THE DEGREE OF REGULARITY OF A QUASICONFORMAL MAPPING

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Abstract. T. Iwaniec has conjectured that the derivative of a locally $\alpha$-Hölder continuous quasiconformal mapping of $\mathbb{R}^n$ is locally integrable to any power $p < \frac{n}{1-\alpha}$. We disprove this conjecture by producing examples of quasiconformal mappings of the plane that are uniformly Hölder continuous with exponent $\frac{1}{2} < \alpha < 1$ but whose derivatives are not locally integrable to the power $\frac{1}{1-\alpha}$.

1. Introduction

Recall that a homeomorphism $f$ of a domain $D \subseteq \mathbb{R}^n$ onto a domain $D' \subseteq \mathbb{R}^n$ is $K$-quasiconformal if $f \in W^{1,n}_{\text{loc}}(D)$ and $|f'(x)|^n \leq Kj_f(x)$ holds a.e. in $D$; here $|f'(x)|$ is the operator norm of the formal derivative $f'(x)$ of $f$. We call $f$ quasiconformal if $f$ is $K$-quasiconformal for some $K$. Gehring proved in his celebrated paper [G3] that, in fact, $f \in W^{1,p}_{\text{loc}}(D)$ for some $p = p(n, K) > n$ (in the plane this result is due to Bojarski [B]) whenever $f$ is a quasiconformal mapping of a domain $D \subseteq \mathbb{R}^n$. Gehring and Reich have conjectured [GR; G2; I, 9.1] that a $K$-quasiconformal mapping $f$ belongs to $W^{1,p}_{\text{loc}}(D)$ for all $1 < p < \frac{nK}{K-1}$.

Since $K$-quasiconformal mappings are locally Hölder continuous [G1] with exponent $\frac{1}{K}$, the familiar Sobolev embedding [GT, Theorem 7.26] indicates that the above local Hölder continuity exponent coincides with the exponent that would be implied by the local integrability of the derivative of $f$ to the power $\frac{nK}{K-1}$. This motivates the following conjecture due to Iwaniec.

Conjecture [I, 9.2]. If a quasiconformal mapping $f$ of a domain $D \subseteq \mathbb{R}^n$ is locally Hölder continuous with exponent $0 < \alpha < 1$, then $f \in W^{1,p}_{\text{loc}}(D)$ for all $1 \leq p < \frac{n}{1-\alpha}$.

Thus the Sobolev embedding is conjectured to be essentially invertible in the class of quasiconformal mappings regardless of the dilatation $K$ of the mapping in question. Unfortunately, this conjecture, which would yield the Gehring and Reich conjecture, is false.

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\textsuperscript{1}K. Astala has recently verified the Gehring and Reich conjecture in the plane.

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Theorem. For each \( \frac{1}{2} < \alpha < 1 \) there is a quasiconformal mapping \( f \) of the plane such that \( f \) is uniformly Hölder continuous with exponent \( \alpha \) but the derivative of \( f \) fails to be locally integrable with exponent \( \frac{1}{1-\alpha} \).

The above theorem immediately disproves Iwaniec's conjecture for \( \frac{1}{2} < \alpha < 1 \). For the other values, one only needs to observe that our theorem guarantees for any \( p > 2 \) the existence of a plane quasiconformal mapping \( f \) that is locally Hölder continuous with exponent \( \frac{1}{2} \) and such that the derivative of \( f \) fails to be integrable to the power \( p \); hence Iwaniec's conjecture fails for \( 0 < \alpha \leq \frac{1}{2} \) as well.

The argument we employ for the proof of our theorem is as follows. First we construct a "bad" quasisymmetric mapping \( g \) of the real line onto a von Koch-type snowflake quasicircle (cf. [A; T2, p. 151]). Then, employing results of Tukia [T1], we obtain a quasiconformal mapping \( f \) of the plane that extends \( g \). This extension will be \( C^1 \)-smooth outside \( \mathbb{R} \), but the derivative of the extension blows up uniformly when we approach \( \mathbb{R} \).

We wish to point out that our construction is fairly standard. Nevertheless, applications of this type seem to have stayed unnoticed; we will employ our construction also in a forthcoming work [KKM] to show sharpness of Radó-type theorems for solutions to degenerate elliptic partial differential equations.

2. Proof of Theorem

Recall that an embedding \( g : \mathbb{R} \to \mathbb{C} \) is quasisymmetric if there is a constant \( C \) such that
\[
|g(x) - g(y)| \leq C |g(x) - g(w)|
\]
whenever \( x, y, w \in \mathbb{R} \) satisfy \( |x - y| \leq |x - w| \). By the Beurling–Ahlfors extension theorem each quasisymmetric \( g \) with \( g(\mathbb{R}) = \mathbb{R} \) extends to a quasiconformal mapping of the plane. The main ingredient in the proof of our theorem is the following similar extension result due to Tukia [T1].

2.1. Theorem [T1]. If \( g : \mathbb{R} \to \mathbb{C} \) is quasisymmetric, then \( g \) has a quasiconformal extension \( f : \mathbb{C} \to \mathbb{C} \) such that \( f \) is \( C^1 \) in \( \mathbb{C} \setminus \mathbb{R} \) and
\[
C^{-1} |f'(x + iy)| \leq \frac{|f(x + y) - f(x - y)|}{|y|} \leq C |f'(x + iy)|
\]
for some fixed constant \( C \) and each \( x + iy \in \mathbb{C} \setminus \mathbb{R} \).

Now we construct an appropriate quasisymmetric \( g \). This construction appears to be folklore (cf. [A; FM; M; T2, p. 151]). Nevertheless, we sketch the necessary steps for the convenience of the reader. Fix \( \frac{1}{4} < t < \frac{1}{2} \), and let \( a_1 = 0, \quad a_2 = t, \quad a_3 = \frac{1}{2} + i(t - \frac{1}{4})^{1/2}, \quad a_4 = 1 - t, \quad a_5 = 1 \). Then the length of each line segment \( \overline{a_j a_{j+1}} \), \( j = 1, 2, 3 \), is \( t \). Next, let \( \sigma_j \), \( j = 1, \ldots, 4 \), be similarities which map the line segment \( \overline{a_1 a_2} \) onto \( \overline{a_j a_{j+1}} \) with \( \sigma_j(a_1) = a_j \).

For a set \( A \) in the plane write \( \sum(A) = \bigcup_1^4 \sigma_j(A) \), and for \( p > 1 \) set \( \sum^p(\overline{a_1 a_5}) = \sum(\sum^{p-1}(\overline{a_1 a_5})) \). Then, for each \( p \geq 1 \), \( \sum^p(\overline{a_1 a_5}) \) consists of \( 4^p \) line segments \( I_k \) of length \( t^p \). Assume that they are in order on \( \sum^p(\overline{a_1 a_5}) \) with \( 0 \in I_1 \). Write \( I_k = [(k - 1)4^{-p}, k4^{-p}] \), and pick a homeomorphism \( h_p : [0, 1] \to \sum^p(\overline{a_1 a_5}) \) such that \( h_p \) is affine in each \( I_k \) with \( h_p(I_k) = J_k \). Then the mappings \( h_p \) converge to a homeomorphism \( h : [0, 1] \to h([0, 1]) := \gamma \), with
$h(4^k x) = (\frac{1}{4})^k h(x)$ whenever $0 \leq x \leq 4^k x \leq 1$. Moreover, $\gamma_t$ is the unique compact set $A \subset \mathbb{C}$ with $\sum(A) = A$ (see [H]) and $\gamma_t$ is the limit of the iterated arcs $\sum^\infty_{t=1} (A_t A_{t+1})$ in the Hausdorff metric. In fact, $\gamma_{1/3}$ is the familiar von Koch curve and, for all $\frac{1}{4} < t < \frac{1}{2}$, $\gamma_t$ is a snowflake-type curve.

Notice that $t\gamma_t$ is a subarc of $\gamma_t$ and likewise $\gamma_t$ is a subarc of $\frac{1}{t}\gamma_t$. Thus, by defining

$$\gamma = \bigcup_{j \geq 0} \left( \frac{1}{t} \right)^j (\gamma_t \cup (-\gamma_t)),$$

we obtain an arc $\gamma$ through $\infty$. To complete the construction, we set

$$g(4^k x) = \left( \frac{1}{t} \right)^k h(x)$$

for $k \geq 1$ and $0 \leq x \leq 1$, and define $g(x)$, $x < 0$, by symmetry. Then $g$ is a homeomorphism of $\mathbb{R}$ onto $\gamma$, and it is straightforward to check (see the calculations in [M, pp. 102-103]; cf. also [FM]) that

$$|x - y|^\alpha/C \leq |g(x) - g(y)| \leq C|x - y|^\alpha$$

for some constant $C$ for all $x, y \in \mathbb{R}$, where $\alpha = \log(\frac{1}{t})/\log 4$. In conclusion, we obtain

2.3. Lemma. For each $\frac{1}{2} < \alpha < 1$ there is an embedding $g : \mathbb{R} \to \mathbb{C}$ such that

$$(2.4) \quad |x - y|^\alpha/C \leq |g(x) - g(y)| \leq C|x - y|^\alpha$$

for some fixed constant $C$ for all $x, y \in \mathbb{R}$.

Proof of Theorem. Fix $\frac{1}{2} < \alpha < 1$, and let $g$ be a mapping as in Lemma 2.3. Then $g$ is quasisymmetric, and hence Theorem 2.1 provides us with a quasiconformal mapping $f$ of the plane extending $g$. Next, from (2.2) and (2.4) we conclude that

$$(2.5) \quad |f'(x + iy)|/C \leq |y|^{\alpha-1} \leq C|f'(x + iy)|$$

for all $x + iy$ in $\mathbb{C} \setminus \mathbb{R}$, where $C$ is a fixed constant. Hence we deduce that $|f'|^{1/(1-\alpha)}$ fails to be locally integrable.

We are left to verify the uniform Hölder continuity of $f$. Fix points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in the upper half plane. Integrating the estimate

$$(2.6) \quad |f'(x + iy)| \leq C|y|^{\alpha-1}$$

we arrive at

$$(2.7) \quad |f(z_j) - f(x_j)| \leq C\alpha^{-1}y_j^\alpha, \quad j = 1, 2.$$  

By (2.6) we may assume that $y_1 \geq y_2$ and that $z_2 \notin B(z_1, y_1/2)$. Then

$$(2.8) \quad y_1 \leq 2|z_1 - z_2|,$$

and the desired estimate

$$(2.9) \quad |f(z_1) - f(z_2)| \leq C|z_1 - z_2|^\alpha$$

for some constant $C$ follows by the triangle inequality from (2.4), (2.7), and (2.8). Analogously, (2.9) holds for $z_1, z_2$ in the lower half plane, and thus the
triangle inequality and the continuity of $f$ verify (2.6) for all $z_1, z_2$ in the plane. The proof is complete. □

2.10. Concluding remarks. (1) Tukia [T2] has used the above construction for $t = \frac{1}{3}$ to produce a quasiconformal group that is not isomorphic to a Möbius group. Recently, Semmes [S] has employed a similar construction in a counterexample related to his work with G. David on strong $A_\infty$-weights.

(2) The proof of our theorem reveals that some global integrability results arrived at in [AK] are sharp. We hope to return to this question in the future.

References


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