manifold and that the projection $p_0^\infty \colon X \to |K_0|$ induces the isomorphisms of homotopy groups of $\dim < n$ and of homotopy groups of ends of $\dim < n$.[4] Since X is homeomorphic to M by [Be, Chapter V, Theorem], we can regard $M = X = \bigcap_{i\in\mathbb{N}} \left(|K_i| \times \prod_{j>i} |L_j|\right) \subset |K_0| \times \prod_{i\in\mathbb{N}} |L_i|$.

Using the free G_i-action on $|L_i|$, we define a free G-action on $|K_0| \times \prod_{i\in\mathbb{N}} |L_i|$ as follows:

$$gx = (x_0;\, g_1x_1,\, g_2x_2,\, \cdots)$$
$$\text{for } \forall g = (g_1, g_2, \cdots) \in G \text{ and } \forall x = (x_0; x_1, x_2, \cdots) \in |K_0| \times \prod_{i\in\mathbb{N}} |L_i|.$$

Since each $g_i \in G_i$ induces the simplicial isomorphism of L_i onto itself, it is easily observed that $g\left(|K_i| \times \prod_{j>i} |L_j|\right) = |K_i| \times \prod_{j>i} |L_j|$ for each $i \in \mathbb{N}$, which implies $gX = X$. Thus we have a free G-action on M as the restriction of the above G-action on $|K_0| \times \prod_{i\in\mathbb{N}} |L_i|$. □

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References

[Be] M. Bestvina, *Characterizing k-dimensional universal Menger compacta*, Mem. Amer. Math. Soc., vol. 71, Amer. Math. Soc., Providence, RI, 1988.

[Dr] A. N. Dranishnikov, *On free actions of zero-dimensional compact groups*, Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), 212–228; English transl., Math. USSR-Izv **32** (1989), 217–232.

[GHW] D. J. Garity, J. P. Henderson, and D. G. Wright, *Menger spaces and inverse limits*, Pacific J. Math. **131** (1988), 249–259.

[Po] L. S. Pontryagin, *Continuous groups*, 4th ed., Nauka, Moscow, 1984; English transl., *Topological groups*, 2nd ed., Gordon and Breach, New York, 1966 .

Institute of Mathematics, University of Tsukuba, Tsukuba-city 305, Japan
E-mail address: `sakaiktr@sakura.cc.tsukuba.ac.jp`

[3]In Theorem 1 of [GHW], if each X_i is locally compact and each p_i is proper, then X is a μ^n-manifold without the condition 2 and it follows from the proof of $(n-1)$-connectedness that the projection of X onto X_1 induces the *monomorphisms* of homotopy groups of $\dim < n$ and of homotopy groups of ends of $\dim < n$. Given a complete metric d_i for each X_i so that $d_{i-1}(p_i(x), p_i(y)) \leqslant d_i(x, y)$ for each $x, y \in X_i$, it suffices to say in the condition 3 that $p_{i+1}h$ is ε_{i+1}-close to f, where $\varepsilon_i > 0$ $(i \in \mathbb{N})$ are prechosen so that $\sum_{i=1}^{\infty} \varepsilon_i \; (= c) < \infty$.

[4]Given $a = (a_1, a_2, \cdots) \in \prod_{i\in\mathbb{N}} |L_i|$, we have the embedding $i_a \colon |K_0| \to M$ defined by $i_a(x) = (x; a_1, a_2, \cdots)$. Since $p_0^\infty i_a = \mathrm{id}$, p_0^∞ induces the *epimorphisms* of homotopy groups and of homotopy groups of ends.