A COMPARISON BETWEEN EULER AND CESÀRO METHODS OF SUMMABILITY

B. KUTTNER AND M. R. PARAMESWARAN

(Communicated by Andrew Bruckner)

Abstract. It is well known that there are sequences that are summable by every Cesàro method $C_r$ ($r > 0$) but are not summable by any Euler method $E_p$ ($0 < p < 1$). It is proved here that on the other hand there are sequences that are summable by every Euler method $E_p$ ($0 < p < 1$) but are not summable by any Cesàro method.

1

We consider the regular Cesàro methods $C_r$ ($r > 0$) and the regular Euler methods $E_p$ ($0 < p < 1$), where the latter is defined as in [5]; that is, the $E_p$-transform of any sequence $\{x_k\}$ is given by $\{t_n(p)\}$ where

$$t_n(p) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} x_k \quad (n = 0, 1, \ldots).$$

(1)

Let $(C_r)$ and $(E_p)$ denote the summability fields of $C_r$ and $E_p$ respectively. It is well known that

$$(E_p) \subset (C_r) \quad \text{and} \quad (C_r) \subset (E_p), \quad 0 < p < 1, \ r > 0$$

(see, for instance, [5, Satz 64.V]). It is also well known that if $0 < p < 1$ and $r > 0$,

$$m \cap \bigcup_{0 < q < 1} (E_q) = m \cap (E_p) \subset m \cap (C_r) = m \cap \bigcup_{s > 0} (C_s),$$

where $m$ is the set of all bounded sequences. Further, it is known that

$$\bigcap_{r > 0} (C_r) \not\subset \bigcup_{0 < p < 1} (E_p);$$

that is, there exist sequences $x$ that are summable by every Cesàro method $C_r$ ($r > 0$) but not by any Euler method $E_p$ [2, p. 251; 1, p. 213]. Such a sequence $x$ can be made to satisfy certain additional conditions too, like being bounded,

Received by the editors December 23, 1992 and, in revised form, February 17, 1993.
1991 Mathematics Subject Classification. Primary 40G05; Secondary 40D09, 40D25.
The research presented here was supported in part by NSERC of Canada.
The first author unfortunately passed away before this paper could be sent for publication.

© 1994 American Mathematical Society
0002-9939/94 $1.00 + $.25 per page

787
or unbounded, or that the series $\sum a_n = \sum (x_n - x_{n-1})$ have gaps of a certain type (see [3]). But the question whether

$$\bigcap_{0 < p < 1} (E_p) \subset (C_r) \quad \text{for some } r > 0$$

seems to have remained open. The object of the present paper is to prove that the answer to the question is in the negative. In fact, we prove rather more.

2

**Theorem.** (a) There exist sequences that are $E_p$-summable for every $0 < p < 1$ but are not summable by any Cesàro method; that is,

$$\bigcap_{0 < p < 1} (E_p) \not\subset \bigcup_{r > 0} (C_r).$$

(b) There exists a sequence $\{V^{(m)}\}$ of row-finite regular matrix methods such that

$$(V^{(1)}) \not\subset \bigcup_{r > 0} (C_r) \quad \text{and} \quad (V^{(1)}) \subset (V^{(2)}) \subset \cdots \subset \bigcap_{0 < p < 1} (E_p).$$

**Proof.** (a) Let the sequence $\{x_n\}$ be defined by

$$x_k = (-1)^k \sum_{r=0}^k c_r[k] \quad (k = 0, 1, \ldots)$$

where $[k]_r := k(k-1) \cdots (k-r+1)$ \{r factors\} (and is interpreted as 0 when $r = 0$ or $k < r$) and where $\{c_r\}$ is some fixed sequence of positive numbers which satisfy a certain condition to be imposed later. For $0 < p < 1$, we have for each $n = 0, 1, \ldots$,

$$t_n(p) = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} (-1)^k \sum_{r=0}^k c_r[k]_r$$

$$= \sum_{k=0}^n c_r \sum_{k=r}^n \binom{n}{k} p^k (1-p)^{n-k} (-1)^k [k]_r$$

$$= \sum_{r=0}^n \binom{n}{k} p^k (1-p)^{n-k} (-1)^k [k]_r$$

since $[k]_r = 0$ for $k < r$. Now

$$\sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} x^k = [px + (1-p)]^n.$$

Differentiating each side of this expression $r$ times (with respect to $x$) and then putting $x = -1$, we see that the inner sum in (3) is equal to

$$(-1)^r[n]_r p^r (1-2p)^{n-r}.$$

Hence

$$t_n(p) = \sum_{r=0}^n c_r (-1)^r[n]_r p^r (1-2p)^{n-r} = \sum_{r=0}^n c_r a_n(p), \quad \text{say.}$$
Now let \( \{a_r\}, \{b_r\} \) be (fixed) sequences of positive numbers decreasing to 0, with \( a_0 < 1/2 \), and let \( \sum b_r < \infty \). We note that for a given \( r \), \( a_{nr}(p) \to 0 \) as \( n \to \infty \), uniformly for \( p \) in any closed interval contained in \((0, 1)\). It is bounded for any fixed \( n \); hence it is bounded for all \( n \), uniformly in \( p \) for \( p \) in any such interval. Hence we can choose \( \{c_r\} \) so that for each \( r \),

\[
c_r |a_{nr}(p)| < b_r \quad (a_r \leq p \leq 1 - a_r, \text{ for all } n).
\]

Now consider an arbitrary fixed value of \( p \), with \( 0 < p < 1 \), and let \( \varepsilon > 0 \) be arbitrarily given. Since \( a_r \to 0 \), we see that \( p \in [a_r, 1 - a_r] \) for sufficiently large \( r \). Choose \( R \) large enough for this to hold for all \( r \geq R \) such that, further, \( \sum_{r=R}^{\infty} b_r < \varepsilon \). Then for all \( n \),

\[
\sum_{r=R}^{\infty} c_r |a_{nr}(p)| < \varepsilon.
\]

But, for fixed \( r \) and \( p \), we have \( a_{nr}(p) \to 0 \) as \( n \to \infty \); so, having fixed \( R \), we have

\[
\sum_{r=0}^{R-1} c_r |a_{nr}(p)| < \varepsilon \quad \text{for all } n \text{ sufficiently large}.
\]

Then it follows from (5) and (6) that \( t_n(p) \to 0 \) as \( n \to \infty \). Thus the sequence \( x = \{x_n\} \) is summable by the method \( E_p \), for arbitrary \( p \in (0, 1) \); that is, \( x \in \bigcap_{0 < \rho < 1} (E_{p}) \).

On the other hand, for each fixed positive integer \( r \),

\[
|x_k| \geq c_r[k]_r = c_r k(k - 1) \cdots (k - r + 1) \geq (1/2)c_r k^r
\]

if \( k \) is sufficiently large. Hence \( \{x_n\} \) does not satisfy the conditions of the limitation theorem for summability by the Cesàro method \( C_r \) [1, pp. 101–102; 5, p. 104]. Since this is true for every positive integer \( r \), the sequence \( \{x_n\} \) is not summable by \( C_r \) for any real value of \( r > 0 \). This proves part (a) of the theorem.

(b) We note that all of the above remains valid if, for each \( r \geq 0 \), we replace \( c_r \) by any smaller positive number \( c'_r \). Hence, if we define the sequences \( x^{(m)}(r) \) \((m = 1, 2, \ldots)\) by setting

\[
x^{(m)}(r) = (-1)^{k} \sum_{r=0}^{k} c_r \left( \frac{r}{r+m} \right) [k]_r \quad (k = 0, 1, \ldots),
\]

then \( x^{(m)} \in \bigcap_{0 < \rho < 1} (E_{p}) \), but \( x^{(m)} \notin \bigcup_{r>0} (C_r) \) \((m = 1, 2, \ldots)\).

Now let \( y = \{y_k\} = a_1 x^{(1)} + \cdots + a_s x^{(s)} \) be an arbitrary given linear combination of the sequences \( x^{(1)}, x^{(2)}, \ldots \), where not all the \( a_j \) are zero. Then for each \( k = 0, 1, \ldots \),

\[
y_k = (-1)^{k} \sum_{r=0}^{k} c_r [k]_r \left( \sum_{j=1}^{s} \frac{ra_j}{r+j} \right) = (-1)^{k} \sum_{r=0}^{k} c_r [k]_r A(r)
\]

where \( A(x) = \sum_{j=1}^{s} x a_j / (x + j) \) \((x \geq 0)\). Now every (nonnegative) solution of the equation \( A(x) = 0 \) is a solution of the equation \( P(x) = 0 \), where \( P(x) = \sum_{j=1}^{s} x a_j \prod_{1 \leq i \leq s, i \neq j} (x+i) \). If the polynomial \( P(x) \) were identically zero, then
\[ P(-j) = 0 \] and hence \[ a_j = 0 \] for \( 1 \leq j \leq s \), contrary to the assumption about the numbers \( a_j \). Hence \( P(x) \) has only a finite number of zeros, so there exists a positive integer \( N \) such that \( P(x) \), and therefore also \( A(x) \), is nonzero and of constant sign in the interval \([N, \infty)\). From the equation (7), we have for each \( k > N \),
\[
y_k = (-1)^k \sum_{r=0}^{N-1} c_r[k] A(r) + (-1)^k \sum_{r=N}^{k} c_r[k] A(r) = S_k + T_k \quad \text{(say)}.
\]

Then \( S_k \) is the sum of \( N \) terms, each of which is \( O(k^{N-1}) \) as \( k \to \infty \), and hence \( S_k = O(k^{N-1}) \). On the other hand, since \( A(r) \) is nonzero and has constant sign for all \( r \geq N \) (and since \( c_r > 0 \) for all \( r \)), we see that
\[
|T_k| \geq c_N[k] |A(N)| = c_N k(k-1)(k-2) \cdots (k-N+1) |A(N)| \\
\geq (1/2)c_N |A(N)| k^N \quad \text{for all sufficiently large } k.
\]

Hence \( T_k \neq O(k^{N-1}) \) as \( k \to \infty \). Since \( S_k = O(k^{N-1}) \), it follows that \( y_k = S_k + T_k \neq O(1) \). That is, no nontrivial linear combination of the sequences \( x^{(1)}, x^{(2)}, \ldots \) yields a bounded sequence. It follows then from Theorem 3 of [4] that for each \( m \geq 1 \) there exists a regular row-finite matrix method \( V^{(m)} \) whose summability field is the smallest sequence space containing \( c \) (the convergent sequences) and the sequences \( x^{(1)}, x^{(2)}, \ldots, x^{(m)} \). Then we have \( (V^{(1)}) \not\subset \bigcup_{r>0}(C_r) \) and \( (V^{(1)}) \subset (V^{(2)}) \subset \cdots \subset \bigcap_{0<p<1}(E_p) \). This completes the proof of the theorem.

**Acknowledgment**

Thanks are offered to the referee, especially for critical remarks which led to a proper presentation of the proof of part (b) of the theorem.

**References**

3. M. R. Parameswaran, *On a comparison between the Cesàro and Borel methods of summa-
5. K. Zeller and W. Beekmann, *Theorie der Limitierungsverfahren*, Springer-Verlag, Heidel-

Department of Mathematics & Astronomy, University of Manitoba, Winnipeg, Canada R3T 2N2

E-mail address, M. R. Parameswaran: param@ccu.umanitoba.ca