NEAR-RINGS ASSOCIATED WITH MATCHED PAIRS
ON RING MODULES

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(Communicated by Lance W. Small)

ABSTRACT. Let $G$ be a module over a ring $R$, let $\mathcal{C} = \{C_i\}, i \in I$, be a family of submodules of $G$, and let $\mathcal{H} = \{H_i\}, i \in I$, where $H_i$ is a subgroup of $\text{Hom}_R(C_i, G)$ with certain properties. To each such pair $(\mathcal{C}, \mathcal{H})$, a near-ring $M(\mathcal{C}, \mathcal{H})$ is associated, which is a generalization of the near-ring of homogeneous functions determined by $(G, R)$. The transfer of information from module properties of $GR$ reflected in $(\mathcal{C}, \mathcal{H})$ to structural properties of $M(\mathcal{C}, \mathcal{H})$ is investigated.

I. INTRODUCTION AND BASIC CONCEPTS

Let $R$ be a ring with identity, and let $G$ be a unitary right $R$-module. Then, under function addition and function composition, the set $M_R(G) := \{f : G \rightarrow G|f(ar) = f(a)r, \forall a \in G, \forall r \in R\}$ is a zero-symmetric near-ring with identity, called the near-ring of homogeneous functions determined by the pair $(R, G)$. The ring $S := \text{End}_R G$ is a subring of $M_R(G)$. The near-rings $M_R(G)$ have been the subject of several investigations. (See [2] and the references given there.) Recently, sub-near-rings of $M_R(G)$ in which the homogeneous functions can be represented locally as endomorphisms of $G$ have been investigated. We proceed to define these subrings.

Let $\mathcal{C} = \{C_\alpha\}, \alpha \in \mathcal{A}$, be a family of submodules of $G$ with the properties:

(i) $\bigcup \mathcal{C} = \bigcup_{\alpha \in \mathcal{A}} C_\alpha = G$, and (ii) for each $s \in S$ and each $C_\alpha \in \mathcal{C}$, there exists $\beta \in \mathcal{A}$ such that $s(C_\alpha) \subseteq C_\beta$. In this case we say $\mathcal{C}$ is a cover for $G$. For each cover $\mathcal{C}$ of $G$ we have a sub-near-ring $PE_R(G, \mathcal{C})$ of $M_R(G)$ given by $PE_R(G, \mathcal{C}) := \{f \in M_R(G)|f|C_\alpha$ extends to some $s \in \text{End}_R G, \forall \alpha \in \mathcal{A}\}$, called the near-ring of piecewise endomorphisms of $G_R$ determined by $\mathcal{C}$ [3, 4]. In [3] it was shown that for finitely generated modules $G$ over a principal ideal domain $D$, if one uses the cover $\mathcal{P}$ of cyclic submodules then $M_D(G) = PE_D(G, \mathcal{P})$, i.e., every homogeneous function is locally an endomorphism of $G$. It is not known if this characterizes finitely generated modules over principal ideal domains.

It is the purpose of this work to initiate an investigation of a generalization of near-rings of piecewise endomorphisms. The main goal of our program of study...
is to develop concepts and techniques which may then be applied to near-rings of piecewise endomorphisms to obtain a better understanding of this situation.

We now present the basic definitions and concepts for our work. Throughout the paper, all rings $R$ will have an identity and all $R$-modules $G$ will be unitary.

Let $\mathcal{C} = \{C_i\}, \ i \in I,$ be a family of submodules of $G,$ where $I$ is an index set and $C_i \neq C_j$ when $i \neq j.$ Let $\mathcal{H} = \{H_i\}, \ i \in I,$ be a family with $H_i$ subgroups of $\text{Hom}_R(C_i, G).$ The pair $(\mathcal{C}, \mathcal{H})$ is called a matched pair on $G$ if: (a) $f \in H_i$ implies $f(C_i) \in \mathcal{C},$ and (b) $f \in H_i$ with $fC_i = C_j$ and $g \in H_j$ implies $gf \in H_i.$

In this definition, as in the sequel unless otherwise stated, unquantified indices like $i$ and $j$ will be understood to range freely over $I.$ We shall find it useful to let $H := \bigcup \mathcal{H}$ and to define the functions $\sigma, \tau : H \to I$ by $\sigma(h) = i$ if $h \in H_i$ and $\tau(h) = j$ if $hC_i = C_j.$ This is well defined because $C_i \neq C_j$ when $i \neq j.$ Also in the sequel we assume without specific mention that we have some matched pair $(\mathcal{C}, \mathcal{H})$ with accompanying index set $I.$

Associated with each matched pair $(\mathcal{C}, \mathcal{H})$ on $G$ is a near-ring $M := M(\mathcal{C}, \mathcal{H})$ which we proceed to define. For our set $M$ we take $M = \times \bigcup H_i = \{s_i, \ i \in I \mid s_i \in H_i\}.$ Addition on $M$ is defined to be componentwise, and multiplication on $M$ is defined for $s, t \in M$ by $(st)_i := s_t(u_i).$

**Theorem 1.1.** $M = M(\mathcal{C}, \mathcal{H})$ is a zero-symmetric abelian near-ring with the property $s(-t) = -(st), \forall s, t \in M.$

**Proof.** Clearly $(M, +)$ is an abelian group. We verify right distributivity and associativity. Right distributivity follows from

$$[(s + t)u]_i = (s + t)_{\tau(u_i)}u_i = (s_{\tau(u_i)} + t_{\tau(u_i)})u_i = s_{\tau(u_i)}u_i + t_{\tau(u_i)}u_i = (su)_i + (tu)_i$$

and associativity from

$$[s(tu)]_i = s_{\tau((tu)_i)}(tu)_i = s_{(\tau(u_i))_{\tau(u_i)}}(tu)_i = (st)_{\tau(u_i)}u_i = [(st)u]_i.$$ 

Finally, if $h, k \in H$ and $hk$ is defined, then $h(-k) = -hk.$ This completes the proof.

**Corollary 1.2.** If $1_{C_i} \in H_i$ for all $i \in I,$ then $M$ is a near-ring with identity $e,$ where $e_i := 1_{C_i}.$

Unless stated to the contrary we always assume $1_{C_i} \in H_i$ for all $i \in I.$ This implies that $e_i \in M$ where $(e^i)_j := 0$ if $j \neq i$ and $(e^i)_i = e_i.$

**Theorem 1.3.** The $e_i$ are mutually orthogonal.

Suppose $\mathcal{P} = \{C_i\}, \ i \in I,$ is the collection of all cyclic submodules of an $R$-module $G.$ If we let $\text{Hom} := \{H_i\}, \ i \in I,$ where $H_i := \text{Hom}_R(C_i, G),$ then $(\mathcal{P}, \text{Hom})$ is a matched pair on $G.$ However, $M(\mathcal{P}, \text{Hom})$ does not necessarily represent a near-ring of functions on $G$ since, for $s \in M(\mathcal{P}, \text{Hom})$ and $x \in C_i \cap C_j,$ $s_i x$ may be different from $s_j x.$ Consequently we define $HF(\mathcal{P}, \text{Hom}) := \{s \in M \mid s_i C_i \cap C_j = s_j C_i \cap C_j, \forall i, j \in I\}.$ Straightforward calculations show that $HF(\mathcal{P}, \text{Hom})$ is a sub-near-ring of $M(\mathcal{P}, \text{Hom}).$ Then $HF(\mathcal{P}, \text{Hom})$ can be thought of as a near-ring of functions of $G$ by
defining \( s(g) = s_t g \) if \( g \in C_i \) and \( s \in HF(\mathcal{P}, \text{Hom}) \). Moreover, since \( s_t \in \text{Hom}_R(C_i, G) \), we have \( s(gr) = (sg)r \), for each \( r \in R \). So, we have a near-ring of homogeneous functions on \( G \). A sub-near-ring, \( PE(\mathcal{P}, \text{Hom}) \) of \( HF(\mathcal{P}, \text{Hom}) \) is defined by \( PE(\mathcal{P}, \text{Hom}) := \{ s \in HF(\mathcal{P}, \text{Hom}) | s_t = \rho|_{C_i} \) for some \( \rho \in \text{End}_R G \). This is a near-ring of piecewise endomorphisms on \( G \) which we denoted previously by \( PE_R(G, \mathcal{P}) \). Thus, in this sense, our near-rings \( M(\mathcal{C}, \mathcal{H}) \) are indeed generalizations of the near-rings of piecewise endomorphisms.

For \( h, g \in H \), if \( hg \) is defined, then one has \( \sigma(hg) = \sigma(g) \) and \( \tau(hg) = \tau(h) \). Now let \( s \in M \), and define \( \text{suppt}(s) = \{ \sigma(s_i)|s_i \neq 0 \} \), \( \text{null}(s) = \{ \sigma(s_i)|s_i = 0 \} \), and \( \text{target}(s) = \{ \tau(s_i)|i \in I \} \). We then have the following lemma, whose proof is left to the reader.

**Lemma 1.4.** For \( s, t \in M \),

(i) \( \text{suppt}(st) \subseteq \text{suppt}(t) \),

(ii) \( \text{null}(st) \supseteq \text{null}(t) \),

(iii) \( \text{target}(st) \subseteq \text{target}(s) \), and

(iv) \( |\text{target}(st)| \leq |\text{target}(t)| \).

As an application of the above we take \( R = \mathbb{Z} \) and \( G = \mathbb{Z}_\mathcal{P} \) and let \( \mathcal{E}_z = \{ C_i \} \) be the set of all ideals of \( Z \) where the indexing set \( I = \mathbb{N}_0 \) and \( C_i = Zi \), the ideal generated by \( i \). Further let \( H_i := \text{Hom}_z(C_i, \mathbb{Z}) \).

**Lemma 1.5.** If \( s, t \in M(\mathbb{C}_z, \mathcal{H}) \) are such that \( |\text{target}(s)| < \infty \) and \( |\text{target}(t)| < \infty \), then \( |\text{target}(s + t)| < \infty \).

**Proof.** Define an equivalence relation \( \sim \) on \( I \) by \( i \sim j \) if and only if \( \tau(s_i) = \tau(t_j) \). By hypothesis, \( \sim \) has finitely many equivalence classes. Let \( J \) be one such class. We show that if \( i \in J \), then there are just finitely many possibilities for \( \tau((s + t)_i) \). If \( \tau(s_i) = j \) then \( s_i(i) = j \) or \( -j \), and likewise if \( \tau(t_j) = k \) then \( t_j(i) = k \) or \( -k \). Hence \( \tau((s + t)_i) = |j \pm k| \).

Recall that an additive subgroup \( A \) of an arbitrary near-ring is invariant if \( NA \subseteq A \) and \( AN \subseteq A \). If the only invariant subgroups of \( N \) are \( \{0\} \) and \( N \), we say \( N \) is invariantly simple. The previous two lemmas show that \( \{s \in M(\mathbb{C}_z, \mathcal{H})| |\text{target}(s)| < \infty \} \) is a nontrivial proper invariant subgroup of \( M(\mathbb{C}_z, \mathcal{H}) \).

**Theorem 1.6.** \( M(\mathbb{C}_z, \mathcal{H}) \) is not invariantly simple.

We remark that this theorem can be generalized to rings in which each nonzero element has a finite number of associates, e.g. \( \mathbb{Z}[x] \), \( \mathbb{Z}[x_1, \ldots, x_n] \), and \( F[x_1, \ldots, x_n] \) where \( F \) is a finite field.

On the other hand we now show that \( M(\mathbb{C}_z, \mathcal{H}) \) is a simple near-ring. We note that this also follows from subsequent results.

**Theorem 1.7.** \( M(\mathbb{C}_z, \mathcal{H}) \) is a simple near-ring.

**Proof.** We adopt the following convention. If \( s \in M(\mathbb{C}_z, \mathcal{H}) \), \( s_t = k \) will mean \( s_t(i) = k \). Now suppose \( s \neq 0 \), say \( s_m = k \). We may assume \( k > 0 \); otherwise use \( -s \). Let \( x \) and \( y \) be defined by \( x_i = m \), \( y_i = 1 \) for \( i > 0 \). Then \( (ysx)_i = 1 \) for \( i > 0 \). Now let \( u \) and \( v \) be defined by \( u_i = i - 1 \), \( v_i = i(i + 1)/2 \), \( i > 0 \). Then \( yxs + u = e \), \( v(ysx + u) = v \), and \( (vu)_i = v_i - 1 \),
i > 0. Hence \([v(y) + u] - vu\] = \(v_i - v_{i-1} = i\), \(i > 0\), so \(e\) is in the ideal generated by \(s\).

We now present a very general way of constructing matched pairs on an \(R\)-module \(G\). To this end let \(\mathcal{A}\) be any class of right \(R\)-modules, and set

\[
\mathcal{G}(\mathcal{A}) := \{\text{Im} f | f \in \text{Hom}_R(A, G), A \in \mathcal{A}\}.
\]

If \(\mathcal{A} = \{A\}\), we write \(\mathcal{G}(A)\) for \(\mathcal{G}({\{A\}})\). Suppose \(\mathcal{G}(\mathcal{A}) = \{C_i\}, i \in I\), where this indexing is done in such a way that \(C_i \neq C_j\) when \(i \neq j\). Two families of sets of mappings are natural candidates to be the second member of the matched pair, viz. \(\text{Hom} := \{\text{Hom}_R(C_i, G)\}, i \in I\), and \(\text{End} := \{\{f| C_i| f \in \text{End}_R G\}\}, i \in I\). However, there are many other possibilities, e.g., if \(T\) is any subring of \(\text{End}_R G\), then \(\{\{f| C_i| f \in T\}\}, i \in I\), is a candidate for \(\mathcal{H}\). We note that if \((\mathcal{C}, \text{Hom})\) is a matched pair, then \(\mathcal{C} = \mathcal{G}(\mathcal{C})\). The next result states a number of obvious facts about matched pairs.

**Theorem 1.8.** (a) Let \((\mathcal{C}, \mathcal{H})\) and \((\mathcal{C}', \mathcal{H}')\) be matched pairs on \(G\). If \(\mathcal{C} \subseteq \mathcal{C}'\) and \(\mathcal{H} \subseteq \mathcal{H}'\), then \(M(\mathcal{C}, \mathcal{H})\) is a sub-near-ring of \(M(\mathcal{C}', \mathcal{H}')\).

(b) \(\mathcal{G}(R)\) is the family of all cyclic submodules of \(G\).

(c) \(\mathcal{G}(\mathcal{F})\), where \(\mathcal{F}\) is the class of simple \(R\)-modules, is the family of simple submodules of \(G\).

Let \((\mathcal{C}, \mathcal{H})\) be a matched pair on \(G\) with relevant indexing set \(I\). We define a relation ~ on \(I\) by \(i \sim j\) if there exist \(h, k \in H\) with \(\sigma(h) = \tau(k) = i\) and \(\tau(h) = \sigma(k) = j\). It is straightforward to verify that ~ is an equivalence relation. We denote the set of equivalence classes by \(I/\sim\) and represent an equivalence class of ~ by \([i]\). We next define a relation ≤ on \(I/\sim\) by \([i] \leq [j]\) if there exists an \(h \in H\) with \(\sigma(h) \in [j]\) and \(\tau(h) \in [i]\). Using the definition of matched pair one verifies that ≤ is a partial order on \(I/\sim\).

**Theorem 1.9.** If every \(h \in H_i \cap \text{End}_R C_i\) is an automorphism of \(C_i\), then every \(h\) with \(\sigma(h) \in [i]\) and \(\tau(h) \in [i]\) is an isomorphism.

**Proof.** If \(\sigma(h), \tau(h) \in [i]\), then there exist \(h_1, h_2 \in H\) with \(\sigma(h_1) = i\), \(\tau(h_1) = \sigma(h), \sigma(h_2) = \tau(h)_1\), and \(\tau(h_2) = i\). This implies that \(h_2h_1\) is an endomorphism of \(C_i\) and, so by hypothesis, an automorphism. But this in turn implies \(h\) is an isomorphism.

**Corollary 1.10.** Let \(G\) be Noetherian. Then every \(h \in H\) with \(\sigma(h), \tau(h) \in [i]\) for some \(i \in I\) is an isomorphism. In particular, if \([i]\) is minimal with respect to ≤, then every nonzero \(h \in H\) with \(\sigma(h) \in [i]\) is an isomorphism.

**Proof.** Since the mappings in \(\mathcal{H}\) are all epimorphisms, the result follows from the above theorem and the well-known theorem that an onto endomorphism of a Noetherian module is an automorphism.

For future use we note that in the case when \(\mathcal{H} = \text{Hom}\), if \(h \in H\) is an isomorphism, then \(h^{-1} \in H\). Hence the above corollary says

**Corollary 1.11.** Let \(G\) be Noetherian, let \((\mathcal{C}, \mathcal{H})\) be a matched pair on \(G\), and let \([i]\) be minimal. If \(h \in H\) is nonzero with \(\sigma(h) \in [i]\), then \(h^{-1} \in H\).

Another easy result in this vein is
Theorem 1.12. If $C_i \in \mathcal{C}$ is minimal in $G$, then $[i]$ is minimal and every $h \in H$ with $\sigma(h), \tau(h) \in [i]$ is an isomorphism.

Convention. Henceforth, for convenience of exposition we take $0 \in I$ and $C_0 = \{0\}$. Note that $[0] = \{0\}$.

We conclude this section by characterizing when $M(\mathcal{C}, \mathcal{H})$ is a ring. From this, we note that, in general, $M(\mathcal{C}, \mathcal{H})$ is not a ring.

Theorem 1.13. Let $(\mathcal{C}, \mathcal{H})$ be a matched pair on $G$. The following are equivalent:

1. $M(\mathcal{C}, \mathcal{H})$ is a ring.
2. For each nonzero $h \in H$, $\sigma(h) = \tau(h)$.
3. $\forall i \in I \setminus \{0\}$, $[i]$ is minimal and $[i] = \{i\}$.

Proof. From the definitions, one observes that (2) and (3) are equivalent, so we show (1) and (2) are equivalent. Let $M(\mathcal{C}, \mathcal{H})$ be a ring, and assume there is $0 \neq h \in H$ with $h : C_j \to C_i$, $i \neq j$. Define $t \in M$ by $t_j = h$ and $t_i = 0$, $i \neq j$. Since $M$ is a ring, $e^t(t + e^t) = e^t + e^t e^t = t$. Let $\tau(t_j + (e^t)_j) = l$. Then for each $x \in C_j$, $[e^t(t + e^t)](x) = (e^t)(h(x) + x) = t_j(x) = h(x)$. If $l \neq i$, we get $0 = h(x)$, which is a contradiction since $0 \neq h \in H$. If $l = i$, we have $h(x) + x = h(x)$, so $x = 0$, again a contradiction since $C_j \neq \{0\}$. Hence if $M(\mathcal{C}, \mathcal{H})$ is a ring, we must have $\sigma(h) = \tau(h)$, for each $0 \neq h \in H$. Conversely, we note that if $\sigma(h) = \tau(h)$ for each nonzero $h \in H$, then $H_i$ is a subring of $\text{End}_R C_i$ and $M(\mathcal{C}, \mathcal{H}) = \times \mathcal{H}_i$ with pointwise addition and multiplication, i.e., $M(\mathcal{C}, \mathcal{H})$ is a ring.

II. Left ideals

Recall that a left ideal $L$ of an arbitrary near-ring $N$ is strictly minimal (strictly maximal) if $L$ is minimal (maximal) as a left invariant subgroup. In this section we identify some minimal left ideals and strictly minimal left ideals of $M = M(\mathcal{C}, \mathcal{H})$.

For any subset $J$ of $I$, the set $A(J) := \{s \in M|s_j = 0 \text{ if } j \in J\}$ is a left ideal of $M$. In particular, every $M e_i$ is a left ideal since $M e_i = A(I\setminus\{i\})$. We investigate when $M e_i$ is minimal, and to this end we transfer the inclusion partial order on $\mathcal{C}$ to $I$, writing $\preceq$ for the resulting partial order on $I$.

Theorem II.1. If $[i]$ is minimal with respect to $\preceq$ and there exists $j \in [i]$ such that $j \not\leq i$, then $M e_i$ is a minimal left ideal of $M$.

Proof. Let $L$ be a nonzero left ideal of $M$ with $L \subseteq M e_i$, and take $0 \neq s \in L$. Then $s_i \neq 0$. We claim that without loss of generality we may assume $\tau(s_i) = j$. For, since $[i]$ is minimal, $\tau(s_i) \in [i]$, so there is some $h \in H$ such that $\tau(h s_i) = j$. If $t \in M$ is such that $(t)_\tau(s_i) = h$, then $t s \in L$ and $\tau((ts)_i) = j$.

Now let $x \in M$ be arbitrary with $s_i + x_i = e_i$. Therefore, $\tau(x_i) \neq i$; otherwise, for each $c \in C_i$, $s_i(c) = c - x_i(c) \in C_i$, which implies $C_j \subseteq C_i$, i.e., $j \leq i$. Let $y \in M$ be defined by $y_i = e_i$ and $y_k = 0$ for $k \neq i$. Then $z := y(s + x) - xy \in L$ and $z_i = y_i(s_i + x_i) - y_{\tau(x_i)} x_i = e_i$, so $z = e_i$. But this means $e_i \in L$ and $L = M e_i$.

Corollary II.2. If $[i]$ is minimal with respect to $\preceq$, then $M e_k$ is a minimal left ideal of $M$ for all but (possibly) one $k \in [i]$. 
Proof. If there is more than one element in \([i]\), then for at most one \(k \in [i]\) can we have that \(j \leq k\) for all \(j \in [i]\).

To illustrate the above, let \(G\) be any torsion free abelian group regarded as a \(\mathbb{Z}\)-module, and consider the matched pair \((\mathcal{C}_G(\mathbb{Z}), \text{Hom})\). Then \(I\) has just two equivalence classes, \([0] = \{0\}\) and \(I \setminus \{0\}\), the latter being minimal. If \(G\) is the infinite cyclic group, then \(G = C_k\). Then \(j \leq k\) for all \(j \in I \setminus \{0\}\), but for each \(j \in I \setminus \{0\}, j \neq k\) we have \(k \nsubseteq j\), so \(Me^j\) is minimal for each \(j \in I \setminus \{0\}, j \neq k\). We note that, in this case, it turns out that \(Me^k\) is also minimal (see Theorem 2.5). By contrast, let \(G := Q\) and \(R := \mathbb{Z}\), and consider the pair \((\mathcal{C}, \mathcal{H})\) where \(\mathcal{C} := \{C_0 := \{0\}, C_1 := Q\}\) and \(\mathcal{H} := \{0, \mathbb{Z}\}\). Then \(M(\mathcal{C}, \mathcal{H}) = Me^1 \cong \mathbb{Z}\), so \(Me^1\) is not minimal; in fact, \(M\) has no minimal left ideal. Note here that \([z] = \{z\}\) for \(i \in I\); so if \([z] = \{i\}\) and \([i]\) is minimal, it need not be the case that \(Me^i\) is minimal.

Theorem II.3. The left ideal \(Me^i\) is strictly minimal if and only if \(h \in H_i, h \neq 0\), implies \(h^{-1} \in H\).

Proof. This follows from the fact that for \(s \in M, Ms\) will be properly contained in \(Me^i\) if and only if \(s = se^i\) and there is no \(t \in M\) such that \(ts = e^i\), i.e., such that \((ts)_i = t_{s(e^i)}s_i = e_i\).

Since \(M = A([i]) \oplus Me^i\), we have the following dual result about maximal left ideals.

Theorem II.4. Let \([i]\) be minimal with respect to \(\preceq\).

(a) If there is \(j \in [i]\) with \(j \nsubseteq i\), then \(A([i])\) is a maximal left ideal of \(M\).

(b) \(A([k])\) is maximal for all but (possibly) one \(k \in [i]\). Moreover, the left ideal \(A([j])\) is strictly maximal if and only if \(h \in H_j, h \neq 0\), implies \(h^{-1} \in H\).

Combining Theorem II.3, the last part of Theorem II.4, and Corollary I.11 we obtain

Theorem II.5. Let \(G\) be Noetherian and \((\mathcal{C}, \mathcal{H})\) be a matched pair on \(G\). If \([i]\) is minimal with respect to \(\preceq\), then \(Me^i\) is strictly minimal and \(A([j])\) is strictly maximal for each \(j \in [i]\).

Recall that a near-ring \(N\) is 2-semisimple provided the intersection of all strictly maximal ideals of \(N\) is \(\{0\}\), i.e., if \(J_2(N) = \{0\}\). When \(G\) is Noetherian and \([i]\) is minimal for each \(i \in I\), then \(J_2(M) \subseteq \bigcap_{i \in I} A([i]) = \{0\}\). Thus we have

Corollary II.6. Under the hypothesis of Theorem II.5, if \([i]\) is minimal with respect to \(\preceq\) for each nonzero \(i \in I\), then \(M\) is 2-semisimple.

III. Two-sided ideals

As noted above, the \(A(J)\) are obvious examples of left ideals in \(M(\mathcal{C}, \mathcal{H})\). In this section we focus on two-sided ideals. We first determine when the \(A(J)\) will be two-sided. We then investigate conditions under which \(M = M(\mathcal{C}, \mathcal{H})\) is a simple near-ring.

We say a nonempty set \(J \subseteq I\) is a sink if \(\sigma(h) \in J\) implies \(\tau(h) \in J\). Note that both \(\{0\}\) and \(I\) are sinks which we refer to as the trivial sinks.
Theorem III.1. For $\emptyset \neq J \subseteq I$, the left ideal $A(J)$ is a two-sided ideal if and only if $J$ is a sink.

Proof. Let $J$ be a sink. We show $A(J)$ is a right ideal. If $s \in A(J)$ and $t \in M$, then for $j \in J$, $(st)_j = skt_j$ where $\tau(t_j) = k$. Since $j \in J$, we have $k \in J$ and thus, since $s \in A(J)$, $sk = 0$. On the other hand, if $J \neq \emptyset$ is not a sink, then there exists $h \in H$ with $j := \sigma(h)$ and $k := \tau(h)$ with $k \notin J$. Let $s \in A(J)$ with $sk = ek$, and let $t \in M$ with $t_j = h$. Then $(st)_j = skt_j = h \neq 0$, so $st \notin A(J)$.

In light of this result it is of some importance to determine the sinks in $I$. The next result gives some information in this regard.

Theorem III.2. (a) If $J$ is any nonempty subset of $I$, then $\Sigma(J) := \{t(A) | A \in H$ and $\sigma(h) \in J\}$ is a sink called the sink generated by $J$.

(b) If $K := \bigcup \{[i] | i \leq J\}$ for some $j \in J \subseteq I\}$, then $K = \Sigma(J)$.

(c) If $\mathcal{A}$ is any class of $R$-modules and $\mathcal{B} \subseteq \mathcal{A}$, then $\Sigma(\mathcal{B}) := \{i \in I | C_i \in \mathcal{C}_G(\mathcal{B})\}$ is a sink with regard to any matched pair $(\mathcal{C}_G(\mathcal{A}), \mathcal{H})$.

(d) If $\mathcal{F}$ is any family of sinks in $I$, then $\bigcup \mathcal{F}$ and $\bigcap \mathcal{F}$ are also sinks.

Proof. (a) Suppose $h \in H$ with $\sigma(h) \in \Sigma(J)$. Thus there exists $h' \in H$ with $\sigma(h') \in J$ and $\tau(h') = \sigma(h)$. But then $hh' \in H$ and $\sigma(hh') = \sigma(h') \in J$. Hence $\tau(h) = \tau(hh') \in \Sigma(J)$ as desired.

(b) If $i \in \Sigma(J)$, then there is some $h \in H$ with $\sigma(h) \in J$ and $\tau(h) = i$. Consequently, $[i] \preceq [\sigma(h)]$, so $i \in K$. Conversely, if $i \in K$, then $[i] \preceq [j]$, for some $j \in J$, which in turn implies that there exists $h \in H$ with $\sigma(h) = j$ and $\tau(h) = i$. Hence $i \in \Sigma(J)$.

(c) If $\sigma(h) \in \Sigma(\mathcal{B})$, then $\tau(h) \in \Sigma(\mathcal{B})$ since $h(C_{\sigma(h)}) \in \mathcal{C}_G(\mathcal{B})$.

(d) This is clear.

We apply this theorem to obtain some specific instances of sinks.

Corollary III.3. The following are sinks for any matched pair $(\mathcal{C}, \mathcal{H})$ on $G$:

(a) $[i] \cup \{0\}$ if $[i]$ is minimal.

(b) $\{i | C_i$ has property $P\}$, where $P$ is any property preserved by homomorphisms, e.g., the property of begin simple or $\{0\}$, singular, finitely generated, cyclic, Noetherian, and Artinian.

Proof. Part (a) follows from III.2(b), and part (b) follows from III.2(c).

Another way of getting sinks in the case $(\mathcal{C}, \text{End})$ is given by

Theorem III.4. Let $F \leq G$ be a fully invariant submodule. Then $\{i \in I | C_i \subseteq F\}$ is a sink with respect to the matched pair $(\mathcal{C}, \text{End})$.

This result, whose straightforward proof is omitted, shows, for example, that there are sinks associated with $\text{Rad} G$ and $\text{Soc} G$ and with the singular submodule $Z(G)$ of $G$.

We now turn to simplicity of $M(\mathcal{C}, \mathcal{H})$. From the above we know if there are two or more nonzero classes in $I \sim$, then $M$ cannot be simple. However, the existence of just one nonzero class does not guarantee simplicity, as the example with $G = Q$ and $R = Z$ discussed after Corollary II.2 shows. We start with a decomposition result.

Lemma III.5. Suppose $I = [0] \cup (U_{k=1}^n [i_k])$ where each $[i_k]$ is minimal. Then $M = \sum_{k=1}^n \bigoplus M_k$ where $M_k := A(I \setminus [i_k])$ is an ideal.
Proof. Note that $I \backslash [i_k]$ is a sink for each $k$, $M = \sum_{k=1}^{n} M_k$, and $M_k \cap (\sum_{k \neq l} M_l) = \{0\}$.

Recall that $\mathcal{P}$ is the class of simple $R$-modules. We now consider the case $(\mathcal{E}_G(\mathcal{P}), \text{Hom})$.

**Theorem III.6.** If $G$ is Noetherian with $\text{Soc} G \neq \{0\}$, then $M(\mathcal{E}_G(\mathcal{P}), \text{Hom})$ is a direct sum of a finite number of ideals, each of which is invariantsy simple as a near-ring.

**Proof.** Since $G$ is Noetherian, there are only finitely many isomorphism types of simple $R$-modules represented in $\mathcal{E}_G(\mathcal{P})$. Each isomorphism class gives rise to a class in $I/\sim$ which is minimal with respect to $\preceq$. Since the union of these classes together with $\{0\}$ is $I$, we can apply the preceding lemma to obtain a decomposition of $M$ into ideals. To complete the proof, it suffices to show if $I = \{0\} \cup [i]$, where $C_j$ is simple for $j \in [i]$, then $M(\mathcal{E}_G(\mathcal{P}), \text{Hom})$ is invariantsy simple. Since $G$ is Noetherian, so is $F := \text{Soc} G$, and therefore $F$ is the direct sum of some of the $C_j$. For ease of notation, we let $F = \sum_{i=1}^{n} C_i$. Denote by $\eta^i : F \to C_i$, $i = 1, 2, \ldots, n$, the projection onto $C_i$. Thus $\sum_{i=1}^{n} \eta^i = 1_F$. Also, $\eta^i$ represents an element of $M$ by defining $(\eta^i)^{-1} := (\eta^i|C_i)$, $i \in I$, and in this context, $\sum_{i=1}^{n} \eta^i = e$, the identity in $M$. Now let $s \in M$, $s \neq 0$, with, say, $s_j \neq 0$. From Corollary I.11, $s_j^{-1} \in H$. Therefore, if $t \in M$ is such that $t_{s_j^{-1}} = s_j^{-1}$, then $e^i = t s_i e^j \notin \langle s \rangle$, the invariant subgroup of $M$ generated by $s$. But from $e^i$ we can easily construct $e^i$, $i = 1, 2, \ldots, n$, by left and right multiplications, again using Corollary I.11. Now $\eta^i = e^i \eta^i$ for $i = 1, 2, \ldots, n$, because, since $C_i$ is simple, $\tau((\eta^i)^{-1})$ is either $i$ or $0$, and therefore $e = \sum_{i=1}^{n} \eta^i \in \langle s \rangle$. This completes the proof.

Since every simple submodule of an $R$-module $G$ is cyclic, we have

**Corollary III.7.** If $G$ is a semisimple, Noetherian module, then $M(\mathcal{E}_G(R), \text{Hom})$ is a direct sum of a finite number of ideals, each of which is invariantly simple as a near-ring.

A special case is the following

**Corollary III.8.** If $V$ is a finite-dimensional vector space over a division ring, then $M(\mathcal{E}_V(D), \text{Hom})$ is invariantly simple.

The above result is not true for an infinite-dimensional vector space, $V$, over a division ring $D$. In fact, if we define, for $f \in M$, $\mathcal{R}(f) := \text{Span}\{ f(C_i)|C_i \in \mathcal{E} \}$ and let $K := \{ f \in M| \text{dim}\mathcal{R}(f) < \infty \}$, then $K$ is an invariant subgroup of $M$, $\{0\} \subseteq K \subseteq M$. However, from the next result, which has wide applicability, we will find that in this case $M(\mathcal{E}_V(D), \text{Hom})$ is simple.

**Theorem III.9.** Let $(\mathcal{E}, \mathcal{H})$ be a matched pair on $G$ with the following properties:

(i) There is only one nonzero class $[i]$.
(ii) If $h \in H$ is nonzero, then $h^{-1} \in H$.
(iii) There is some $F \subseteq G$ such that $|\{ i \in I|C_i \subseteq F \}| = |I|$.
(iv) There exists a $C \in \mathcal{E}$ with $C \not\subseteq F$.

Then $M(\mathcal{E}, \mathcal{H})$ is a simple near-ring.
Proof. For convenience we let \( C_1 := C \) and \( J := \{ i \in I | C_i \subseteq F \} \). From (iii) there is a bijection between \( I \) and \( J \), so without loss of generality we let \( \Phi : I \to J \) be a bijection with \( \Phi(0) = 0 \). Suppose \( s \) is a nonzero element of \( M \). We show that the ideal generated by \( s \), which we denote by \( T \), is \( M \). We first show that \( e^1 \in T \). Indeed, if \( s_i \neq 0 \), then we know there exist \( u, v \in M \) with \( \tau(v_1) = i \) and \( v_j = 0 \) for \( j \neq 1 \) and \( u{\tau(s_i)} = v_1^{-1}s_i^{-1} \), \( u_j = 0 \) otherwise. Then \( usv = e^1 \). Next, let \( t \in M \) be such that \( \tau(t_1) = 1 \) if \( i \neq 0 \). Hence \( e^1 t = t \). Now choose \( x \in M \) so that \( \tau(t_1 + x_i) = \Phi(i) \). We claim that \( \tau(x_i) \notin J \) for all \( i \neq 0 \). To establish this for a particular \( i \), let \( g \in C_i \) be such that \( t_i(g) \notin F \), which is possible because \( C_i \subseteq F \). Then \( (t_i + x_i)(g) = t_i(g) + x_i(g) \in \Phi(i) \subseteq J \). This implies \( x_i(g) \notin F \), which proves our assertion. Finally, choose \( y \in M \) with \( y_j = \frac{(t_{\Phi^{-1}(j)} + x_{\Phi^{-1}(j)})^{-1}}{j \notin J \text{ or } j = 0} \). Then \( yx = 0 \) and \( y(t + x) = e \), so \( e = y(t + x) - yx \in T \), i.e. \( T = M \).

We remark that Theorem I.7 follows from Theorem III.9. Further we have

Corollary III.10. Let \( V \) be a vector space over a division ring \( D \). Then \( M(\mathbb{C}(D), \mathbb{H}) \) is a simple near-ring.

Proof. The finite-dimensional case is that of Corollary III.8. For the infinite-dimensional case, let \( \mathcal{B} = \{ b_k \} \), \( k \in \mathcal{K} \), be a basis for \( V \). Then the cardinality of the set of cyclic submodules in \( V' := \text{Span}(\mathcal{B} \setminus \{ b_k \}) \) is the same as that of the set of all cyclic submodules in \( V \). So, by applying Theorem I.12 and choosing \( F := V' \), \( C := b_kD \), we see that the hypotheses of Theorem III.9 are satisfied. Thus \( M(\mathbb{C}(D), \mathbb{H}) \) is simple.

As a further application of Theorem III.9 we let \( K \) be any infinite field and let \( R := K[x] \) and \( G := K[x] \). With \( \mathcal{E} := \mathbb{E}(R) \) and \( \mathcal{H} := \mathbb{H} \), it is clear that (i), (ii), and (iii) of Theorem III.9 are satisfied if we take \( F := xK[x] \). Note that for each \( \alpha \in K \), \( (x^2 + \alpha x)K[x] \subseteq F \), but \( K[x] \notin F \).

We wish to characterize when \( M = M(\mathbb{E}(R), \mathbb{H}) \) is a simple near-ring. As we have noted above, if \( M \) is simple, then there is only one nonzero class with respect to \( \sim \), i.e., \( |I/\sim| = 2 \). So, we focus now on \( |I/\sim| = 2 \). In certain situations this condition is also sufficient.

Theorem III.11. Let \( \mathcal{E} = \mathbb{E}(R) \), \( \mathcal{H} = \mathbb{H} \), and suppose \( G \) has finite length. Then the following are equivalent:

1. \( M \) is simple.
2. \( M \) is invariantly simple.
3. \( |I/\sim| = 2 \).

In this case, \( G \) is semisimple.

Proof. We know (2) \( \Rightarrow \) (1) and (1) \( \Rightarrow \) (3) , so it remains to show (3) \( \Rightarrow \) (2). To this end let \( C_j \in \mathcal{E} \). Since \( \mathcal{E} = \mathbb{E}(R) \), \( C_j \) is a cyclic \( R \)-module. We show each \( C_j \) is simple. Assume the contrary. Then there is a cyclic submodule \( C_i \) of \( C_j \), \( C_i \subsetneq C_j \). Now, \( C_i \in \mathcal{E} \) and \( |I/\sim| = 2 \) implies there exists \( h \in H \), \( h : C_j \to C_i \). Moreover, since \( G \) is of finite length, \( G \) is Noetherian, so \( h^{-1} \in H \). But then, considering \( h|C_i \), we get \( h(C_i) = C_k \subseteq C_i \), otherwise \( \text{Ker}(h) \neq \{0\} \). In this manner we get an infinite descending chain in \( G \), contrary to \( G \) being of finite length. Since each cyclic is simple, \( \text{Soc} G \neq \{0\} \), so now as in the proof of Theorem III.6, we get that \( M \) is invariantly simple.

To show that \( G \) is semisimple, let \( a \in G \). If \( G = ar \), then we are finished. If not, there exists \( b \in G \setminus ar \). Then \( ar \cap br = \{0\} \) since they are both simple.
If \( G = aR \oplus bR \), then again we are finished. If not, we find \( c \in G \setminus aR \oplus bR \). This process must end since \( G \) is of finite length.

**Corollary III.12.** If \( R \) is an Artinian ring and \( G \) is finitely generated, then the following are equivalent:

1. \( M \) is simple.
2. \( M \) is invariantly simple.
3. \( |I/\sim| = 2 \).

Further, if \( G \) is a faithful \( R \)-module, then \( R \) is a semisimple ring.

**Proof.** Since \( R \) is Artinian, \( R \) is Noetherian, and \( G \) is finitely generated, \( G \) has finite length. Thus the first sentence follows from Theorem III.11. Suppose now \( G \) is faithful. For each \( g \in G \setminus \{0\} \), \( gr \) is simple, so \( rR(g) = \{r \in R|gr = 0\} \) is a maximal right ideal of \( R \). Therefore, \( J(R) \subseteq \bigcap_{g \in R} rR(g) = A(G) = \{0\} \), hence the result.

We remark that Theorems I.6 and I.7 show that the above corollary is not true if we take \( R \) to be Noetherian. We are thus led to the following problem which remains open.

**Problem.** Let \( G \) be Noetherian with matched pair \((G_G(R), \text{Hom})\) and \( |I/\sim| = 2 \). Is \( M(G_G(R), \text{Hom}) \) a simple near-ring?

We conclude this section and the paper with one further special case. We start here with a few remarks which provide a summary of the above, indicating the present status of the problem. First, we note that all of the nonzero \( C_i \in \mathcal{F} \) are isomorphic, and if \( h: C_i \rightarrow C_j \) is any isomorphism, then \( h \in H \). Moreover, if \( h \in H \), \( h \neq 0 \), then \( h^{-1} \in H \).

Suppose any \( C \in \mathcal{F} \) is minimal; then it is simple, and hence every \( C \in \mathcal{F} \) is simple. So this is the semisimple case, and since \( G \) is Noetherian, \( G \) is the direct sum of finitely many simple submodules. This case was handled in Theorem III.6. So, henceforth, we assume every nonzero \( C \in \mathcal{F} \) contains an infinite descending chain of elements of \( \mathcal{F} \). If any one (and therefore all) of them contains \( |I| \) elements of \( \mathcal{F} \), then we can apply Theorem III.9. We therefore assume that no element of \( \mathcal{F} \) contains \( |I| \) elements of \( \mathcal{F} \). This implies that \( G \) is not cyclic, so every cyclic is proper.

Since \( G \) is Noetherian, \( G \) is the sum (albeit not necessarily direct) of finitely many elements of \( \mathcal{F} \). For convenience we take \( G = \sum_{i=1}^{n} C_i \). Let \( J_i := \{j \in I|C_j \subseteq C_i\} \), \( i = 1, 2, \ldots, n \), and let \( C_{j_i} \) be chosen such that \( j_i \notin J_i \), i.e., \( C_{j_i} \not\subseteq C_i \).

Let \( T \) be any nonzero ideal of \( M \). Then, for each \( l \in I \), \( e^l \in T \). In fact, let \( 0 \neq s \in T \). For some \( i \in I \), \( s_i \neq 0 \). Furthermore, there exists \( h \in H \), \( h : C_i \rightarrow C_j \). We let \( u \in M \) be defined by \( u_i = h \) and \( u_j = 0 \), \( j \neq l \), and let \( v \in M \) be defined by \( v_{r(s_i)} = h^{-1}s_i^{-1} \) and \( 0 \) otherwise. Then \( vse^l \in T \) and \( vse^l = e^l \). Note that here we have used only the fact that \( T \) is an invariant subgroup.

Now, for \( i = 1, 2, \ldots, n \) define \( \delta^i \in M \) by \( (\delta^i)_j = e_j \) if \( j \in J_i \) and \( 0 \) otherwise. We show \( \delta^i \in T \). Indeed, let \( s \in M \) with \( r(s_j) = j_i \) if \( j \in J_i \) and \( 0 \) otherwise. Then \( e^lse = s \in T \) since \( e^l \in T \). Now choose \( x \in M \) such that \( s_j + x_j = e_j \) if \( j \in J_i \) and \( 0 \) otherwise. If \( c \in C_j \) is such that \( s_j(c) \notin C_i \) (which is possible because \( s_j(C_j) = C_j \not\subseteq C_i \)), then \( s_j(c) + x_j(c) = c \). Therefore, \( x_j(c) \notin C_i \), i.e., \( \tau(x_j) \notin J_i \). From this we get \( \delta^i(s + x) - \delta^ix = \delta^i \in T \).
We now make an additional assumption, namely, if $C_i$, $C_j \in \mathcal{C}$ with $C_i = g_i R$ and $C_j = g_j R$, then there exists $h: C_i \to C_j$ with $h(g_i) = g_j$. From this, we show $e \in T$. Let $K := \{g_i | i \in I\}$ be a set of generators for $\mathcal{C}$, i.e., $C_i = g_i R$. Let $j \in I$, $j \neq 0$, and suppose $g_j = f_{j1} + \cdots + f_{jn}$, where $f_{ji} \in C_i$. Each $f_{ji}$ is a generator for a $C_{ji}$, although we do not necessarily have $f_{ji} \in K$. However, we do have $ji \in J_i$. Let $h_{ji} \in H$ be that isomorphism of $C_j$ onto $C_{ji}$ which takes $g_j$ to $f_{ji}$, and let $(\gamma^i)j = h_{ji}$, for each $j \in I$, $j \neq 0$. Then $\delta^i \gamma^i = \gamma^i \in T$. We claim that $\sum_{i=1}^n \gamma^i = e$. Indeed, $(\sum_{i=1}^n \gamma^i)(g_j) = (h_{j1} + \cdots + h_{jn})(g_j) = f_{j1} + \cdots + f_{jn} = g_j$. Hence, $e \in T$, and consequently $M$ is simple. We have established

**Theorem III.13.** Let $G$ be a noncyclic Noetherian $R$-module with matched pair $(\mathcal{C}_G(R), \text{Hom})$ satisfying $|I| \sim | = 2$. If for each $C_i$, $C_j \in \mathcal{C}$, $C_i = g_i R$, $C_j = g_j R$, there exists $h: C_i \to C_j$ with $h(g_i) = g_j$, then $M(\mathcal{C}_G(R), \text{Hom})$ is a simple near-ring.

**Acknowledgment**

This paper was written while the second author was a visiting professor at Texas A & M University in 1991. He wishes to express his gratitude for the financial assistance and hospitality provided by Texas A & M.

**References**


