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THE DEGREE OF REGULARITY OF A QUASICONFORMAL MAPPING

PEKKA KOSKELA

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ABSTRACT. T. Iwaniec has conjectured that the derivative of a locally α-Hölder continuous quasiconformal mapping of $\mathbb{R}^n$ is locally integrable to any power $p < \frac{n}{1-\alpha}$. We disprove this conjecture by producing examples of quasiconformal mappings of the plane that are uniformly Hölder continuous with exponent $\frac{1}{2} < \alpha < 1$ but whose derivatives are not locally integrable to the power $\frac{1}{1-\alpha}$.

1. INTRODUCTION

Recall that a homeomorphism f of a domain $D \subset \mathbb{R}^n$ onto a domain $D' \subset \mathbb{R}^n$ is K-quasiconformal if $f \in W^{1,n}_{\mathrm{loc}}(D)$ and $|f'(x)|^n \leq K J_f(x)$ holds a.e. in D; here $|f'(x)|$ is the operator norm of the formal derivative $f'(x)$ of f. We call f quasiconformal if f is K-quasiconformal for some K. Gehring proved in his celebrated paper [G3] that, in fact, $f \in W^{1,p}_{\mathrm{loc}}(D)$ for some $p = p(n, K) > n$ (in the plane this result is due to Bojarski [B]) whenever f is a quasiconformal mapping of a domain $D \subset \mathbb{R}^n$. Gehring and Reich have conjectured [GR; G2; I, 9.1] that a K-quasiconformal mapping f belongs to $W^{1,p}_{\mathrm{loc}}(D)$ for all $p < \frac{nK}{K-1}$.[1]

Since K-quasiconformal mappings are locally Hölder continuous [G1] with exponent $\frac{1}{K}$, the familiar Sobolev embedding [GT, Theorem 7.26] indicates that the above local Hölder continuity exponent coincides with the exponent that would be implied by the local integrability of the derivative of f to the power $\frac{nK}{K-1}$. This motivates the following conjecture due to Iwaniec.

Conjecture [I, 9.2]. If a quasiconformal mapping f of a domain $D \subset \mathbb{R}^n$ is locally Hölder continuous with exponent $0 < \alpha < 1$, then $f \in W^{1,p}_{\mathrm{loc}}(D)$ for all $1 \leq p < \frac{n}{1-\alpha}$.

Thus the Sobolev embedding is conjectured to be essentially invertible in the class of quasiconformal mappings regardless of the dilatation K of the mapping in question. Unfortunately, this conjecture, which would yield the Gehring and Reich conjecture, is false.

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[1]K. Astala has recently verified the Gehring and Reich conjecture in the plane.

Theorem. *For each $\frac{1}{2} < \alpha < 1$ there is a quasiconformal mapping f of the plane such that f is uniformly Hölder continuous with exponent α but the derivative of f fails to be locally integrable with exponent $\frac{1}{1-\alpha}$.*

The above theorem immediately disproves Iwaniec's conjecture for $\frac{1}{2} < \alpha < 1$. For the other values, one only needs to observe that our theorem guarantees for any $p > 2$ the existence of a plane quasiconformal mapping f that is locally Hölder continuous with exponent $\frac{1}{2}$ and such that the derivative of f fails to be integrable to the power p; hence Iwaniec's conjecture fails for $0 < \alpha \leq \frac{1}{2}$ as well.

The argument we employ for the proof of our theorem is as follows. First we construct a "bad" quasisymmetric mapping g of the real line onto a von Koch-type snowflake quasicircle (cf. [A; T2, p. 151]). Then, employing results of Tukia [T1], we obtain a quasiconformal mapping f of the plane that extends g. This extension will be C^1-smooth outside $\mathbb{R}$, but the derivative of the extension blows up uniformly when we approach $\mathbb{R}$.

We wish to point out that our construction is fairly standard. Nevertheless, applications of this type seem to have stayed unnoticed; we will employ our construction also in a forthcoming work [KKM] to show sharpness of Radó-type theorems for solutions to degenerate elliptic partial differential equations.

2. Proof of Theorem

Recall that an embedding $g : \mathbb{R} \to \mathbb{C}$ is quasisymmetric if there is a constant C such that

$$|g(x) - g(y)| \leq C|g(x) - g(w)|$$

whenever $x, y, w \in \mathbb{R}$ satisfy $|x - y| \leq |x - w|$. By the Beurling–Ahlfors extension theorem each quasisymmetric g with $g(\mathbb{R}) = \mathbb{R}$ extends to a quasiconformal mapping of the plane. The main ingredient in the proof of our theorem is the following similar extension result due to Tukia [T1].

2.1. **Theorem** [T1]. *If $g : \mathbb{R} \to \mathbb{C}$ is quasisymmetric, then g has a quasiconformal extension $f : \mathbb{C} \to \mathbb{C}$ such that f is C^1 in $\mathbb{C} \setminus \mathbb{R}$ and*

$$C^{-1}|f'(x+iy)| \leq \frac{|f(x+y) - f(x-y)|}{|y|} \leq C|f'(x+iy)| \tag{2.2}$$

for some fixed constant C and each $x + iy \in \mathbb{C} \setminus \mathbb{R}$.

Now we construct an appropriate quasisymmetric g. This construction appears to be folklore (cf. [A; FM; M; T2, p. 151]). Nevertheless, we sketch the necessary steps for the convenience of the reader. Fix $\frac{1}{4} < t < \frac{1}{2}$, and let $a_1 = 0$, $a_2 = t$, $a_3 = \frac{1}{2} + i(t - \frac{1}{4})^{1/2}$, $a_4 = 1 - t$, $a_5 = 1$. Then the length of each line segment $\overline{a_j a_{j+1}}$, $j = 1, 2, 3$, is t. Next, let σ_j, $j = 1, \ldots, 4$, be similarities which map the line segment $\overline{a_1 a_5}$ onto $\overline{a_j a_{j+1}}$ with $\sigma_j(a_1) = a_j$.

For a set A in the plane write $\sum(A) = \bigcup_1^4 \sigma_j(A)$, and for $p > 1$ set $\sum^p(\overline{a_1 a_5}) = \sum(\sum^{p-1}(\overline{a_1 a_5}))$. Then, for each $p \geq 1$, $\sum^p(\overline{a_1 a_5})$ consists of 4^p line segments J_k of length t^p. Assume that they are in order on $\sum^p(\overline{a_1 a_5})$ with $0 \in I_1$. Write $I_k = [(k-1)4^{-p}, k4^{-p}]$, and pick a homeomorphism $h_p : [0, 1] \to \sum^p(\overline{a_1 a_5})$ such that h_p is affine in each I_k with $h_p(I_k) = J_k$. Then the mappings h_p converge to a homeomorphism $h : [0, 1] \to h([0, 1]) := \gamma_t$ with

$h(4^k x) = (\frac{1}{t})^k h(x)$ whenever $0 \le x \le 4^k x \le 1$. Moreover, γ_t is the unique compact set $A \subset \mathbb{C}$ with $\sum(A) = A$ (see [H]) and γ_t is the limit of the iterated arcs $\sum^p(\overline{a_1 a_5}) = \sum(\sum^{p-1}(\overline{a_1 a_5}))$ in the Hausdorff metric. In fact, $\gamma_{1/3}$ is the familiar von Koch curve and, for all $\frac{1}{4} < t < \frac{1}{2}$, γ_t is a snowflake-type curve.

Notice that $t\gamma_t$ is a subarc of γ_t and likewise γ_t is a subarc of $\frac{1}{t}\gamma_t$. Thus, by defining

$$\gamma = \bigcup_{j \ge 0} \left(\frac{1}{t}\right)^j (\gamma_t \cup (-\gamma_t)),$$

we obtain an arc γ through ∞. To complete the construction, we set

$$g(4^k x) = \left(\frac{1}{t}\right)^k h(x)$$

for $k \ge 1$ and $0 \le x \le 1$, and define $g(x)$, $x < 0$, by symmetry. Then g is a homeomorphism of $\mathbb{R}$ onto γ, and it is straightforward to check (see the calculations in [M, pp. 102–103]; cf. also [FM]) that

$$|x-y|^\alpha / C \le |g(x) - g(y)| \le C|x-y|^\alpha$$

for some constant C for all $x, y \in \mathbb{R}$, where $\alpha = \log(\frac{1}{t})/\log 4$. In conclusion, we obtain

2.3. **Lemma.** *For each $\frac{1}{2} < \alpha < 1$ there is an embedding $g : \mathbb{R} \to \mathbb{C}$ such that*

$$|x-y|^\alpha / C \le |g(x) - g(y)| \le C|x-y|^\alpha \tag{2.4}$$

for some fixed constant C for all $x, y \in \mathbb{R}$.

Proof of Theorem. Fix $\frac{1}{2} < \alpha < 1$, and let g be a mapping as in Lemma 2.3. Then g is quasisymmetric, and hence Theorem 2.1 provides us with a quasiconformal mapping f of the plane extending g. Next, from (2.2) and (2.4) we conclude that

$$|f'(x+iy)|/C \le |y|^{\alpha-1} \le C|f'(x+iy)| \tag{2.5}$$

for all $x+iy$ in $\mathbb{C} \setminus \mathbb{R}$, where C is a fixed constant. Hence we deduce that $|f'|^{1/(1-\alpha)}$ fails to be locally integrable.

We are left to verify the uniform Hölder continuity of f. Fix points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in the upper half plane. Integrating the estimate

$$|f'(x+iy)| \le C|y|^{\alpha-1} \tag{2.6}$$

we arrive at

$$|f(z_j) - f(x_j)| \le C\alpha^{-1} y_j^\alpha, \qquad j = 1, 2. \tag{2.7}$$

By (2.6) we may assume that $y_1 \ge y_2$ and that $z_2 \notin B(z_1, y_1/2)$. Then

$$y_1 \le 2|z_1 - z_2|, \tag{2.8}$$

and the desired estimate

$$|f(z_1) - f(z_2)| \le C|z_1 - z_2|^\alpha \tag{2.9}$$

for some constant C follows by the triangle inequality from (2.4), (2.7), and (2.8). Analogously, (2.9) holds for z_1, z_2 in the lower half plane, and thus the

triangle inequality and the continuity of f verify (2.6) for all z_1, z_2 in the plane. The proof is complete. □

2.10. *Concluding remarks.* (1) Tukia [T2] has used the above construction for $t = \frac{1}{3}$ to produce a quasiconformal group that is not isomorphic to a Möbius group. Recently, Semmes [S] has employed a similar construction in a counterexample related to his work with G. David on strong A_∞-weights.

(2) The proof of our theorem reveals that some global integrability results arrived at in [AK] are sharp. We hope to return to this question in the future.

References

[A] K. Astala, *Selfsimilar zippers*, Holomorphic Functions and Moduli I (D. Drasin, F. W. Gehring, C. M. Earle, I. Kra, A. Marden, eds.), Springer-Verlag, New York, 1988, pp. 61–74.

[AK] K. Astala and P. Koskela, *Quasiconformal mappings and global integrability of the derivative*, J. Analyse Math. **57** (1991), 203–220.

[B] B. Bojarski, *Generalized solutions of a system of differential equations of first order and elliptic type with discontinuous coefficients*, Mat. Sb. **85** (1957), 451–503.

[FM] K. J. Falconer and D. T. Marsh, *Classification of quasi-circles by Hausdorff dimension*, Nonlinearity **2** (1989), 489–493.

[G1] F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.

[G2] ——, *Open problems*, Proc. Romanian–Finnish Seminar on Teichmuller Spaces and Quasiconformal Mappings, Romania, 1969, p. 306.

[G3] ——, *The L^P-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. **130** (1973), 265–277.

[GR] F. W. Gehring and E. Reich, *Area distortion under quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **388** (1966), 1–14.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin and Heidelberg, 1983.

[H] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

[I] T. Iwaniec, *L^P-theory of quasiregular mappings*, Quasiconformal Space Mappings. A Collection of Surveys 1960–90 (M. Vuorinen, ed.), Lecture Notes in Math., vol. 1508, Springer-Verlag, Berlin, 1992, pp. 39–64.

[KKM] T. Kilpeläinen, P. Koskela, and O. Martio, *On the fusion problem for second order elliptic partial differential equations*, preprint, 1993.

[M] J. McKemie, *Quasiconformal groups with small dilatation*, Ann. Acad. Sci. Fenn. Ser. A I Math. **12** (1987), 95–118.

[S] S. Semmes, *Bilipschitz mappings and strong A_∞-weights*, Ann. Acad. Sci. Fenn. A I Math. **18** (1993), 211–248.

[T1] P. Tukia, *Extension of Lipschitz embeddings of the real line into the plane*, Ann. Acad. Sci. Fenn. Ser. A I Math. **6** (1981), 89–94.

[T2] ——, *A quasiconformal group not isomorphic to a Möbius group*, Ann. Acad. Sci. Fenn. Ser. A I Math. **6** (1981), 149–160.

Department of Mathematics, University of Jyväskylä, P.O. Box 35, SF-40351 Jyväskylä, Finland

Current address: Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

E-mail address: koskela@math.lsa.umich.edu