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REMARKS ON THE $(C, -1)$-SUMMABILITY OF THE DISTRIBUTION OF ZEROS OF ORTHOGONAL POLYNOMIALS

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ABSTRACT. Given $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_{n-1}$, two interlacing sequences of real numbers, the rectangular diagram for these numbers is a continuous piecewise linear function with slopes ± 1 and with n local minima at the points x_i and $n-1$ local maxima at the points y_j. Recently, S. Kerov determined the asymptotic behavior of the rectangular diagrams associated with the zeros of two consecutive orthogonal polynomials for which the coefficients in the three-term recurrence relation converge. The purpose of this note is to show how this result of S. Kerov and even some of its generalizations follow directly from certain $(C, -1)$-summability results on distribution of zeros of orthogonal polynomials proved by us some time ago.

1. INTRODUCTION

Let α be a non-negative finite Borel measure with finite moments on the real line, and let $P_n(\alpha)$ be the corresponding *monic* orthogonal polynomials with zeros $x_{1,n}(\alpha) < x_{2,n}(\alpha) < \cdots < x_{n,n}(\alpha)$. Consider the distribution function

$$F_n(\alpha, x) = \sum_{j=1}^{n} U(x - x_{j,n}(\alpha)),$$

where U is the Heaviside function defined by

$$U(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

Then $F_n(\alpha)$ is a nondecreasing piecewise constant function with jumps of size 1 at each zero $x_{j,n}(\alpha)$.

Definition. Given $0 \leq a < \infty$ and $b \in \mathbb{R}$, we say that the orthogonality measure $\alpha \in \mathrm{M}(b, a)$ if $\lim_{n\to\infty} a_n(\alpha) = a/2$ and $\lim_{n\to\infty} b_n(\alpha) = b$ where $a_n(\alpha) > 0$ and $b_n(\alpha) \in \mathbb{R}$ are the recursion coefficients in three-term recurrence

$$(x - b_n(\alpha))P_n(\alpha, x) = P_{n+1}(\alpha, x) + a_n^2(\alpha)P_{n-1}(\alpha, x). \tag{1}$$

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Define the function $\omega_n(\alpha) : \mathbb{R} \mapsto \mathbb{R}$ by

$$\begin{aligned}\omega_n(\alpha, t) &\stackrel{\text{def}}{=} \int_{x_{1,n}(\alpha)}^{t} [2F_n(\alpha, x) - 2F_{n-1}(\alpha, x) - 1]\, dx \\ &\quad - \int_{x_{1,n}(\alpha)}^{x_{n,n}(\alpha)} [F_n(\alpha, x) - F_{n-1}(\alpha, x) - 1]\, dx .\end{aligned} \tag{2}$$

Then, obviously,

$$\omega_n'(\alpha, t) = 2F_n(\alpha, t) - 2F_{n-1}(\alpha, t) - 1, \qquad t \notin \{x_{j,n}(\alpha)\} \cup \{x_{j,n-1}(\alpha)\}.$$

In particular, taking into account the interlacing property $x_{j,n}(\alpha) < x_{j,n-1}(\alpha) < x_{j+1,n}(\alpha)$,

$$\omega_n'(\alpha, t) = \begin{cases} -1, & t < x_{1,n}(\alpha), \\ 2j - 2(j-1) - 1 = 1, & x_{j,n}(\alpha) < t < x_{j,n-1}(\alpha), \\ 2j - 2j - 1 = -1, & x_{j,n-1}(\alpha) < t < x_{j+1,n}(\alpha), \\ 1, & t > x_{n,n}(\alpha), \end{cases}$$

so that $\omega_n(\alpha)$ is continuous and piecewise linear with slopes ± 1, with minima at $x_{j,n}(\alpha)$, and maxima at $x_{j,n-1}(\alpha)$, and

$$\omega_n(\alpha, t) = \left| t - x_{n,n}(\alpha) + \int_{x_{1,n}(\alpha)}^{x_{n,n}(\alpha)} [F_n(\alpha, x) - F_{n-1}(\alpha, x)]\, dx \right| , \qquad t \notin [x_{1,n}(\alpha), x_{n,n}(\alpha)], \tag{3}$$

that is, using S. Kerov's terminology [2, (i) and (ii) on p. 1, and formula (1.6) on p. 2] (cf. [3], [4], and [5]), $\omega_n(\alpha)$ is precisely the *rectangular diagram* describing the separation of the zeros of $P_{n-1}(\alpha)$ and $P_n(\alpha)$.

Notice that

$$\begin{aligned}\int_{x_{1,n}(\alpha)}^{x_{n,n}(\alpha)} F_n(\alpha, x)\, dx &= \sum_{j=1}^{n} \int_{x_{1,n}(\alpha)}^{x_{n,n}(\alpha)} U(x - x_{j,n}(\alpha))\, dx \\ &= \sum_{j=1}^{n} (x_{n,n}(\alpha) - x_{j,n}(\alpha)) = n\, x_{n,n}(\alpha) - \sum_{j=1}^{n} x_{j,n}(\alpha),\end{aligned}$$

and, similarly,

$$\begin{aligned}\int_{x_{1,n}(\alpha)}^{x_{n,n}(\alpha)} F_{n-1}(\alpha, x)\, dx &= \sum_{j=1}^{n-1} \int_{x_{1,n}(\alpha)}^{x_{n,n}(\alpha)} U(x - x_{j,n-1}(\alpha))\, dx \\ &= \sum_{j=1}^{n-1} (x_{n,n}(\alpha) - x_{j,n-1}(\alpha)) = (n-1)x_{n,n}(\alpha) - \sum_{j=1}^{n-1} x_{j,n-1}(\alpha),\end{aligned}$$

so that

$$
\begin{aligned}
\omega_n(\alpha, t) &= \int_{x_{1,n}(\alpha)}^{t} [2F_n(\alpha, x) - 2F_{n-1}(\alpha, x) - 1]\, dx \\
&\quad - x_{1,n}(\alpha) + \sum_{j=1}^{n} x_{j,n}(\alpha) - \sum_{j=1}^{n-1} x_{j,n-1}(\alpha) \\
&= \int_{x_{1,n}(\alpha)}^{t} [2F_n(\alpha, x) - 2F_{n-1}(\alpha, x) - 1]\, dx \\
&\quad - x_{1,n}(\alpha) + b_{n-1}(\alpha), \qquad t \in \mathbb{R},
\end{aligned} \tag{4}
$$

and

$$
\omega_n(\alpha, t) = \left| t + \sum_{j=1}^{n-1} x_{j,n-1}(\alpha) - \sum_{j=1}^{n} x_{j,n}(\alpha) \right| = |t - b_{n-1}(\alpha)|, \qquad t \notin [x_{1,n}(\alpha), x_{n,n}(\alpha)], \tag{5}
$$

where we used the formula

$$
\sum_{j=1}^{n} x_{j,n}(\alpha) = \sum_{j=0}^{n-1} b_j(\alpha),
$$

which follows directly from the Jacobi–matrix interpretation of orthogonal polynomials.

S. Kerov [2, Theorem 1, p. 2] (cf. [3, Theorem 1] and [4, Theorem 1.5]) proved the following

Theorem 1. *Suppose* $\alpha \in \mathrm{M}(b, a)$ *with* $a > 0$. *Then*

$$
\lim_{n\to\infty} \omega_n(\alpha, t) = \begin{cases} \frac{2a}{\pi}\left(\frac{t-b}{a}\arcsin\left(\frac{t-b}{a}\right) + \sqrt{1-\left(\frac{t-b}{a}\right)^2}\right), & \\ \qquad\qquad t \in [b-a, b+a], \\ |t-b|, \qquad\qquad t \notin [b-a, b+a], \end{cases}
$$

uniformly for $t \in \mathbb{R}$.

The purpose of this note is to show how Theorem 1 and even some of its generalizations follow directly from certain $(C, -1)$–summability results on the distribution of zeros of orthogonal polynomials proved by us some time ago (see, e.g., [6, 10]).

2. A generalization of Kerov's theorem

First, we point out that [6, Theorem 10, p. 350] can be generalized from continuously differentiable functions to absolutely continuous ones (and from $\mathrm{M}(b, a)$ with $a > 0$ to $\mathrm{M}(b, a)$ with $a \geq 0$) as follows.

Theorem 2. *Suppose* $\alpha \in \mathrm{M}(b, a)$ *with* $a \geq 0$. *If* f *is absolutely continuous in an interval, say,* Δ *such that* $\mathrm{supp}(\alpha) \subseteq \Delta$, *then*

$$
\lim_{n\to\infty} \sum_{j=1}^{n} f(x_{j,n}(\alpha)) - \sum_{j=1}^{n-1} f(x_{j,n-1}(\alpha)) = \begin{cases} \frac{1}{\pi}\int_{b-a}^{b+a} \frac{f(t)\,dt}{\sqrt{a^2-(t-b)^2}}, & \text{if } a > 0, \\ f(b), & \text{if } a = 0. \end{cases} \tag{6}
$$

For $a > 0$, this is a straightforward consequence of [6, Theorem 10, p. 350]. Nevertheless, the simplicity of the following proof deserves special attention. For the proof of Theorem 2, we need the following

Lemma 3. *Let ρ be given by $2\rho(z) = z + \sqrt{z^2 - 1}$ where we choose that branch of the square root which is positive for $z > 1$. If $\alpha \in \mathrm{M}(b, a)$ with $a \geq 0$, then the corresponding monic orthogonal polynomials $P_n(\alpha)$ satisfy*

$$\lim_{n\to\infty} \frac{P_n(\alpha, z)}{P_{n-1}(\alpha, z)} = \begin{cases} a\rho(\frac{z-b}{a}), & \text{if } a > 0, \\ z - b, & \text{if } a = 0, \end{cases} \tag{7}$$

uniformly on every compact set outside the support of α.

Proof of Lemma 3. For $a > 0$ this Poincaré type lemma has been proved in [7, Theorem 4.1.13, p. 33], whereas for $a = 0$ we argue as follows. Consider the functionals $G_n(\alpha)$ given by

$$G_n(\alpha, g) = \sum_{k=1}^{n} \lambda_{k,n}(\alpha) g(x_{k,n}(\alpha)) p_{n-1}^2(\alpha, x_{k,n}(\alpha))$$

where $p_n(\alpha)$ denotes the orthonormal polynomial associated with α and $\lambda_{k,n}(\alpha)$ are the Cotes numbers of the corresponding Gauss–Jacobi quadrature formula (cf. [8, formula (3.4.1), p. 47]). For $i = 0, 1$, and 2, let g_i be given by $g_i(x) \equiv x^i$. Then, by the recurrence formula (1) and the Gauss–Jacobi quadrature formula, we have $G_n(\alpha, g_0) \equiv 1$, $G_n(\alpha, g_1) \equiv b_{n-1}(\alpha)$, and $G_n(\alpha, g_2) \equiv b_{n-1}^2(\alpha) + a_{n-1}^2(\alpha)$, so that $\lim_{n\to\infty} G_n(\alpha, g_i) = g_i(b)$ for $i = 0, 1$, and 2. Hence, by the Bohman–Korovkin theorem on the convergence of monotone (or positive) operators (cf. [1, p. 67]), we have $\lim_{n\to\infty} G_n(\alpha, g) = g(b)$ for every continuous function g. In particular, if g_z is given by $g_z(x) = (z - x)^{-1}$, then, by the Stieltjes–Vitali theorem, $\lim_{n\to\infty} G_n(\alpha, g_z) = g_z(b)$ locally uniformly outside the derived set of the set of all zeros of all $p_n(\alpha)$'s. According to [7, Theorem 3.3.8, p. 24], for $\alpha \in \mathrm{M}(b, a)$, the derived set of the set of all zeros of all $p_n(\alpha)$'s is precisely the support of α.[1] Finally, we observe that $G_n(\alpha, g_z) \equiv \frac{P_{n-1}(\alpha, z)}{P_n(\alpha, z)}$ (cf. [8, formulas (3.3.9), (3.3.10), and (3.4.7), pp. 47–48]). Hence, (7) follows for $a = 0$ as well. □

Proof of Theorem 2. We have

$$\sum_{j=1}^{n} f(x_{j,n}(\alpha)) - \sum_{j=1}^{n-1} f(x_{j,n-1}(\alpha)) = \sum_{j=2}^{n} \int_{x_{j-1,n-1}(\alpha)}^{x_{j,n}(\alpha)} f'(t)\,dt + f(x_{1,n}(\alpha)),$$

so that

$$\left| \sum_{j=1}^{n} f(x_{j,n}(\alpha)) - \sum_{j=1}^{n-1} f(x_{j,n-1}(\alpha)) \right| \leq \int_\Delta |f'(t)|\,dt + \sup_{t\in\Delta} |f(t)|, \tag{8}$$

since the zeros $\{x_{j,n-1}(\alpha)\}_{j=1}^{n-1}$ and $\{x_{j,n}(\alpha)\}_{j=1}^{n}$ interlace and belong to Δ. Hence, we need to prove (6) for sufficiently smooth, say, entire functions only.

[1] For measures not belonging to $\mathrm{M}(b, a)$ this no longer holds; a simple counterexample being the Lebesgue measure supported in the intervals $[-2, -1]$ and $[1, 2]$.

However, if f is entire, then we can write

$$\begin{aligned}
&\sum_{j=1}^{n} f(x_{j,n}(\alpha)) - \sum_{j=1}^{n-1} f(x_{j,n-1}(\alpha)) \\
&= \frac{1}{2\pi i}\int_{\Gamma} f(z)\left[\frac{P_n'(\alpha, z)}{P_n(\alpha, z)} - \frac{P_{n-1}'(\alpha, z)}{P_{n-1}(\alpha, z)}\right] dz \\
&= \frac{1}{2\pi i}\int_{\Gamma} f(z)\left[\log\frac{P_n(\alpha, z)}{P_{n-1}(\alpha, z)}\right]' dz
\end{aligned} \tag{9}$$

where Γ is a Jordan curve containing Δ in its interior. Hence, by Lemma 3 and the Stieltjes–Vitali theorem, (6) follows for entire functions, and then, by (8), for all absolutely continuous functions as well. □

A generalized Kerov function is probably best formulated in terms of a natural generalization of $\omega_n(\alpha, t)$ in (2) given by

$$\Omega_n(\alpha, g) \stackrel{\text{def}}{=} \int_{x_{1,n}(\alpha)}^{x_{n,n}(\alpha)} [2F_n(\alpha, x) - 2F_{n-1}(\alpha, x) - 1]\, g(x)\, dx. \tag{10}$$

Then, using (4), for $t \in [x_{1,n}(\alpha), x_{n,n}(\alpha)]$, Kerov's *rectangular diagram* $\omega_n(\alpha)$ can be written as

$$\omega_n(\alpha, t) = \Omega_n(\alpha, g_t) - x_{1,n}(\alpha) + b_{n-1}(\alpha) \tag{11}$$

where g_t is the characteristic function of $(-\infty, t)$, and Theorem 1 is a partial case of the following

Theorem 4. *Suppose* $\alpha \in \mathrm{M}(b, a)$ *with* $a \geq 0$. *If* g *is integrable in an interval, say,* Δ *such that* $\operatorname{supp}(\alpha) \subseteq \Delta$, *then*

$$\lim_{n\to\infty} \Omega_n(\alpha, g) = \begin{cases} \frac{2}{\pi}\int_{b-a}^{b+a} g(t)\arcsin\left(\frac{t-b}{a}\right) dt - \int_{\xi_L(\alpha)}^{b-a} g(t)\, dt + \int_{b+a}^{\xi_R(\alpha)} g(t)\, dt, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } a > 0, \\ -\int_{\xi_L(\alpha)}^{b} g(t)\, dt + \int_{b}^{\xi_R(\alpha)} g(t)\, dt, \quad \text{if } a = 0, \end{cases} \tag{12}$$

where $\xi_L(\alpha) \stackrel{\text{def}}{=} \lim_{n\to\infty} x_{1,n}(\alpha)$ *and* $\xi_R(\alpha) \stackrel{\text{def}}{=} \lim_{n\to\infty} x_{n,n}(\alpha)$.

Proof of Theorem 2. Let G be an antiderivative of g. Then

$$\begin{aligned}
\int_{x_{1,n}(\alpha)}^{x_{n,n}(\alpha)} F_n(\alpha, x) g(x)\, dx &= \sum_{j=1}^{n} \int_{x_{1,n}(\alpha)}^{x_{n,n}(\alpha)} U(x - x_{j,n}(\alpha)) g(x)\, dx \\
&= \sum_{j=1}^{n} [G(x_{n,n}(\alpha)) - G(x_{j,n}(\alpha))] \\
&= nG(x_{n,n}(\alpha)) - \sum_{j=1}^{n} G(x_{j,n}(\alpha)),
\end{aligned}$$

and, similarly,

$$\begin{aligned}\int_{x_{1,n}(\alpha)}^{x_{n,n}(\alpha)} F_{n-1}(\alpha, x) g(x)\, dx &= \sum_{j=1}^{n-1} \int_{x_{1,n}(\alpha)}^{x_{n,n}(\alpha)} U(x - x_{j,n-1}(\alpha)) g(x)\, dx \\ &= \sum_{j=1}^{n-1} [G(x_{n,n}(\alpha)) - G(x_{j,n-1}(\alpha))] \\ &= (n-1) G(x_{n,n}(\alpha)) - \sum_{j=1}^{n-1} G(x_{j,n-1}(\alpha)) ,\end{aligned}$$

so that

$$\Omega_n(\alpha, g) = G(x_{1,n}(\alpha)) + G(x_{n,n}(\alpha)) - 2 \left(\sum_{j=1}^{n} G(x_{j,n}(\alpha)) - \sum_{j=1}^{n-1} G(x_{j,n-1}(\alpha)) \right) .$$

Hence, if $a > 0$, then by Theorem 2,

$$\begin{aligned}\lim_{n\to\infty} \Omega_n(\alpha, g) &= G(\xi_L(\alpha)) + G(\xi_R(\alpha)) - \frac{2}{\pi} \int_{b-a}^{b+a} \frac{G(t)\, dt}{\sqrt{a^2 - (t-b)^2}} \\ &= \frac{2}{\pi} \int_{b-a}^{b+a} g(t) \arcsin\left(\frac{t-b}{a}\right) dt \\ &\quad + G(\xi_L(\alpha)) - G(b-a) + G(\xi_R(\alpha)) - G(b+a),\end{aligned}$$

whereas, for $a = 0$, Theorem 2 gives

$$\begin{aligned}\lim_{n\to\infty} \Omega_n(\alpha, g) &= G(\xi_L(\alpha)) + G(\xi_R(\alpha)) - 2G(b) \\ &= G(\xi_L(\alpha)) - G(b) + G(\xi_R(\alpha)) - G(b)\end{aligned}$$

which is equivalent to (12). □

3. Unbounded recurrence coefficients

Definition. The function $\varphi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is called a *log-slowly* varying function, if it is an increasing function for which

$$\lim_{x\to+\infty} \varphi(x) = \infty \qquad \text{and} \qquad \lim_{x\to+\infty} \frac{\varphi(x+y)}{\varphi(x)} = 1, \qquad \forall y \in \mathbb{R},$$

hold.

When the recurrence coefficients are unbounded, that is, when

$$\lim_{n\to\infty} (a_n(\alpha) + |b_n(\alpha)|) = \infty ,$$

then an interesting class of recurrence coefficients is the class for which the asymptotic behavior $\lim_{n\to\infty} a_n(\alpha)/\varphi(n) = a/2 < \infty$ and $\lim_{n\to\infty} b_n(\alpha)/\varphi(n) = b \in \mathbb{R}$ hold, where φ is a log-slowly varying function. For such recurrence coefficients we have the following analog of Lemma 3.

Lemma 5. *Suppose* $\lim_{n\to\infty} a_n(\alpha)/\varphi(n) = a/2 \geq 0$ *and* $\lim_{n\to\infty} b_n(\alpha)/\varphi(n) = b \in \mathbb{R}$, *where* φ *is a log-slowly varying function. If* $[A, B]$ *is the convex hull of* $[b-a, b+a] \cup \{0\}$ *then the corresponding monic orthogonal polynomials satisfy*

$$\lim_{n\to\infty} \frac{1}{\varphi(n)} \frac{P_n(\alpha, \varphi(n)z)}{P_{n-1}(\alpha, \varphi(n)z)} = \begin{cases} a\rho(\frac{z-b}{a}), & \textit{if } a > 0, \\ z - b, & \textit{if } a = 0, \end{cases} \tag{13}$$

uniformly on every compact set outside $[A, B]$. *In addition, the smallest and largest zeros satisfy*

$$\lim_{n\to\infty} \frac{x_{1,n}(\alpha)}{\varphi(n)} = A \quad \textit{and} \quad \lim_{n\to\infty} \frac{x_{n,n}(\alpha)}{\varphi(n)} = B, \tag{14}$$

respectively.

Proof of Lemma 5. The ratio asymptotic behavior in (13) is given in [10, Theorem 2.1, p. 8]. To prove (14), first we observe that

$$\min(0, b-a) \leq \liminf_{n\to\infty} \frac{x_{1,n}(\alpha)}{\varphi(n)} \leq \limsup_{n\to\infty} \frac{x_{n,n}(\alpha)}{\varphi(n)} \leq \max(0, b+a)$$

which follows immediately from the Gershgorin bounds

$$\min_{0\leq i\leq n-1} (b_i(\alpha) - a_i(\alpha) - a_{i+1}(\alpha)) \leq x_{j,n}(\alpha) \leq \max_{0\leq i\leq n-1} (b_i(\alpha) + a_i(\alpha) + a_{i+1}(\alpha)),$$

where $a_0 = 0$ (cf. [11, p. 437 and p. 455]). The ratio asymptotic behavior (13) is equivalent to

$$\lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{j,n}(\alpha) p_{n-1}^2(\alpha, x_{j,n}(\alpha)) f(x_{j,n}(\alpha)/\varphi(n))$$
$$= \begin{cases} \frac{2}{\pi a^2} \int_{b-a}^{b+a} f(t)\sqrt{a^2 - (t-b)^2}\, dt, & \text{if } a > 0, \\ f(b), & \text{if } a = 0, \end{cases}$$

for every continuous function f in Δ, where $\lambda_{j,n}(\alpha)$ are the Christoffel numbers and $p_n(\alpha)$ are the orthonormal polynomials (see [10, p. 9]). This means that every point in the interval $[b-a, b+a]$ is a limit point of the set of contracted zeros $\{x_{j,n}(\alpha)/\varphi(n)\}$. In particular, this implies

$$\liminf_{n\to\infty} \frac{x_{1,n}(\alpha)}{\varphi(n)} \leq b - a \quad \text{and} \quad \limsup_{n\to\infty} \frac{x_{n,n}(\alpha)}{\varphi(n)} \geq b + a.$$

If $0 \in [b-a, b+a]$, then $A = b - a$ and $B = b + a$ and the result in (14) follows. If $0 < b - a$ then $A = 0$ and $B = b + a$, so that the result already holds for the largest zeros. For the smallest zeros we use the interlacing property to observe that $x_{1,n}(\alpha) < x_{1,1}(\alpha)$. Hence, since φ is increasing, we obtain

$$\liminf_{n\to\infty} \frac{x_{1,n}(\alpha)}{\varphi(n)} \leq 0,$$

which gives the desired result for the smallest zeros. Similarly, when $0 > b+a$, then $A = b - a$ and $B = 0$ so that the result holds for the smallest zeros, whereas for the largest zeros the interlacing property gives $x_{n,n}(\alpha) \geq x_{1,1}(\alpha)$, from which

$$\limsup_{n\to\infty} \frac{x_{n,n}(\alpha)}{\varphi(n)} \geq 0,$$

giving the desired result again. □

The analog of the $(C, -1)$-summability result of Theorem 2 is given by

Theorem 6. *Suppose* $\lim_{n\to\infty} a_n(\alpha)/\varphi(n) = a/2 \geq 0$ *and* $\lim_{n\to\infty} b_n(\alpha)/\varphi(n) = b \in \mathbb{R}$, *where* φ *is a log-slowly varying function. Let* f *be absolutely continuous in an interval* Δ. *If, for every* $n > 0$, $[x_{1,n}(\alpha)/\varphi(n), x_{n,n}(\alpha)/\varphi(n)] \subseteq \Delta$, *then*

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(x_{j,n}(\alpha)/\varphi(n)) - \sum_{j=1}^{n-1} f(x_{j,n-1}(\alpha)/\varphi(n))$$
$$= \begin{cases} \frac{1}{\pi}\int_{b-a}^{b+a} \frac{f(t)\,dt}{\sqrt{a^2-(t-b)^2}}, & \text{if } a > 0, \\ f(b), & \text{if } a = 0. \end{cases}$$

For continuously differentiable functions f and $a > 0$, Theorem 6 is given in [10, Theorem 2.4, p. 14]. The extension to absolutely continuous functions and $a \geq 0$ can be made just as in the proof of Theorem 2 by using Lemma 5. Note that by (14), $(A, B) \subseteq \Delta$.

The analog of Theorem 4 for log-slowly varying recurrence coefficients is in terms of

$$(15)\quad \Omega_n^{\varphi}(\alpha, g) \stackrel{\text{def}}{=} \frac{1}{\varphi(n)} \int_{x_{1,n}(\alpha)}^{x_{n,n}(\alpha)} [2F_n(\alpha, x) - 2F_{n-1}(\alpha, x) - 1]\, g\left(\frac{x}{\varphi(n)}\right) dx$$

which is a natural analog of (10).

Theorem 7. *Suppose* $\lim_{n\to\infty} a_n(\alpha)/\varphi(n) = a/2 \geq 0$ *and* $\lim_{n\to\infty} b_n(\alpha)/\varphi(n) = b \in \mathbb{R}$, *where* φ *is a log-slowly varying function. Let* g *be integrable in an interval* Δ. *If, for every* $n > 0$, $[x_{1,n}(\alpha)/\varphi(n), x_{n,n}(\alpha)/\varphi(n)] \subseteq \Delta$, *then*

$$\lim_{n\to\infty} \Omega_n^{\varphi}(\alpha, g) = \begin{cases} \frac{2}{\pi}\int_{b-a}^{b+a} g(t) \arcsin\left(\frac{t-b}{a}\right) dt - \int_A^{b-a} g(t)\,dt + \int_{b+a}^{B} g(t)\,dt, \\ \qquad\qquad\qquad\qquad \text{if } a > 0, \\ -\int_A^b g(t)\,dt + \int_b^B g(t)\,dt, \quad \text{if } a = 0, \end{cases}$$

where $[A, B]$ *is the convex hull of* $[b-a, b+a] \cup \{0\}$.

The proof of Theorem 7 goes along the lines of the proof of Theorem 4 with minor modifications.

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