PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 3, November 1994

AUSLANDER'S δ-INVARIANTS OF GORENSTEIN LOCAL RINGS

SONGQING DING

(Communicated by Eric Friedlander)

ABSTRACT. Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring with associated graded ring $G(R)$. It is conjectured that for any integer $n > 0$, Auslander's δ-invariant $\delta(R/\mathfrak{m}^n)$ of $R/\mathfrak{m}^n$ equals 1 if and only if $\mathfrak{m}^n$ is contained in a parameter ideal of R. In an earlier paper we showed that the conjecture holds if $G(R)$ is Cohen-Macaulay. In this paper we prove that the conjecture has an affirmative answer if $\operatorname{depth} G(R) = \dim R - 1$ and R is gradable. We also prove that if R is not regular and $\operatorname{depth} G(R) \geq \dim R - 1$, then $\delta(R/\mathfrak{m}^2) = 1$ if and only if R has minimal multiplicity.

INTRODUCTION

Throughout this paper we assume that $(R, \mathfrak{m}, k)$ is a commutative Noetherian Gorenstein local ring (or a graded Gorenstein k-algebra with unique maximal graded ideal $\mathfrak{m}$), and all modules are finitely generated. For an R-module M, there is an exact sequence of R-modules

$$0 \longrightarrow Y_M \longrightarrow X_M \xrightarrow{\varphi} M \longrightarrow 0$$

where X_M is a maximal Cohen-Macaulay module, Y_M is a module of finite projective dimension, and φ is *right minimal*, i.e., all endomorphisms $\alpha\colon X_M \to X_M$ with $\varphi \circ \alpha = \varphi$ are isomorphisms (see [1, 2]). This sequence is called the minimal Cohen-Macaulay approximation of M. It is uniquely determined (up to isomorphism) by M. The theory of Cohen-Macaulay approximations was initiated by Auslander and Buchweitz. The rank of a maximal free direct summand of X_M is denoted by $\delta_R(M)$ (or simply $\delta(M)$) and is called the δ-invariant of M (over R) by Auslander.

One of the main questions is how the δ-invariant of M reflects the structures of M and of R. If M is a module of finite projective dimension, the minimal Cohen-Macaulay approximation of M is just the minimal free resolution of M. If an R-module M has a factor module which is of finite projective dimension, then $\delta(M) > 0$ [1, 3]. Thus the δ-invariant of M can be regarded as a measure of how far the module M is from being of finite projective dimension. It was first conjectured that for an R-module M, $\delta(M) > 0$ if and only if M has a factor module of finite projective dimension. Unfortunately this is not the

Received by the editors July 22, 1992.
1991 *Mathematics Subject Classification*. Primary 13H10, 13C14, 13A15.

case in general (see the example in §2). However, the modules of the form $M = R/\mathfrak{m}^n$, $n \geq 1$, seem special. Following [3] and [5] we showed in [4] that if the associated graded ring $G(R)$ of R is Cohen-Macaulay then $\delta(R/\mathfrak{m}^n) = 1$ if and only if $\mathfrak{m}^n$ is contained in a parameter ideal of R. Here an ideal I is called a parameter ideal if it is generated by a system of parameters of R. We then conjectured that this is true in general, i.e., without the assumption on $G(R)$. In this paper we give further evidence for the conjecture. We show that if R is a graded Gorenstein k-algebra such that $\operatorname{depth} G(R) \geq \dim R - 1$, then the conjecture holds for R. We call a local ring $(R, \mathfrak{m}, k)$ gradable if the completion $\hat{R}$ of R at its maximal ideal $\mathfrak{m}$ is is omorphic to the completion of a graded k-algebra at the graded maximal ideal. Since the δ-invariants behave well under faithfully flat ring extensions [1], we have that the conjecture holds for a Gorenstein local ring R if R is gradable and $\operatorname{depth} G(R) \geq \dim R - 1$.

It is known that $\delta(R/\mathfrak{m}) = 1$ if and only if R is a regular local ring [3]. We show that for a Gorenstein local ring R with $\operatorname{depth} G(R) \geq \dim R - 1$, $\delta(R/\mathfrak{m}^2) = 1$ if and only if $\mathfrak{m}^2$ is contained in a parameter ideal of R. If R is not regular, then by a result of Sally [6], R has minimal multiplicity.

The results in this paper are based on a particular description of the minimal Cohen-Macaulay approximation of a module M with $\operatorname{depth} M = \dim R - 1$. We give the description in §1. If $\operatorname{depth} G(R) = \dim R - 1$, then by a result of [4], we can always reduce to the case where the dimension of R is 1. We thus devote §2 to giving various properties and examples of the δ-invariants of cyclic modules over a 1-dimensional Gorenstein local ring. Using the properties we develop in §2, we prove the main results in §3.

1. A construction of minimal Cohen-Macaulay approximation

In this section we assume that $(R, \mathfrak{m}, k)$ is a Gorenstein local ring. We denote by $\mu(M)$ the minimal number of generators of an R-module M. Let M be an R-module with $\operatorname{depth} M = \dim R - 1$. The following description of the minimal Cohen-Macaulay approximation of M seems well known.

Proposition 1.1 [1]. *Let M be an R-module with* $\operatorname{depth} M = \dim R - 1$, *and let $0 \to Y_M \to X_M \to M \to 0$ be the minimal Cohen-Macaulay approximation of M. Then $Y_M \cong R^n$, where $n = \mu(\operatorname{Ext}^1(M, R))$.*

Proof. Since M is not a maximal Cohen-Macaulay module, we have

$$\operatorname{Ext}^1(M, R) \neq (0).$$

Let $\xi_1, \dots, \xi_n$ be a minimal set of generators of $\operatorname{Ext}^1(M, R)$, and let E be the extension of M by R^n corresponding to the element $(\xi_1, \dots, \xi_n)$ of $\operatorname{Ext}^1(M, R^n) \cong \bigoplus_n \operatorname{Ext}^1(M, R)$. Applying the functor $\operatorname{Hom}(\ , R)$ to the exact sequence

$$0 \longrightarrow R^n \longrightarrow E \longrightarrow M \longrightarrow 0 \tag{1}$$

yields a long exact sequence

$$\begin{aligned} 0 \ \to\ & \operatorname{Hom}(M, R) \to \operatorname{Hom}(E, R) \to \operatorname{Hom}(R^n, R) \\ & \xrightarrow{\psi} \operatorname{Ext}^1(M, R) \to \operatorname{Ext}^1(E, R) \to 0, \end{aligned}$$

and $\mathrm{Ext}^i(E, R) = (0)$ for $i \geq 2$. By the choice of E, the map ψ is surjective. Therefore, $\mathrm{Ext}^1(E, R) = (0)$. Hence E is a maximal Cohen-Macaulay R-module. Since $n = \mu(\mathrm{Ext}^1(M, R))$, (1) in fact is the minimal Cohen-Macaulay approximation of M. □

As an application of Proposition 1.1, we have the following result on the δ-invariant of M.

Proposition 1.2. *Let M be an R-module with* $\operatorname{depth} M = \dim R - 1$. *Then*

$$\delta(M) = \mu(M) + \mu(\mathrm{Ext}^1(M, R)) - \mu(\mathrm{Hom}(\Omega(M), R))$$

where $\Omega(M)$ is the first syzygy of M.

Proof. We put $s = \mu(M)$, $n = \mu(\mathrm{Ext}^1(M, R))$, and $t = \mu(\mathrm{Hom}(\Omega(M), R)$. We have the short exact sequence $0 \to \Omega(M) \to R^s \to M \to 0$. Let $0 \to R^n \to X_M \to M \to 0$ be the minimal Cohen-Macaulay approximation of M. We can construct the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & R^n & = & R^n & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \Omega(M) & \to & F & \to & X_M & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \Omega(M) & \to & R^s & \to & M & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Thus we have $F \cong R^n \oplus R^s$. On the other hand, $\Omega(M)$ is a maximal Cohen-Macaulay R-module. Applying the functor $\mathrm{Hom}(\ , R)$ to the middle row, we obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}(X_M, R) \longrightarrow \mathrm{Hom}(F, R) \longrightarrow \mathrm{Hom}(\Omega(M), R) \longrightarrow 0.$$

Therefore, $F \cong R^t \oplus R^{\delta(M)}$. Hence, we get $\delta(M) + t = s + n$. □

2. One-dimensional rings

In this section $(R, \mathfrak{m}, k)$ will always be a 1-dimensional Gorenstein local ring. For any R-module M, we denote by $\mathrm{soc}(M)$ the socle of M, i.e., $\mathrm{soc}(M) = \{x \in M \mid \mathfrak{m}x = 0\}$. If M is of finite length, we denote by $l(M)$ the length of M. We put $M^* = \mathrm{Hom}(M, R)$. In this section we always assume that I is a regular ideal of R, i.e., the ideal I contains a regular element of R. In this section we illustrate and apply the results in §1 to the cyclic modules over R.

First we have

Proposition 2.1. *Let I be an ideal of R. Then*

$$\delta(R/I) = 1 + l(\mathrm{soc}(R/I)) - \mu(I^*).$$

In particular, we have $\delta(R/I) > 0$ if and only if $l(\mathrm{soc}(R/I)) = \mu(I^*)$.

Proof. By Proposition 1.2 it suffices to show that $l(\mathrm{soc}(M)) = \mu(\mathrm{Ext}^1(M, R))$ for an R-module M. Let N be the maximal finite length submodule of M. Applying the functor $\mathrm{Hom}(\ , R)$ to the exact sequence $0 \to N \to M \to M/N \to 0$, we obtain an isomorphism $\mathrm{Ext}^1(N, R) \cong \mathrm{Ext}^1(M, R)$ since M/N is a maximal Cohen-Macaulay module. Therefore, $l(\mathrm{soc}(M)) = \mu(\mathrm{Ext}^1(N, R)) = \mu(\mathrm{Ext}^1(M, R))$. Since $\delta(R/I) \leq 1$ for any ideal I of R [1], we get the second statement. □

Let $x \in \mathfrak{m}$ be an arbitrary R-regular element. The formula in Proposition 2.1 can be written in the following form.

Corollary 2.2. *Let $x \in \mathfrak{m}$ be R-regular. Then*

$$\delta(R/I) = 1 + l(\mathrm{soc}(R/I)) - l(\mathrm{soc}(I/xI)),$$

and $\delta(R/I) = 1$ if and only if $l(\mathrm{soc}(R/I)) = l(\mathrm{soc}(I/xI))$.

Proof. We need only show that $\mu(I^*) = l(\mathrm{soc}(I/xI))$ for any regular element x of R. The exact sequence $0 \to R \xrightarrow{x} R \to R/xR \to 0$ gives rise to an exact sequence $0 \to I^* \xrightarrow{x} I^* \to \mathrm{Hom}_{R/xR}(I/xI, R/xR)) \to 0$. Since $x \in \mathfrak{m}$, we have $\mu(I^*) = \mu(\mathrm{Hom}(I/xI, R/xR)) = l(\mathrm{soc}(I/xI))$. □

As a consequence of the Proposition 2.1 we have

Corollary 2.3. *Let I be a Gorenstein ideal of R. Then $\delta(R/I) = 1$ if and only if I is a principal ideal.*

Proof. Since R/I has a simple socle, we have $\delta(R/I) = 1$ if and only if I^* is generated by one element. Hence I^* is a free R-module of rank one. Applying the functor $\mathrm{Hom}(\ , R)$ to the exact sequence $0 \to I \to R \to R/I \to 0$ gives an exact sequence $0 \to R \to I^* \cong R \to \mathrm{Ext}^1(R/I, R) \to 0$. Therefore, $\mathrm{Ext}^1(R/I, R) \cong R/(x)$ for some regular element x of R. Since I is a Gorenstein ideal, we get $R/I \cong \mathrm{Ext}^1(R/I, R)$, and this implies that $I = (x)$. □

Now we consider another case. Let $Q(R)$ be the total ring of quotients of R with respect to the R-regular elements, and let $I^{-1} = \{z \in Q(R) \mid zI \subset R\}$. Then we know that there exists a natural R-isomorphism $\phi_I \colon I^{-1} \to I^*$ given by sending $z \in I^{-1}$ to the morphism in I^* defined as multiplication by z. Also we have $I \subseteq II^{-1} \subseteq R$. If $II^{-1} = R$, then I is a principal ideal and so $\delta(R/I) = 1$. We now consider the case where $II^{-1} = I$. We first give the following criterion.

Proposition 2.4. *$\delta(R/I) = 1$ if and only if $R \subset \mathfrak{m}I^{-1}$.*

Proof. We have the following commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\alpha} & I^{-1} & \longrightarrow & \mathrm{coker}\,\alpha & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^* & \longrightarrow & I^* & \longrightarrow & \mathrm{Ext}^1(R/I, R) & \longrightarrow & 0
\end{array}$$

where α is the natural inclusion and all the vertical arrows are isomorphisms. By Proposition 2.1, $\delta(R/I) = 1$ if and only if $\mu(I^*) = \mu(\mathrm{Ext}^1(R/I, R))$. Hence we have $\delta(R/I) = 1$ if and only if $\mathrm{Im}\,\alpha \subset \mathfrak{m}I^{-1}$. □

As a consequence we have

Corollary 2.5. *If $II^{-1} = I$, then $\delta(R/I) = 0$.*

Proof. We must show that $R \not\subset \mathfrak{m}I^{-1}$. Suppose $R \subset \mathfrak{m}I^{-1}$; then we have $I \subset \mathfrak{m}I^{-1}I \subset \mathfrak{m}I$ since $II^{-1} = I$. This implies that $I = \mathfrak{m}I$, which is impossible by Nakayama's Lemma. □

Remark. The condition $II^{-1} = I$ in fact characterizes the conductors of the overrings of R in $Q(R)$. Recall that for an overring R' with $R \subset R' \subset Q(R)$, the conductor $\mathbf{c}$ of R' in R is defined by $\mathbf{c} = \{z \in Q(R) \mid zR' \subset R\}$. Here $\mathbf{c}$ is an ideal of R. It is easy to check that if $\mathbf{c}$ is the conductor of an overring R' in R, then $\mathbf{c}c^{-1} = \mathbf{c}$. Conversely, if $II^{-1} = I$, then I is the conductor of $\phi_I^{-1}(\mathrm{End}(I))$ in R, where $\phi_I\colon I^{-1} \to I^*$ is the natural isomorphism and $\mathrm{End}(I) \subset I^*$.

Now we return to the question in the introduction. If I is contained in an ideal J of finite projective dimension (in our situation J has to be a principal ideal), then we know that $\delta(R/I) = 1$. The converse is not true in general.

Example. Let $R = k[[t^3, t^4]]$ where k is a field, and let $I = (t^8 + t^9, t^{10})$. It is not hard to check by direct calculation that

(a) I is not contained in any principal ideal of R, and
(b) $l(\mathrm{soc}(R/I)) = l(\mathrm{soc}(I/t^3I)) = 2$.

Therefore, $\delta(R/I) = 1 + l(\mathrm{soc}(R/I)) - l(\mathrm{soc}(I/t^3I)) = 1$ by Corollary 2.2, but R/I has no factor module of finite projective dimension.

3. The conjecture

Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring (or a graded Gorenstein k-algebra with a unique graded maximal ideal $\mathfrak{m}$). We showed in [4] that if the associated graded ring $G(R)$ of R is Cohen-Macaulay, then for any integer $n > 0$, $\delta(R/\mathfrak{m}^n) = 1$ if and only if there exists an R-sequence $\mathbf{x}$ such that $\mathfrak{m}^n \subset (\mathbf{x})$. Rings with $G(R)$ Cohen-Macaualy include hypersurface rings and homogeneous Gorenstein k-algebras. (A graded k-algebra is called homogeneous if it is generated by degree 1 elements over k.)

Conjecture. Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring (or a graded Gorenstein k-algebra with a unique graded maximal ideal $\mathfrak{m}$). For any integer $n > 0$, $\delta(R/\mathfrak{m}^n) = 1$ if and only if there is an R-sequence $\mathbf{x}$ such that $\mathfrak{m}^n \subset (\mathbf{x})$.

In this section we show two results related to the conjecture. In both cases we assume that R satisfies the condition $\operatorname{depth} G(R) \geq \dim R - 1$. We first use the criterion for $\delta(R/I) = 1$ that we developed in §2 to prove the one-dimensional case. Then we reduced the general case to the one-dimensional case by the result of [4].

Theorem 3.1. *Let R be a 1-dimensional graded Gorenstein k-algebra. For any graded ideal I of R, $\delta(R/I) = 1$ if and only if $I \subset (x)$, where x is a homogeneous regular element of R. In particular, the conjecture holds for R.*

Proof. According to Proposition 2.4, $\delta(R/I) > 0$ if and only if $R \subset \mathfrak{m}I^{-1}$. In particular, $\delta(R/I) > 0$ implies that $1 \in \mathfrak{m}I^{-1}$. Suppose $1 = \sum_{i=1}^{l} z_i x_i / y_i$ where x_i, y_i, and z_i are homogeneous elements of degree at least one and $x_i/y_i \in I^{-1}$ for all i. By comparing degrees we know that there is an i such that

$\deg(z_i x_i/y_i) = \deg z_i + \deg x_i - \deg y_i = 0$. Therefore, we have $1 = u z_i x_i/y_i$ with $u \in k$. This implies that $x_i/y_i = 1/(u z_i)$. Hence we get $I \subset (z_i)$. □

Remark. Let R be a 1-dimensional complete local Gorenstein domain containing an algebraically closed field k. It is known that R is the completion of a graded k-algebra with respect to its irrelevant maximal ideal [7]. Therefore, the conjecture holds for such rings.

In [4] we showed the following, which allows us to use a reduction argument.

Lemma 3.2 [4]. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring (or a graded Gorenstein k-algebra). Let $x \in \mathfrak{m}\backslash\mathfrak{m}^2$ be R-regular. Set $\overline{R} = R/xR$. Suppose the induced map $\overline{x}\colon \mathfrak{m}^{i-1}/\mathfrak{m}^i \to \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is injective for $i \geq 0$. Then $\delta_R(R/\mathfrak{m}^i) = 1$ if and only if $\delta_{\overline{R}}(R/(\mathfrak{m}^i, x)) = 1$ for $i \geq 1$.*

Combining this lemma and Theorem 3.1 we now obtain

Corollary 3.3. *Let $(R, \mathfrak{m}, k)$ be a graded Gorenstein k-algebra. Let $G(R)$ be the associated graded ring of R. If* $\operatorname{depth} G(R) \geq \dim R - 1$, *then the conjecture holds for R.*

Recall that a local ring $(R, \mathfrak{m}, k)$ is said to be gradable if the completion $\widehat{R}$ of R at its maximal ideal $\mathfrak{m}$ is isomorphic to the completion of a graded k-algebra at its unique maximal graded ideal. Since the δ-invariants behave well under faithfully flat ring extensions [1], we have

Corollary 3.4. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring. If R is gradable and* $\operatorname{depth} G(R) \geq \dim R - 1$, *then the conjecture holds for R.*

Example. Let $R = k[[x, y, z]]/I$ where $I = (x^3 + y^9, x^2z^4 + y^7)$. Then $G(R) = k[x, y, z]/J$, with $J = (x^3, x^2z^4, xy^7, y^{14})$, is not Cohen-Macaulay. However, it is easily checked that R is gradable. This gives an example where $G(R)$ is not Cohen-Macaulay, but the conjecture holds for R. In this case $\delta(R/\mathfrak{m}^i) = 0$ for $i \leq 7$ and $\delta(R/\mathfrak{m}^8) = 1$.

It is known that $\delta(R/\mathfrak{m}) = 1$ implies that $\mathfrak{m}$ is contained in a parameter ideal of R. Therefore, R is a regular local ring. In dealing with the case where $\delta(R/\mathfrak{m}^2) = 1$, we have

Theorem 3.5. *Let $(R, \mathfrak{m}, k)$ be a 1-dimensional Gorenstein local ring. Then $\delta(R/\mathfrak{m}^2) = 1$ if and only if $\mathfrak{m}^2 \subset (x)$ for some $x \in R$.*

Proof. The "if" part is obvious. Now let $\delta(R/\mathfrak{m}^2) > 0$. Set $I = \mathfrak{m}^2$. By Proposition 2.4 we have that $\delta(R/I) > 0$ if and only if $R \subset \mathfrak{m}I^{-1}$ if and only if $1 = \sum r_i u_i$, where $r_i \in \mathfrak{m}$ and $u_i \in I^{-1}$. We define

$$l = \min\left\{ n \mid 1 = \sum_{i=1}^{n} r_i u_i,\ r_i \in \mathfrak{m},\ u_i \in I^{-1} \right\}.$$

To show that $\mathfrak{m}^2 \in (x)$ for some R-regular element x is equivalent to showing that $l = 1$. We now assume that $l > 1$ and $\mu(\mathfrak{m}) \geq 3$ and derive a contradiction. (If $\mu(\mathfrak{m}) = 2$, then R is a hypersurface and we know the conjecture is true.)

Let $t \in \mathfrak{m}\backslash\mathfrak{m}^2$ be R-regular. Then u_i can be written in the form $u_i = z_i/t^2$ with $z_i \in R$. We assume that all z_i are in $\mathfrak{m}$ (otherwise, we have $l = 1$). Thus we have

$$1 = \frac{\sum r_i z_i}{t^2} + \frac{\sum s_i x_i}{t} \tag{2}$$

where z_i/t^2, x_i/t are in I^{-1}. We assume that $r_i, s_i, z_i, x_i \in \mathfrak{m}$ (otherwise, $l = 1$) and $z_i, s_i x_i \notin tR$.

Now consider $z_i/t^2 \in I^{-1}$ with $z_i \in \mathfrak{m}$ and $z_i \notin tR$. We have $z_i\mathfrak{m}/t = z_i t\mathfrak{m}/t^2 \subset R$, i.e., $z_i\mathfrak{m} \subset tR$. Therefore, under the reduction map $R \to \overline{R} = R/tR$, we have $\overline{z_i} \in \operatorname{soc}(R/tR)$. Let z_j/t^2 be another element in I^{-1} with $z_j \in \mathfrak{m}$ and $z_j \notin tR$. The same argument shows that $\overline{z_j} \in \operatorname{soc}(R/tR)$. Since R/tR has simple socle, we get $z_j = az_i + ty_j$ where $a \in R$ and $y_j \in R$. This implies that $y_i/t \in I^{-1}$. Hence (2) can be written in the form

$$1 = \frac{rz}{t^2} + \frac{\sum_{i=1}^{l-1} s_i x_i}{t} \tag{3}$$

where z/t^2, x_i/t are in I^{-1}, $r, s_i \in \mathfrak{m}$, and $z, s_i x_i \notin tR$. We assume that z and x_i are in $\mathfrak{m}$, and both rz/t^2 and $\sum s_i x_i/t$ are not in $\mathfrak{m}$ (otherwise, we are done).

Since $\sum s_i x_i \notin tR$, we may assume that $s_1 x_1 \notin tR$. However, we have $s_1 x_1 \mathfrak{m} \subset tR$, that is, under the reduction map $R \to R/tR$, $\overline{s_1 x_1} \in \operatorname{soc}(R/tR)$. Since R is Gorenstein, we get $s_1 x_1 = az + ty$ with $a, y \in R$ and so

$$\frac{s_1 x_1}{t} = \frac{atz}{t^2} + y.$$

If $y \in \mathfrak{m}$, we get

$$1 - y = \frac{(r + at)z}{t^2} + \frac{\sum_{i=2}^{l-1} s_i x_1}{t}, \tag{4}$$

contrary to the minimality of l. Therefore, y is a unit in R and (3) then can be written in the form

$$1 = \frac{rz}{t^2} + \frac{ux}{t} \tag{5}$$

where z/t^2, x/t are in I^{-1}, $z, x, r, u \in \mathfrak{m}$, and $z, x \notin tR$. We now consider the two induced linear maps

$$\overline{x} : \mathfrak{m}/\mathfrak{m}^2 \to R/tR \quad \text{and} \quad \overline{z} : \mathfrak{m}/\mathfrak{m}^2 \to R/t^2R.$$

We have that $\operatorname{Im}\overline{x} \subset \operatorname{soc}(R/tR)$ and $\operatorname{Im}\overline{z} \subset \operatorname{soc}(R/t^2R)$. Therefore,

$$\dim_k \operatorname{Ker}\overline{x} \geq \mu(\mathfrak{m}) - 1 \quad \text{and} \quad \dim_k \operatorname{Ker}\overline{z} \geq \mu(\mathfrak{m}) - 1.$$

Since $\mu(\mathfrak{m}) \geq 3$, we have $\operatorname{Ker}\overline{x} \cap \operatorname{Ker}\overline{z} \neq (0)$. Thus there exists $s \in \mathfrak{m}\backslash\mathfrak{m}^2$ such that $zs \in t^2R$ and $xs \in tR$. Suppose $zs = t^2y$, $xs = tw$. Then $w \in \mathfrak{m}$. If y is a unit of R, then s is R-regular and $z/t^2 = y/s$. This implies that $\mathfrak{m}^2 \subset (s)$ and $l = 1$. Suppose $y \in \mathfrak{m}$; then multiplying both sides of (5) by s yields $s = ry + uw \in \mathfrak{m}^2$. This is a contradiction since $s \in \mathfrak{m}\backslash\mathfrak{m}^2$, and the proof is complete. $\square$

Combining this result and Lemma 3.2 we obtain the following.

Corollary 3.6. *Let* $(R, \mathfrak{m}, k)$ *be a Gorenstein local ring. Suppose that*

$$\operatorname{depth} G(R) \geq \dim R - 1.$$

Then $\delta(R/\mathfrak{m}^2) = 1$ *if and only if* $\mathfrak{m}^2$ *is contained in a parameter ideal of* R. *In particular, if* R *is not regular, then* $\delta(R/\mathfrak{m}^2) = 1$ *if and only if* R *has minimal multiplicity.*

Acknowledgment

I would like to thank Professor Maurice Auslander for introducing me to this subject and for stimulating discussions during the preparation of this paper.

References

1. M. Auslander, *Minimal Cohen-Macaulay approximations* (in preparation).
2. M. Auslander and R. O. Buchweitz, *The homological theory of maximal Cohen-Macaulay approximations*, Mém. Soc. Math. France (N.S.), no. 38, Soc. Math. France, Paris, 1989, pp. 5–37.
3. S. Ding, *A note on the index of Cohen-Macaulay local rings*, Comm. Algebra **21** (1993), 53–71.
4. ———, *The associated graded ring and the index of a Gorenstein local ring*, Proc. Amer. Math. Soc. (to appear)
5. J. Herzog, *On the index of a homogeneous Gorenstein ring*, preprint, 1992.
6. J. Sally, *Tangent cones at Gorenstein singularities*, Comput. Math. Appl., vol. 40, Academic Press, New York, 1980, pp. 169–175.
7. K. Watenabe, *Some examples of one dimensional Gorenstein domains*, Nagoya Math. J. **49** (1973), 101–109.

Department of Mathematics, Texas Tech University, Lubbock, Texas 79409-1042
Current address: Department of Mathematics, Michigan State University, East Lansing, Michigan 48824
E-mail address, Michigan University: `ding@math.msu.edu`
E-mail address, Texas Tech University: `sding@ttmath.ttu.edu`