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A CHARACTERIZATION OF σ-SYMMETRICALLY POROUS SYMMETRIC CANTOR SETS

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ABSTRACT. The purpose of this paper is to characterize those symmetric Cantor sets which are σ-symmetrically porous in terms of a defining sequence of deleted proportions. In contrast to other notions of porosity, a symmetric Cantor set can be σ-symmetrically porous without being symmetrically porous.

1. INTRODUCTION

If A is a subset of the real line $\mathbf{R}$ and $x \in \mathbf{R}$, then the *porosity of A at x* is defined to be

$$\limsup_{r \to 0^+} \frac{\lambda(A, x, r)}{r},$$

where $\lambda(A, x, r)$ is the length of the longest open interval contained in either $(x, x+r) \cap A^c$ or $(x-r, x) \cap A^c$ and A^c denotes the complement of A. A set is said to be *porous at x* if it has positive porosity at x and is called a *porous set* if it is porous at each of its points. The *symmetric porosity of A at x* is defined as

$$\limsup_{r \to 0^+} \frac{\gamma(A, x, r)}{r},$$

where $\gamma(A, x, r)$ is the supremum of all positive numbers h such that there is a positive number t with $t + h \leq r$ such that both of the intervals $(x - t - h, x - t)$ and $(x + t, x + t + h)$ lie in A^c. (A notation that we find useful is to set $\mathrm{sp}(A, x, r) = \gamma(A, x, r)/r$.) A set A is *symmetrically porous* if it has positive symmetric porosity at each of its points. A set A is σ-porous (σ-symmetrically porous) if it is a countable union of porous (symmetrically porous) sets. The concepts of σ-porosity and σ-symmetric porosity have been shown to be distinct in [4] and [8].

The notions of porosity and σ-porosity have proved quite useful in real analysis as a means of describing the smallness of sets that arise as exceptional sets to some type of nice behavior. A survey of porosity as well as these types of

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applications may be found in [9]. Recently, the more restrictive concepts of symmetric porosity and σ-symmetric porosity have begun to find similar uses. For example, Zajíček [10] improved upon a theorem from [1], by showing that the exceptional set to a certain nice behavior of continuous functions, which was known to be σ-porous, is actually σ-symmetrically porous; and Evans [3] has recently extended this result of Zajíček to a wider class of functions.

The purpose of this paper is to characterize those symmetric Cantor sets which are σ-symmetrically porous in terms of a defining sequence of deleted proportions. (Symmetric Cantor sets are quite useful for constructing examples of pathological behavior. Examples of such constructions can be found in [4] and [2].) In [7] and [6] necessary and sufficient conditions were established for a symmetric Cantor set to be porous. There it was shown that such a set is porous if and only if it is σ-porous. With symmetric porosity the situation is markedly different: a symmetric Cantor set can be σ-symmetrically porous without being symmetrically porous. We shall give an example of such a symmetric Cantor set at the end of this paper. Interestingly, it turns out that a symmetric Cantor set is σ-symmetrically porous if and only if it is σ-porous. We begin by establishing the necessary notation in the following section.

2. Preliminary notation

First we define the class of symmetric Cantor sets in $[0, 1]$. Let Σ denote the set of all finite sequences of 0's and 1's, and let Σ^* denote the set of all infinite sequences of 0's and 1's. If $\sigma \in \Sigma$, we denote the length of σ by $|\sigma|$ and will write σ in expanded form as $\sigma(1)\sigma(2)\sigma(3)\cdots\sigma(n)$. If $\sigma \in \Sigma^*$ and $n \in \mathbf{N}$, then $\sigma|_n$ will denote $\sigma(1)\sigma(2)\sigma(3)\cdots\sigma(n)$. If $0 \le \alpha_n < 1$ for all $n = 0, 1, \ldots,$ then $\{\alpha_n\}$ determines a symmetric Cantor set, $\mathscr{C}(\alpha_n)$, in $[0, 1]$. If $\alpha_n \neq 0$, we identify the complementary intervals and the noncomplementary intervals to this Cantor set using subscripts from Σ in the usual way; i.e., $I_\varnothing = (\frac{1}{2} - \alpha_0/2, \frac{1}{2} + \alpha_0/2)$, J_0 and J_1 are the right- and left-hand components of the complement of $I_\varnothing$, respectively, I_0 and I_1 are the open intervals of length $\alpha_1(1-\alpha_0)/2$ centered in J_0 and J_1, respectively, and so on. If one of the $\alpha_n = 0$, we proceed as above with the exception that if $|\sigma| = n$, then I_σ is a "marking" of the center point of J_σ (not a interval) and $J_{\sigma 0}$ and $J_{\sigma 1}$ intersect in I_σ. The Cantor set defined by the sequence $\{\alpha_n\}$ is then

$$C(\alpha_n) = \bigcap_{n=1}^{\infty} \bigcup_{|\sigma|=n} J_\sigma .$$

Note that

$$|J_\sigma| = \prod_{n=0}^{|\sigma|-1} \left(\frac{1-\alpha_n}{2}\right) \quad \text{and} \quad |I_\sigma| = \alpha_{|\sigma|}|J_\sigma|,$$

where $|H|$ is used to denote the length of an interval H. In the obvious manner each $\sigma \in \Sigma^*$ determines a unique point in $\mathscr{C}(\alpha_n)$. We shall denote this point by x_σ.

Finally, we adopt the notation $d(x, I)$ for the distance from a point x to an interval I.

3. A σ-SYMMETRIC POROSITY CHARACTERIZATION

We shall prove the following characterization.

Theorem 1. *The symmetric Cantor set $\mathscr{C}(\alpha_n)$ is σ-symmetrically porous if and only if* $\limsup \alpha_n > 0$.

Proof. If $\mathscr{C}(\alpha_n)$ is σ-symmetrically porous, then it is σ-porous and hence by [7, Theorem TH] $\limsup \alpha_n > 0$. Now suppose $\limsup \alpha_n > 0$. We denote the union of the complementary intervals of stage n by $I_{n,0}$; that is, $I_{n,0} = \bigcup_{|\sigma|=n} I_\sigma$. Let s_n denote the length of any J_σ, where $|\sigma| = n$, and define $I^*_{n,1}$ to be the union of those intervals of length $3\alpha_n s_n/4$ which are centered between adjacent components of $I_{n,0}$ and which do not intersect $I_{n,0}$. Note that the components of $I^*_{n,1}$ either contain or are contained in the components of $\bigcup_{m<n} I_{m,0}$. We let

$$I_{n,1} = \left(\bigcup_{m<n} I_{m,0}\right) \cup I^*_{n,1}.$$

Proceeding inductively, for $k \geq 2$ we define $I_{n,k}$ to be the union of those intervals of length $(\alpha_n/2 + \alpha_n/2^{k+1})s_n$ which are centered between adjacent components of $\bigcup_{i<k} I_{n,i}$ and which do not intersect $\bigcup_{i<k} I_{n,i}$.

Let $\alpha = \limsup \alpha_n$, and assume $\alpha_{n_k} \to \alpha$. Note that there is an $M \in \mathbf{N}$ depending only on α such that $I_{n_k,M} \neq \varnothing$, but $I_{n_k,m} = \varnothing$ whenever $m > M$.

If $\alpha > \frac{1}{2}$, then Proposition 1 in [5] assures that $\mathscr{C}(\alpha_n)$ is symmetrically porous, so we assume here that $\alpha \leq \frac{1}{2}$.

Claim 1. *For any interval $I \subset [0, 1]$ and for every $k_0 \in \mathbf{N}$*

$$I = I \cap \left[\left(\bigcup_{m=0}^{M} \bigcup_{k=k_0}^{\infty} I_{n_k,m}\right) \cup E_{I,k_0}\right]$$

where E_{I,k_0} is symmetrically porous.

Proof of Claim 1. Let $E_{I,k_0} = I \setminus \bigcup_{m=0}^{M} \bigcup_{k=k_0}^{\infty} I_{n_k,m}$. Suppose $x \in E_{I,k_0}$. Note that if $x \notin \mathscr{C}(\alpha_n)$, then $x \in I_{n_0,0}$ for some n_0, and hence $x \in I_{n_k,1}$ whenever $n_k > n_0$. Thus, $E_{I,k_0} \subseteq \mathscr{C}(\alpha_n)$. Let $\varepsilon > 0$ be given. There is a $K \geq k_0$ such that $|J_\sigma| < \varepsilon/2$ whenever $|\sigma| = n_k$ and $k \geq K$. Let $k \geq K$. As $x \in \mathscr{C}(\alpha_n)$ there is a $\sigma \in \Sigma$ with $|\sigma| = n_k$ with $x \in J_\sigma$, $x \notin \bigcup_{m=0}^{M} I_{n_k,m}$. Hence, x lies between two component intervals of $\bigcup_{m=0}^{M} I_{n_k,m}$; denote that component to the left of x by I_l and that to the right of x by I_r. The definition of M entails that the distance between I_l and I_r does not exceed $\alpha/2 \cdot |J_\sigma|$. Also, $\min\{|I_l|, |I_r|\} \geq (\alpha/2 + \alpha/2^M)|J_\sigma|$. Setting

$$h = \min\{|I_l| + d(x, I_l),\ |I_r| + d(x, I_r)\},$$

we compute

$$\operatorname{sp}(E_{I,k_0}, x, h) \geq \frac{\alpha|J_\sigma|/2^M}{h} \geq \frac{\alpha/2^M}{\alpha/2 + \alpha} + \frac{1}{3 \cdot 2^{M-1}},$$

from which it follows that E_{I,k_0} is symmetrically porous. This completes the proof of Claim 1.

Claim 2. *For every* $i_0 > 0$ *and* $k_0 \in \mathbf{N}$,

$$I_{n_{k_0}, i_0} = \left[\bigcup_{m=0}^{M} \bigcup_{k=k_0+1}^{\infty} I_{n_k, m} \right] \cup E_{k_0, i_0}$$

where E_{k_0, i_0} *is symmetrically porous.*

Proof of Claim 2. This claim follows immediately from Claim 1.

Claim 3. *Each set* $(\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} I_{n_k, i}) \cap \mathscr{C}(\alpha_n)$ *is* σ*-symmetrically porous.*

Proof of Claim 3. Denote $S_i = (\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} I_{n_k, i}) \cap \mathscr{C}(\alpha_n)$. As a first case consider $i > 1$ and suppose $x \in S_i$. Then $x \in I_{n_k, i}$ for infinitely many k. Fix one such k. Then there is a $\sigma \in \Sigma$ with $|\sigma| = n_k$ and $x \in J_\sigma$. The component of $I_{n_k, i}$ containing x has length $(\alpha_{n_k}/2 + \alpha_{n_k}/2^{i+1})|J_\sigma|$. This component is centered between two components, I_l and I_r, of $\bigcup_{j<i} I_{n_k, i}$ and the length of each of I_l and I_r is at least $(\alpha_{n_k}/2 + \alpha_{n_k}/2^i)|J_\sigma|$. Let $h = \min\{|I_l| + d(x, I_l), |I_r| + d(x, I_r)\}$. Then $h < (1 - \alpha_{n_k})|J_\sigma|/2$ and we compute:

$$\operatorname{sp}(S_i, x, h) \geq \frac{(\alpha_{n_k}/2 + \alpha_{n_k}/2^i) - (\alpha_{n_k}/2 + \alpha_{n_k}/2^{i+1})}{(1 - \alpha_{n_k})/2} = \frac{\alpha_{n_k}}{2^i(1 - \alpha_{n_k})},$$

which tends to $\alpha/2^i(1-\alpha) > 0$ as $k \to +\infty$.

The case when $i = 1$ is similar if one notices that for $x \in I_{n_k, 0} \cap \mathscr{C}(\alpha_n)$, x lies in some component of $I^*_{n_k, 0}$ which has length $3\alpha_{n_k}|J_\sigma|/4$. Such a component of $I^*_{n_k, 0}$ is centered between two intervals complementary to $\mathscr{C}(\alpha_n)$ at the n_kth stage. Hence, these intervals have length $\alpha_{n_k}|J_\sigma|$. The symmetric porosity computation then proceeds as that in the case above. Claim 3 is thereby established.

We are now in a position to complete the proof of Theorem 1. Suppose $x \in \mathscr{C}(\alpha_n)$ but $x \notin E_{[0,1],0}$ (see Claim 1). It follows from Claim 1 that $\mathscr{I} = \{i : x \in I_{n_k, i} \text{ for some } k\} \neq \varnothing$. For $i \in \mathscr{I}$ define $k(i) = \{k : x \in I_{n_k, i}\}$. If $k(i_0)$ is infinite for some $i_0 \in \mathscr{I}$, then

$$x \in \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} I_{n_k, i_0}.$$

If $k(i)$ is finite for each $i \in \mathscr{I}$, we set $k^* = \max \bigcup_{i \in \mathscr{I}} k(i)$. As $\mathscr{I}$ has fewer than M elements, k^* is finite, and there is a largest $i^* \in \mathscr{I}$ with $k^* \in k(i^*)$. Now, by Claim 2

$$I_{n_{k^*}, i^*} = \left(\bigcup_{m=0}^{M} \bigcup_{k=k^*+1}^{\infty} I_{n_k, m} \right) \cup E_{k^*, i^*}.$$

As $x \in I_{n_{k^*}, i^*}$ but $x \notin \bigcup_{m=0}^{M} \bigcup_{k=k^*+1}^{\infty} I_{n_k, m}$, it follows that $x \in E_{k^*, i^*}$. In any case, then

$$x \in \left[\bigcup_{m=0}^{M} \left(\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} I_{n_k, m} \right) \right] \cup \left[\bigcup_{m=0}^{M} \bigcup_{k=1}^{\infty} E_{m, k} \right].$$

Hence,

$$\mathscr{C}(\alpha_n) = \left[\bigcup_{m=0}^{M}\left(\bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty} I_{n_k,m}\right)\right] \cup \left[\bigcup_{m=0}^{M}\bigcup_{k=1}^{\infty} E_{m,k}\right] \cup E_{[0,1],0}.$$

This decomposition of $\mathscr{C}(\alpha_n)$ as a countable union of symmetrically porous sets completes the proof.

Combining this result with those in [6] and [7] we now know that the following are equivalent statements for a symmetric Cantor set $\mathscr{C}(\alpha_n)$:

(1) $\mathscr{C}(\alpha_n)$ is porous.
(2) $\mathscr{C}(\alpha_n)$ is σ-porous.
(3) $\mathscr{C}(\alpha_n)$ is σ-symmetrically porous.
(4) $\limsup \alpha_n > 0$.

We conclude this note by observing that the statement "$\mathscr{C}(\alpha_n)$ is symmetrically porous" cannot be added to this list.

Example 1. There is a σ-symmetrically porous symmetric Cantor set $\mathscr{C}(\alpha_n)$ which is not symmetrically porous.

Proof of Example 1. Let $S = \{\frac{1}{2}k^2 + \frac{9}{2}k : k = 0, 1, 2, \dots\}$ and for each $n = 0, 1, 2, \dots$ set

$$\alpha_n = \begin{cases} \frac{1}{10} & \text{if } n \in S, \\ 0 & \text{otherwise}; \end{cases}$$

that is,

$$\{\alpha_n\} = \{\tfrac{1}{10}, 0, 0, 0, 0, \tfrac{1}{10}, 0, 0, 0, 0, 0, \tfrac{1}{10}, 0, 0, 0, 0, 0, 0, \tfrac{1}{10}, \dots\}.$$

We shall show that $\mathscr{C}(\alpha_n)$ is not symmetrically porous at the point $x_\sigma \in \mathscr{C}(\alpha_n)$ where for each natural number n

$$\sigma(n) = \begin{cases} 1 & \text{if } n-1 \in S \text{ or } n-4 \in S, \\ 0 & \text{otherwise}. \end{cases}$$

For any two contiguous intervals I, I' we adopt the notation

$$r(I, I') \equiv \sup_{h>0}\left\{\frac{|\{t : 0 < t < h,\ x_\sigma - t \in I,\ x_\sigma + t \in I'\}|}{h}\right\}.$$

Suppose that $\mathscr{C}(\alpha_n)$ has symmetric porosity $\alpha > 0$ at x_σ. Then there exists a sequence $\{I_{\gamma_n}, I_{\tau_n}\}$ of contiguous intervals such that:

(1) γ_n, $\tau_n \in \Sigma$, with $|\gamma_n|$, $|\tau_n| \in S$;
(2) $|\gamma_n| \le |\tau_n|$;
(3) $\min\{|\gamma_n|, |\tau_n|\} \to \infty$, as $n \to \infty$;
(4) $\min\{d(x_\sigma, I_{\gamma_n}), d(x_\sigma, I_{\tau_n}\} \to 0$ as $n \to \infty$; and
(5) $r(I_{\gamma_n}, I_{\tau_n}) \to \alpha$ as $n \to \infty$.

Claim 1. *Except for finitely many* n, $|\gamma_n| = |\tau_n|$.

Proof of Claim 1. Suppose $|\gamma_n| < |\tau_n|$. Let $N = |\gamma_n|$. As $x_\sigma \in J_{\sigma|_N 1001}$, x_σ is separated from I_{γ_n} by $J_{\sigma|_N 1000}$ if I_{γ_n} is left of x_σ and by $J_{\sigma|_N 11}$ if I_{γ_n} is right of x_σ. In either case, $d(x_\sigma, I_{\gamma_n}) \ge |J_{\gamma_n 0000}|$. As $|\tau_n| > N$,

$$|I_{\tau_n}| \le \tfrac{1}{10}2^{-(|\tau_n|-N)}|J_{\sigma|_N}| = \tfrac{1}{10}2^{4-(|\tau_n|-|\gamma_n|)}|J_{\sigma|_N 0000}|.$$

Hence, $r(I_{\gamma_n}, I_{\tau_n}) \leq \frac{1}{10}2^{4-(|\tau_n|-|\gamma_n|)}$, and in view of (5) this can occur for only finitely many n, completing the proof of Claim 1.

Hence we have that $|\gamma_n| = |\tau_n|$ for all sufficiently large n. Since $|\gamma_n| \in S$, there is a nonnegative integer k_n such that $S(k_n) \equiv \frac{1}{2}k_n^2 + \frac{9}{2}k_n = |\gamma_n|$.

Claim 2. *Except for finitely many* n,

$$\gamma_n|_{S(k_n-1)} = \sigma|_{S(k_n-1)} = \tau_n|_{S(k_n-1)}.$$

Proof of Claim 2. If $\gamma_n|_{S(k_n-1)} \neq \sigma|_{S(k_n-1)}$, then $d(x_\sigma, I_{\gamma_n}) > |I_{\sigma|_{S(k_n-1)}}| = |I_{\gamma_n|_{S(k_n-1)}}|$. But clearly

$$\frac{|I_{\gamma_n}|}{|I_{\gamma_n|_{S(k_n-1)}}|} \to 0 \quad \text{as } n \to \infty,$$

again contradicting (5) and establishing the claim.

Thus we may assume that both I_{γ_n} and I_{τ_n} lie in $J_{\sigma|_{S(k_n-1)}}$ and $|\gamma_n| = S(k_n) = |\tau_n|$. However, for any two such intervals it is easy to see that $I_{\gamma_n} \cap (2x_\sigma - I_{\tau_n}) = \varnothing$, and thus $r(I_{\gamma_n}, I_{\tau_n}) = 0$. This contradiction completes the proof of the example.

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