A CHARACTERIZATION OF $\sigma$-SYMMETRICALLY POROUS SYMMETRIC CANTOR SETS

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Abstract. The purpose of this paper is to characterize those symmetric Cantor sets which are $\sigma$-symmetrically porous in terms of a defining sequence of deleted proportions. In contrast to other notions of porosity, a symmetric Cantor set can be $\sigma$-symmetrically porous without being symmetrically porous.

1. Introduction

If $A$ is a subset of the real line $\mathbb{R}$ and $x \in \mathbb{R}$, then the porosity of $A$ at $x$ is defined to be

$$\limsup_{r \to 0^+} \frac{\lambda(A, x, r)}{r},$$

where $\lambda(A, x, r)$ is the length of the longest open interval contained in either $(x, x + r) \cap A^c$ or $(x - r, x) \cap A^c$ and $A^c$ denotes the complement of $A$. A set is said to be porous at $x$ if it has positive porosity at $x$ and is called a porous set if it is porous at each of its points. The symmetric porosity of $A$ at $x$ is defined as

$$\limsup_{r \to 0^+} \frac{\gamma(A, x, r)}{r},$$

where $\gamma(A, x, r)$ is the supremum of all positive numbers $h$ such that there is a positive number $t$ with $t + h \leq r$ such that both of the intervals $(x - t - h, x - t)$ and $(x + t, x + t + h)$ lie in $A^c$. (A notation that we find useful is to set $sp(A, x, r) = \gamma(A, x, r)/r$.) A set $A$ is symmetrically porous if it has positive symmetric porosity at each of its points. A set $A$ is $\sigma$-porous ($\sigma$-symmetrically porous) if it is a countable union of porous (symmetrically porous) sets. The concepts of $\sigma$-porosity and $\sigma$-symmetric porosity have been shown to be distinct in [4] and [8].

The notions of porosity and $\sigma$-porosity have proved quite useful in real analysis as a means of describing the smallness of sets that arise as exceptional sets to some type of nice behavior. A survey of porosity as well as these types of
applications may be found in [9]. Recently, the more restrictive concepts of symmetric porosity and $\sigma$-symmetric porosity have begun to find similar uses. For example, Zajíček [10] improved upon a theorem from [1], by showing that the exceptional set to a certain nice behavior of continuous functions, which was known to be $\sigma$-porous, is actually $\sigma$-symmetrically porous; and Evans [3] has recently extended this result of Zajíček to a wider class of functions.

The purpose of this paper is to characterize those symmetric Cantor sets which are $\sigma$-symmetrically porous in terms of a defining sequence of deleted proportions. (Symmetric Cantor sets are quite useful for constructing examples of pathological behavior. Examples of such constructions can be found in [4] and [2].) In [7] and [6] necessary and sufficient conditions were established for a symmetric Cantor set to be porous. There it was shown that such a set is porous if and only if it is $\sigma$-porous. With symmetric porosity the situation is markedly different: a symmetric Cantor set can be $\sigma$-symmetrically porous without being symmetrically porous. We shall give an example of such a symmetric Cantor set at the end of this paper. Interestingly, it turns out that a symmetric Cantor set is $\sigma$-symmetrically porous if and only if it is $\sigma$-porous. We begin by establishing the necessary notation in the following section.

2. Preliminary notation

First we define the class of symmetric Cantor sets in $[0, 1]$. Let $\Sigma$ denote the set of all finite sequences of 0’s and 1’s, and let $\Sigma^*$ denote the set of all infinite sequences of 0’s and 1’s. If $\sigma \in \Sigma$, we denote the length of $\sigma$ by $|\sigma|$ and will write $\sigma$ in expanded form as $\sigma(1)\sigma(2)\sigma(3) \cdots \sigma(n)$. If $\sigma \in \Sigma^*$ and $n \in \mathbb{N}$, then $\sigma|_n$ will denote $\sigma(1)\sigma(2)\sigma(3) \cdots \sigma(n)$. If $0 \leq \alpha_n < 1$ for all $n = 0, 1, \ldots$, then $\{\alpha_n\}$ determines a symmetric Cantor set, $\mathcal{C}(\alpha_n)$, in $[0, 1]$. If $\alpha_n \neq 0$, we identify the complementary intervals and the noncomplementary intervals to this Cantor set using subscripts from $\Sigma$ in the usual way; i.e., $I_\sigma = (\frac{1}{2} - \alpha_0/2, \frac{1}{2} + \alpha_0/2)$, $J_0$ and $J_1$ are the right- and left-hand components of the complement of $I_\sigma$, respectively, $I_0$ and $I_1$ are the open intervals of length $\alpha_1(1 - \alpha_0)/2$ centered in $J_0$ and $J_1$, respectively, and so on. If one of the $\alpha_n = 0$, we proceed as above with the exception that if $|\sigma| = n$, then $I_\sigma$ is a “marking” of the center point of $J_\sigma$ (not a interval) and $J_\sigma 0$ and $J_\sigma 1$ intersect in $I_\sigma$. The Cantor set defined by the sequence $\{\alpha_n\}$ is then

$$C(\alpha_n) = \bigcap_{n=1}^{\infty} \bigcup_{|\sigma|=n} J_\sigma.$$  

Note that

$$|J_\sigma| = \prod_{n=0}^{(|\sigma|-1)} \left( \frac{1 - \alpha_n}{2} \right) \quad \text{and} \quad |I_\sigma| = \alpha_{|\sigma|}|J_\sigma|,$$

where $|H|$ is used to denote the length of an interval $H$. In the obvious manner each $\sigma \in \Sigma^*$ determines a unique point in $\mathcal{C}(\alpha_n)$. We shall denote this point by $x_\sigma$.

Finally, we adopt the notation $d(x, I)$ for the distance from a point $x$ to an interval $I$. 

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3. A $\sigma$-SYMMETRIC POROSITY CHARACTERIZATION

We shall prove the following characterization.

**Theorem 1.** The symmetric Cantor set $\mathcal{C}(\alpha_n)$ is $\sigma$-symmetrically porous if and only if $\limsup \alpha_n > 0$.

**Proof.** If $\mathcal{C}(\alpha_n)$ is $\sigma$-symmetrically porous, then it is $\sigma$-porous and hence by [7, Theorem TH] $\limsup \alpha_n > 0$. Now suppose $\limsup \alpha_n > 0$. We denote the union of the complementary intervals of stage $n$ by $I_{n,0}$; that is, $I_{n,0} = \bigcup_{|\sigma|=n} I_\sigma$. Let $s_n$ denote the length of any $I_\sigma$, where $|\sigma| = n$, and define $I_{n,1}^*$ to be the union of those intervals of length $3\alpha_n s_n/4$ which are centered between adjacent components of $I_{n,0}$ and which do not intersect $I_{n,0}$. Note that the components of $I_{n,1}^*$ either contain or are contained in the components of $\bigcup_{m<n} I_{m,0}$. We let

$$I_{n,1} = \left( \bigcup_{m<n} I_{m,0} \right) \cup I_{n,1}^*.$$ 

Proceeding inductively, for $k \geq 2$ we define $I_{n,k}$ to be the union of those intervals of length $(\alpha_n/2 + \alpha_n/2^{k+1})s_n$ which are centered between adjacent components of $\bigcup_{i<k} I_{i,i}$ and which do not intersect $\bigcup_{i<k} I_{i,i}$.

Let $\alpha = \limsup \alpha_n$, and assume $\alpha_k \to \alpha$. Note that there is an $M \in \mathbb{N}$ depending only on $\alpha$ such that $I_{n,k,M} \neq \emptyset$, but $I_{n,k,m} = \emptyset$ whenever $m > M$.

If $\alpha > 1/2$, then Proposition 1 in [5] assures that $\mathcal{C}(\alpha_n)$ is symmetrically porous, so we assume here that $\alpha \leq 1/2$.

**Claim 1.** For any interval $I \subset [0, 1]$ and for every $k_0 \in \mathbb{N}$

$$I = I \cap \left[ \left( \bigcup_{m=k_0}^M \bigcup_{k=k_0}^{\infty} I_{n_k,m} \right) \cup E_I, k_0 \right]$$

where $E_I, k_0$ is symmetrically porous.

**Proof of Claim 1.** Let $E_I, k_0 = I \bigcap \bigcup_{m=0}^M \bigcup_{k=k_0}^{\infty} I_{n_k,m}$. Suppose $x \in E_I, k_0$. Note that if $x \notin \mathcal{C}(\alpha_n)$, then $x \in I_{n_0,0}$ for some $n_0$, and hence $x \in I_{n_0,1}$ whenever $n_k > n_0$. Thus, $E_I, k_0 \subseteq \mathcal{C}(\alpha_n)$. Let $\varepsilon > 0$ be given. There is a $K \geq k_0$ such that $|J_\sigma| < \varepsilon/2$ whenever $|\sigma| = n_k$ and $k \geq K$. Let $k \geq K$. As $x \in \mathcal{C}(\alpha_n)$ there is a $\sigma \in \Sigma$ with $|\sigma| = n_k$ with $x \in J_\sigma$, $x \notin \bigcup_{m=0}^{n_k} I_{n_k,m}$. Hence, $x$ lies between two component intervals of $\bigcup_{m=0}^{n_k} I_{n_k,m}$; denote that component to the left of $x$ by $I_l$ and that to the right of $x$ by $I_r$. The definition of $M$ entails that the distance between $I_l$ and $I_r$ does not exceed $\alpha/2 \cdot |J_\sigma|$. Also, $\min\{|I_l|, |I_r|\} \geq (\alpha/2 + \alpha/2^M)|J_\sigma|$. Setting

$$h = \min\{|I_l| + d(x, I_l), |I_r| + d(x, I_r)|,$$

we compute

$$\text{sp}(E_I, k_0, x, h) \geq \frac{\alpha |J_\sigma|/2^M}{h} \geq \frac{\alpha/2^M}{\alpha/2 + \alpha} + \frac{1}{3 \cdot 2^{M-1}},$$

from which it follows that $E_I, k_0$ is symmetrically porous. This completes the proof of Claim 1.
Claim 2. For every $i_0 > 0$ and $k_0 \in \mathbb{N}$,

$$I_{n_{k_0}, i_0} = \left( \bigcup_{m=0}^{M} \bigcup_{k=k_0+1}^{\infty} I_{n_k, m} \right) \cup E_{k_0, i_0}$$

where $E_{k_0, i_0}$ is symmetrically porous.

Proof of Claim 2. This claim follows immediately from Claim 1.

Claim 3. Each set $(\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} I_{n_k, i}) \cap C(\alpha_n)$ is $\sigma$-symmetrically porous.

Proof of Claim 3. Denote $S_i = \left( \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} I_{n_k, i} \right) \cap C(\alpha_n)$. As a first case consider $i > 1$ and suppose $x \in S_i$. Then $x \in I_{n_k, i}$ for infinitely many $k$. Fix one such $k$. Then there is a $\sigma \in \Sigma$ with $|\sigma| = n_k$ and $x \in J_\sigma$. The component of $I_{n_k, i}$ containing $x$ has length $(\alpha_n/2 + \alpha_n/2^{i+1})|J_\sigma|$. This component is centered between two components, $I_l$ and $I_r$, of $\bigcup_{j<i} I_{n_k, i}$ and the length of each of $I_l$ and $I_r$ is at least $(\alpha_n/2 + \alpha_n/2^i)|J_\sigma|$. Let $h = \min\{|I_l| + d(x, I_l), |I_r| + d(x, I_r)|$. Then $h < (1 - \alpha_n)|J_\sigma|/2$ and we compute:

$$sp(S_i, x, h) \geq \frac{(\alpha_n/2 + \alpha_n/2^i) - (\alpha_n/2 + \alpha_n/2^{i+1})}{(1 - \alpha_n)/2} = \alpha_n
\frac{2^i(1 - \alpha_n)}{(1 - \alpha_n)^2}$$

which tends to $\alpha/2^i(1 - \alpha) > 0$ as $k \to +\infty$.

The case when $i = 1$ is similar if one notices that for $x \in I_{n_k, 0} \cap C(\alpha_n)$, $x$ lies in some component of $I_{n_k, 0}$ which has length $3\alpha_n|J_\sigma|/4$. Such a component of $I_{n_k, 0}$ is centered between two intervals complementary to $C(\alpha_n)$ at the $n_k$th stage. Hence, these intervals have length $\alpha_n|J_\sigma|$. The symmetric porosity computation then proceeds as that in the case above. Claim 3 is thereby established.

We are now in a position to complete the proof of Theorem 1. Suppose $x \in C(\alpha_n)$ but $x \notin E_{[0, 1], 0}$ (see Claim 1). It follows from Claim 1 that $\mathcal{F} = \{ i : x \in I_{n_k, i} \text{ for some } k \} \neq \emptyset$. For $i \in \mathcal{F}$ define $k(i) = \{ k : x \in I_{n_k, i} \}$. If $k(i_0)$ is infinite for some $i_0 \in \mathcal{F}$, then

$$x \in \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} I_{n_k, i_0}.$$ If $k(i)$ is finite for each $i \in \mathcal{F}$, we set $k^* = \max_{i \in \mathcal{F}} k(i)$. As $\mathcal{F}$ has fewer than $M$ elements, $k^*$ is finite, and there is a largest $i^* \in \mathcal{F}$ with $k^* \in k(i^*)$. Now, by Claim 2

$$I_{n_{k^*}, i^*} = \left( \bigcup_{m=0}^{M} \bigcup_{k=k^*+1}^{\infty} I_{n_k, m} \right) \cup E_{k^*, i^*}.$$ As $x \in I_{n_{k^*}, i^*}$ but $x \notin \bigcup_{m=0}^{M} \bigcup_{k=k^*+1}^{\infty} I_{n_k, m}$, it follows that $x \in E_{k^*, i^*}$. In any case, then

$$x \in \left[ \bigcup_{m=0}^{M} \left( \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} I_{n_k, m} \right) \right] \cup \left[ \bigcup_{m=0}^{M} \bigcup_{k=1}^{\infty} E_{m, k} \right].$$
Hence,

\[ \mathcal{C}(\alpha_n) = \bigcup_{m=0}^{M} \left( \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} I_{n_k,m} \right) \cup \bigcup_{m=0}^{M} \bigcup_{k=1}^{\infty} E_{m,k} \cup E_{[0,1],0} . \]

This decomposition of \( \mathcal{C}(\alpha_n) \) as a countable union of symmetrically porous sets completes the proof.

Combining this result with those in [6] and [7] we now know that the following are equivalent statements for a symmetric Cantor set \( \mathcal{C}(\alpha_n) \):

1. \( \mathcal{C}(\alpha_n) \) is porous.
2. \( \mathcal{C}(\alpha_n) \) is \( \sigma \)-porous.
3. \( \mathcal{C}(\alpha_n) \) is \( \sigma \)-symmetrically porous.
4. \( \limsup \alpha_n > 0 \).

We conclude this note by observing that the statement "\( \mathcal{C}(\alpha_n) \) is symmetrically porous" cannot be added to this list.

**Example 1.** There is a \( \sigma \)-symmetrically porous symmetric Cantor set \( \mathcal{C}(\alpha_n) \) which is not symmetrically porous.

**Proof of Example 1.** Let \( S = \{\frac{1}{2}k^2 + \frac{9}{2}k : k = 0, 1, 2, \ldots \} \) and for each \( n = 0, 1, 2, \ldots \) set

\[ \alpha_n = \begin{cases} \frac{1}{10} & \text{if } n \in S, \\ 0 & \text{otherwise}; \end{cases} \]

that is,

\[ \{\alpha_n\} = \{\frac{1}{10}, 0, 0, 0, 0, \frac{1}{10}, 0, 0, 0, 0, 0, \frac{1}{10}, 0, 0, 0, 0, 0, \frac{1}{10}, \ldots \} . \]

We shall show that \( \mathcal{C}(\alpha_n) \) is not symmetrically porous at the point \( x_\sigma \in \mathcal{C}(\alpha_n) \) where for each natural number \( n \)

\[ \sigma(n) = \begin{cases} 1 & \text{if } n-1 \in S \text{ or } n-4 \in S, \\ 0 & \text{otherwise}. \end{cases} \]

For any two contiguous intervals \( I, I' \) we adopt the notation

\[ r(I, I') = \sup_{h>0} \left\{ \frac{\left|\{t: 0 < t < h, x_\sigma - t \in I, x_\sigma + t \in I'\}\right|}{h} \right\} . \]

Suppose that \( \mathcal{C}(\alpha_n) \) has symmetric porosity \( \alpha > 0 \) at \( x_\sigma \). Then there exists a sequence \( \{I_{\gamma_n}, I_{\tau_n}\} \) of contiguous intervals such that:

1. \( \gamma_n, \tau_n \in \Sigma \), with \( |\gamma_n|, |\tau_n| \in S \);
2. \( |\gamma_n| \leq |\tau_n| \);
3. \( \min\{|\gamma_n|, |\tau_n|\} \to \infty \), as \( n \to \infty \);
4. \( \min\{d(x_\sigma, I_{\gamma_n}), d(x_\sigma, I_{\tau_n})\} \to 0 \) as \( n \to \infty \); and
5. \( r(I_{\gamma_n}, I_{\tau_n}) \to \alpha \) as \( n \to \infty \).

**Claim 1.** Except for finitely many \( n \), \( |\gamma_n| = |\tau_n| \).

**Proof of Claim 1.** Suppose \( |\gamma_n| < |\tau_n| \). Let \( N = |\gamma_n| \). As \( x_\sigma \in J_{|\gamma_n|10001} \), \( x_\sigma \) is separated from \( I_{\gamma_n} \) by \( J_{|\gamma_n|10001} \) if \( I_{\gamma_n} \) is left of \( x_\sigma \) and by \( J_{|\gamma_n|11} \) if \( I_{\gamma_n} \) is right of \( x_\sigma \). In either case, \( d(x_\sigma, I_{\gamma_n}) \geq |J_{|\gamma_n|0000}| \). As \( |\tau_n| > N \),

\[ |I_{\tau_n}| \leq \frac{1}{10} 2^{-(|\tau_n|-N)} |J_{|\gamma_n|}| = \frac{1}{10} 2^{4-(|\tau_n|-|\gamma_n|)} |J_{|\gamma_n|0000}| . \]
Hence, \( r(I_{\gamma_n}, I_{\tau_n}) \leq \frac{1}{10} 2^{4-(|\gamma_n| - |\tau_n|)} \), and in view of (5) this can occur for only finitely many \( n \), completing the proof of Claim 1.

Hence we have that \( |\gamma_n| = |\tau_n| \) for all sufficiently large \( n \). Since \( |\gamma_n| \in S \), there is a nonnegative integer \( k_n \) such that \( S(k_n) \equiv \frac{1}{2} k_n + \frac{9}{2} k_n = |\gamma_n| \).

**Claim 2.** Except for finitely many \( n \),

\[ \gamma_n |_{S(k_n-1)} = \sigma |_{S(k_n-1)} = \tau_n |_{S(k_n-1)} . \]

**Proof of Claim 2.** If \( \gamma_n |_{S(k_n-1)} \neq \sigma |_{S(k_n-1)} \), then \( d(x_{\sigma}, I_{\gamma_n}) > |I_{\sigma} |_{S(k_n-1)} | = |I_{\gamma_n} |_{S(k_n-1)} | \). But clearly

\[ \frac{|I_{\gamma_n}|}{|I_{\gamma_n} |_{S(k_n-1)}} \to 0 \quad \text{as} \quad n \to \infty , \]

again contradicting (5) and establishing the claim.

Thus we may assume that both \( I_{\gamma_n} \) and \( I_{\tau_n} \) lie in \( J_{\sigma} \) and \( |\gamma_n| = S(k_n) = |\tau_n| \). However, for any two such intervals it is easy to see that \( I_{\gamma_n} \cap (2x_{\sigma} - I_{\tau_n}) = \emptyset \), and thus \( r(I_{\gamma_n}, I_{\tau_n}) = 0 \). This contradiction completes the proof of the example.

**References**