FIXED POINT ITERATION PROCESSES
FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. Let $X$ be a uniformly convex Banach space which satisfies Opial's
condition or has a Fréchet differentiable norm, $C$ a bounded closed convex
subset of $X$, and $T: C \to C$ an asymptotically nonexpansive mapping. It is
then shown that the modified Mann and Ishikawa iteration processes defined by

$$x_{n+1} = t_n T^n x_n + (1-t_n)x_n \quad \text{and} \quad x_{n+1} = t_n T^n (s_n T^n x_n + (1-s_n)x_n) + (1-t_n)x_n,$$

respectively, converge weakly to a fixed point of $T$.

1. Introduction

Let $C$ be a nonempty subset of a Banach space $X$. A mapping $T: C \to C$
is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of
positive numbers with $\lim_{n \to \infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y$ in $C$ and $n = 1, 2, \ldots$. This class of mappings, as a natural
extension to that of nonexpansive mappings, was introduced by Goebel and
Kirk [4] in 1972. They proved that if $C$ is a bounded closed convex subset of
a uniformly convex Banach space $X$, then every asymptotically nonexpansive
self-mapping $T$ of $C$ has a fixed point. This existence result was recently
generalized in [14] to a nearly uniformly convex (NUC) Banach space setting
(see [5] for definition).

The study of iterative construction for fixed points of asymptotically nonex-
pansive mappings began in 1978. Bose [1] first proved that if $C$ is a bounded
closed convex subset of a uniformly convex Banach space $X$ which satisfies
Opial's condition [7] and if $T: C \to C$ is an asymptotically nonexpansive
mapping, then $\{T^n x\}$ converges weakly to a fixed point of $T$ provided $T$
is asymptotically regular at $x$, i.e., $\lim_{n \to \infty} \|T^n x - T^{n+1} x\| = 0$. This conclusion
is still valid [8, 14] if Opial's condition of $X$ is replaced by the condition that
$X$ has a Fréchet differentiable norm. Furthermore, in both cases, asymptotic
regularity of $T$ at $x$ can be weakened to weak asymptotic regularity of $T$ at
$x$, i.e., $\text{w-lim}_{n \to \infty} (T^n x - T^{n+1} x) = 0$ (see [12, 13]).

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Recently, Schu [10] considered the following modified Mann iteration process:
\[
(M) \quad x_{n+1} = t_n T^n x_n + (1 - t_n) x_n, \quad n \geq 1,
\]
where \( \{t_n\} \) is a sequence of real numbers in \((0, 1)\) which is bounded away from both 0 and 1, i.e., \( a \leq t_n \leq b \) for all \( n \) and some \( 0 < a \leq b < 1 \). He verified that if \( C \) is a bounded closed convex subset of a Banach space \( X \) satisfying Opial’s condition and if \( T: C \rightarrow C \) is an asymptotically nonexpansive mapping such that \( \sum_{n=1}^{\infty} (k_n - 1) \) converges, then the modified Mann iteration process (M) converges weakly to a fixed point of \( T \). Unfortunately, Schu’s theorem does not apply to the \( L^p \) spaces if \( p \neq 2 \) since none of these spaces satisfy Opial’s condition (cf. [7]).

In this paper we first show that Schu’s theorem remains true if the assumption that \( X \) satisfies Opial’s condition is replaced by the one that \( Y \) has a Fréchet differentiable norm. This result (Theorem 3.1) applies to the \( L^p \) spaces for \( 1 < p < \infty \) since each of these spaces is uniformly convex and uniformly smooth. We then prove the weak convergence of the modified Ishikawa iteration process (cf. Ishikawa [6]):
\[
(I) \quad x_{n+1} = t_n T^n (s_n T^n x_n + (1 - s_n) x_n) + (1 - t_n) x_n, \quad n \geq 1,
\]
in a uniformly convex Banach space which either satisfies Opial’s condition or has a Fréchet differentiable norm.

2. Preliminaries and lemmas

Let \( X \) be a Banach space. Recall that \( X \) is said to satisfy Opial’s condition [7] if for each sequence \( \{x_n\} \) in \( X \) the condition \( x_n \rightharpoonup x \) weakly implies
\[
\lim_{n \to \infty} \|x_n - x\| < \lim_{n \to \infty} \|x_n - y\| \quad \text{for all } y \in X \text{ different from } x.
\]
It is known [7] that each \( L^p \) \((1 \leq p < \infty)\) enjoys this property, while \( L^2 \) does not unless \( p = 2 \). It is also known [3] that any separable Banach space can be equivalently renormed so that it satisfies Opial’s condition. Recall also that \( X \) is said to have a Fréchet differentiable norm if, for each \( x \) in \( S(X) \), the unit sphere of \( X \), the limit
\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]
exists and is attained uniformly in \( y \in S(X) \). In this case, we have
\[
(2.1) \quad \frac{1}{2} \|x\|^2 + \langle h, J(x) \rangle \leq \frac{1}{2} \|x + h\|^2 \leq \frac{3}{2} \|x\|^2 + \langle h, J(x) \rangle + b(\|h\|)
\]
for all \( x, h \in X \), where \( J \) is the normalized duality map from \( X \) to \( X^* \) defined by
\[
J(x) = \{x^* \in X^*: \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},
\]
\( \langle \cdot, \cdot \rangle \) is the duality pairing between \( X \) and \( X^* \), and \( b \) is a function defined on \([0, \infty)\) such that \( \lim_{t \to 0} b(t)/t = 0 \).

Suppose now that \( C \) is a bounded closed convex subset of a Banach space \( X \) and \( \{T_n\} \) is a sequence of Lipschitzian self-mappings of \( C \) such that the set \( F \) of common fixed points of \( \{T_n\} \) is nonempty. Denote by \( L_n \) the Lipschitz constant of \( T_n \). In the sequel, we always assume \( L_n \geq 1 \) for all \( n \geq 1 \) and use the notations \( \lim = \limsup \), \( \lim = \liminf \), \( \rightharpoonup \) for weak convergence, \( \to \) for strong convergence, and \( F(T) \) for the set of fixed points of \( T \).
For a given $x_1 \in C$, we recurrently define the sequence $\{x_n\}$ by
\[ x_{n+1} = T_n x_n, \quad n \geq 1. \]

**Lemma 2.1.** Suppose that $\sum_n (L_n - 1)$ converges. Then for each $f \in F$, $\lim_n \|x_n - f\|$ exists.

**Proof.** For all $n, m \geq 1$, we have
\[
\|x_{n+m+1} - f\| = \|T_{n+m} x_{n+m} - f\| \leq L_{n+m} \|x_{n+m} - f\| \leq \left( \prod_{j=n}^{n+m} L_j \right) \|x_n - f\|.
\]
Since $\sum_n (L_n - 1)$ converges, it follows that
\[
\lim_{m \to \infty} \|x_{n+m+1} - f\| \leq \left( \prod_{j=n}^{\infty} L_j \right) \|x_n - f\|.
\]
Consequently,
\[
\lim_n \|x_n - f\| \leq \lim_n \|x_n - f\|.
\]
This proves the lemma. \(\square\)

**Lemma 2.2.** Suppose that $X$ is uniformly convex and $\sum_n (L_n - 1)$ converges. Then $\lim_{n \to \infty} \|tx_n + (1-t)f_1 - f_2\|$ exists for every $f_1, f_2 \in F$ and $0 \leq t \leq 1$.

**Proof.** We follow an idea of Reich [9]. Set
\[
a_n = a_n(t) = \|tx_n + (1-t)f_1 - f_2\|, \quad S_{n,m} = T_{n+m-1} T_{n+m-2} \cdots T_n,
\]
and
\[
b_{n,m} = \|S_{n,m}(tx_n + (1-t)f_1) - (tx_{n+m} + (1-t)f_1)\|.
\]
Then, observing $S_{n,m} x_n = x_{n+m}$, we get
\[
a_{n+m} = \|tx_{n+m} + (1-t)f_1 - f_2\| \\
\leq b_{n,m} + \|S_{n,m}(tx_n + (1-t)f_1) - f_2\| \\
\leq b_{n,m} + \left( \prod_{j=n}^{n+m-1} L_j \right) a_n \leq b_{n,m} + H_n a_n,
\]
where $H_n = \prod_{j=n}^{\infty} L_j$. By a result of Bruck [2], we have
\[
b_{n,m} \leq H_n g^{-1}(\|x_n - f_1\| - H_n^{-1} \|S_{n,m} x_n - f_1\|) \\
\leq H_n g^{-1}(\|x_n - f_1\| - \|x_{n+m} - f_1\| + (1 - H_n^{-1})d),
\]
where $g: [0, \infty) \to [0, \infty)$, $g(0) = 0$, is a strictly increasing continuous function depending only on $d$, the diameter of $C$. Since $\lim_{n \to \infty} H_n = 1$, it follows from Lemma 2.1 that $\lim_{n \to \infty, m \to \infty} b_{n,m} = 0$. Therefore,
\[
\lim_{m \to \infty} \lim_{n \to \infty, m \to \infty} a_m \leq \lim_{n \to \infty, m \to \infty} b_{n,m} + \lim_{n \to \infty} H_n a_n = \lim_{n \to \infty} a_n.
\]
This completes the proof. \(\square\)
Lemma 2.3. Suppose that $X$ is a uniformly convex Banach space with a Fréchet differentiable norm and that $\sum_n (L_n - 1)$ converges. Then for every $f_1, f_2 \in F$, $\lim_{n \to \infty} \langle x_n, J(f_1 - f_2) \rangle$ exists; in particular,

$$\langle p - q, J(f_1 - f_2) \rangle = 0$$

for all $p, q \in \omega_\omega(x_n)$. Here, $\omega_\omega(x_n)$ denotes the weak $\omega$-limit set of $\{x_n\}$, i.e., $\omega_\omega(x_n) = \{y \in X : y = w^{-}\lim_{k \to \infty} x_{n_k} \text{ for some } n_k \uparrow \infty\}$.

Proof. Taking $x = f_1 - f_2$ and $h = t(x_n - f_1)$ in (2.1), we get

$$\frac{1}{2} \|f_1 - f_2\|^2 + t \langle x_n - f_1, J(f_1 - f_2) \rangle \leq \frac{1}{2} \|tx_n + (1-t)f_1 - f_2\|^2$$

$$\leq \frac{1}{2} \|f_1 - f_2\|^2 + t \langle x_n - f_1, J(f_1 - f_2) \rangle + b(t\|x_n - f_1\|).$$

It follows from Lemma 2.2 that

$$\frac{1}{2} \|f_1 - f_2\|^2 + t \liminf_{n \to \infty} \langle x_n - f_1, J(f_1 - f_2) \rangle \leq \frac{1}{2} \|f_1 - f_2\|^2 + t \lim_{n \to \infty} \langle x_n - f_1, J(f_1 - f_2) \rangle + o(t).$$

This yields

$$\liminf_{n \to \infty} \langle x_n - f_1, J(f_1 - f_2) \rangle \leq \lim_{n \to \infty} \langle x_n - f_1, J(f_1 - f_2) \rangle + o(1).$$

Letting $t \to 0^+$, we see that $\lim_{n \to \infty} \langle x_n - f_1, J(f_1 - f_2) \rangle$ exists. \(\square\)

We also need the following known lemmas.

Lemma 2.4 (cf. Schu [10]). Let $X$ be a uniformly convex Banach space, $\{t_n\}$ a sequence of real numbers in $(0, 1)$ bounded away from 0 and 1, and $\{x_n\}$ and $\{y_n\}$ sequences of $X$ such that $\lim_{n \to \infty} \|x_n\| \leq a$, $\lim_{n \to \infty} \|y_n\| \leq a$, and $\lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = a$ for some $a \geq 0$. Then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

Lemma 2.5 [11]. Let $X$ be a normed space, $C$ a convex subset of $X$, and $T : C \rightarrow C$ a uniformly $L$-Lipschitzian mapping, i.e., $\|T^n x - T^n y\| \leq L \|x - y\|$ for all $x, y$ in $C$ and $n = 1, 2, \ldots$. For any given $x_1$ in $C$ and sequences $\{t_n\}$ and $\{s_n\}$ in $[0, 1]$, define $\{x_n\}$ by

$$x_{n+1} = t_n T^n (s_n T^n x_n + (1 - s_n) x_n) + (1 - t_n) x_n, \quad n \geq 1.$$

Then we have

$$\|x_n - T x_n\| \leq c_n + c_{n-1} L(1 + 3L + 2L)^2$$

for all $n \geq 2$, where $c_n = \|x_n - T^n x_n\|$.

Lemma 2.6 [14]. Suppose that $C$ is a bounded closed convex subset of a uniformly convex Banach space and $T : C \rightarrow C$ is an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at the origin, i.e., for any sequence $\{x_n\}$ in $C$, the conditions $x_n \to x_0$ and $x_n - T x_n \to 0$ imply $x_0 - T x_0 = 0$.

3. Weak convergence

In this section we prove the weak convergence of the modified Mann and the modified Ishikawa iteration processes in a uniformly convex Banach space which satisfies Opial's condition or has a Fréchet differentiable norm.
Theorem 3.1. Let $X$ be a uniformly convex Banach space with a Fréchet differentiable norm, $C$ a bounded closed convex subset of $X$, and $T : C \to C$ an asymptotically nonexpansive mapping such that $\sum_n(k_n - 1)$ converges. Then for each $x_1 \in C$, the sequence $\{x_n\}$ defined by the modified Mann iteration process (M) with $\{t_n\}$ a sequence of real numbers bounded away from 0 and 1 converges weakly to a fixed point of $T$.

Proof. Set $T_n = t_nT^n + (1 - t_n)I$. (Here $I$ is the identity operator of $X$.) Then it is easily seen that $x_{n+1} = T_nx_n$, $F(T_n) \supseteq F(T)$, and $T_n$ is Lipschitzian with constant $L_n = t_nk_n + (1 - t_n) \geq 1$. Since $L_n - 1 = t_n(k_n - 1) \leq k_n - 1$ and $\sum_n(k_n - 1)$ converges, $\sum_n(L_n - 1)$ also converges. It thus follows from Lemma 2.3 that

\begin{equation}
(p - q, J(f_1 - f_2)) = 0
\end{equation}

for all $p, q \in \omega_w(x_n)$ and $f_1, f_2 \in F(T)$. Moreover, for $f \in F(T)$, we have

\[\lim_{n \to \infty} \|T^n x_n - f\| \leq \lim_{n \to \infty} k_n\|x_n - f\| = \lim_{n \to \infty} \|x_n - f\|\]

and

\[\lim_{n \to \infty} \|t_n(T^n x_n - f) + (1 - t_n)(x_n - f)\| = \lim_{n \to \infty} \|x_{n+1} - f\|.
\]

It follows from Lemma 2.4 that $\lim_{n \to \infty} \|T^n x_n - x_n\| = 0$, which implies by Lemma 2.5 that $\lim_{n \to \infty} \|x_n - T x_n\| = 0$, which in turn implies by Lemma 2.6 that $\omega_w(x_n)$ is contained in $F(T)$. So to show that $\{x_n\}$ converges weakly to a fixed point of $T$, it suffices to show that $\omega_w(x_n)$ consists of just one point. To this end, let $p, q$ be in $\omega_w(x_n)$. Then since $p, q$ belong to $F(T)$, it follows from (3.1) that

\[\|p - q\|^2 = (p - q, J(p - q)) = 0.
\]

Therefore, $p = q$ and the proof is complete. □

Remark. We do not know whether Theorem 3.1 remains valid if $k_n$ is allowed to approach 1 slowly enough so that $\sum_n(k_n - 1)$ diverges.

Next, we consider the modified Ishikawa iteration process (I) described in §1.

Theorem 3.2. Let $X$ be a uniformly convex Banach space which satisfies Opial's condition or has a Fréchet differentiable norm, $C$ a bounded closed convex subset of $X$, and $T : C \to C$ an asymptotically nonexpansive mapping such that $\sum_n(k_n - 1)$ converges. Suppose that $x_1$ is a given point in $C$ and $\{t_n\}$ and $\{s_n\}$ are real sequences such that $\{t_n\}$ is bounded away from 0 and 1 and $\{s_n\}$ is bounded away from 1. Then the sequence $\{x_n\}$ defined by the modified Ishikawa iteration process (I) converges weakly to a fixed point of $T$.

Proof. Define a mapping $T_n : C \to C$ by

\[T_n x = t_nT^n(s_nT^n x + (1 - s_n)x) + (1 - t_n)x, \quad x \in C.
\]

Then it is easily seen that $x_{n+1} = T_nx_n$, $F(T_n) \supseteq F(T)$, and $T_n$ is Lipschitzian with constant $L_n = 1 + t_nk_n(1 + s_nk_n - s_n) - t_n \geq 1$ for $k_n \geq 1$. Since $L_n - 1 = t_n(1 + s_nk_n)(k_n - 1) \leq (1 + L)(k_n - 1)$, where $L = \sup_{n \geq 1} k_n$, we see that $\sum_n(L_n - 1)$ converges. Now repeating the arguments in the proof of Theorem 3.1, we arrive at the following conclusions:

(i) $\lim_{n \to \infty} \|x_n - f\|$ exists for every $f \in F(T)$.

(ii) $\langle p - q, J(f_1 - f_2) \rangle = 0$ for every $p, q \in \omega_w(x_n)$ and $f_1, f_2 \in F(T)$.

(iii) $\lim_{n \to \infty} \|x_n - T^n y_n\| = 0$ with $y_n = s_nT^n x_n + (1 - s_n)x_n$.
Since
\[\|T^nx_n - x_n\| \leq \|T^nx_n - T^ny_n\| + \|T^ny_n - x_n\|\]
\[\leq k_n\|x_n - y_n\| + \|T^ny_n - x_n\|\]
\[= k_n s_n\|T^nx_n - x_n\| + \|T^ny_n - x_n\|,\]
we have
\[\|T^nx_n - x_n\| \leq \frac{1}{1 - k_n s_n}\|T^ny_n - x_n\|,
from which, together with the facts that \{s_n\} is bounded away from 1 and \{k_n\}
converges to 1, we conclude that \(\lim_{n \to \infty} \|T^nx_n - x_n\| = 0\). By Lemma 2.5, we
have the following result:
(iv) \(\lim_{n \to \infty} \|x_n^* - T^nx_n\| = 0\).

It follows from (iv) and Lemma 2.6 that \(\omega_w(x_n) \subset F(T)\). So to show the
theorem, it suffices to show that \(\omega_w(x_n)\) is a singleton. To this end, we suppose
first that \(X\) satisfies Opial’s condition. Let \(p, q\) be in \(\omega_w(x_n)\) and \(\{x_n\}\) and
\(\{x_m\}\) be subsequences of \(\{x_n\}\) chosen so that \(x_n \to p\) and \(x_m \to q\). If
\(p \neq q\), then Opial’s condition of \(X\) implies that
\[\lim_{n \to \infty} \|x_n - p\| = \lim_{i \to \infty} \|x_{n_i} - p\| < \lim_{i \to \infty} \|x_{n_i} - q\| = \lim_{j \to \infty} \|x_{m_j} - q\|\]
\[< \lim_{j \to \infty} \|x_{m_j} - p\| = \lim_{n \to \infty} \|x_n - p\|.
This contradiction proves the theorem in case \(X\) satisfies Opial’s condition.
Next, we assume that \(X\) has a Fréchet differentiable norm. Then since \(\omega_w(x_n)\)
\(\subset F(T)\), as in the proof of Theorem 3.1, we derive from (ii) that for every \(p, q\) in \(\omega_w(x_n)\)
\[\|p - q\|^2 = \langle p - q, J(p - q) \rangle = 0.
This completes the proof. \(\square\)

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