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HÖLDER-CONTINUITY OF THE SOLUTIONS FOR OPERATORS WHICH ARE A SUM OF SQUARES OF VECTOR FIELDS PLUS A POTENTIAL

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ABSTRACT. In this paper we study the local Hölder-regularity of weak solutions to $\mathscr{L}u + Vu = 0$ where $\mathscr{L}$ is a Hörmander hypoelliptic operator and the potential V belongs to a new class of functions which is the natural extension of Morrey spaces to this situation. We improve a recent result of Citti, Garofalo, and Lanconelli.

1. INTRODUCTION

The regularity properties of solutions of the Schrödinger equation

(1) $\qquad \mathscr{L}u + Vu = 0$ in an open bounded subset of Ω of R^n

has been intensively studied in the last few years.

When $\mathscr{L}$ is a second-order uniformly elliptic operator in divergence form, Hölder regularity was initially obtained if $V \in L^p(\Omega)$, $p > n/2$. This condition on V is the best possible in the scale of L^p spaces (see [LU]) but, in general, it is possible to weaken it. Indeed in 1982 Aizenman and Simon (see [AS]) proved Harnack's inequality for positive solutions of (1) assuming ($L = \Delta$ and) V in the Stummel-Kato class $S(\Omega)$, i.e., assuming that

$$\lim_{r\to 0}\left(\sup_{x\in\Omega}\int_{|x-y|<r}\frac{|V(y)|}{|x-y|^{n-2}}dy\right) = 0.$$

Let us explicitly note that Aizenman and Simon's result improve the previous one in [S], since $L^p(\Omega) \subset S(\Omega)$ for every $p > n/2$ and $\bigcup_{p>n/2} L^p(\Omega) \neq S(\Omega)$.

The proof in [AS] relies on some probabilistic techniques. However, in 1986, Chiarenza, Fabes, and Garofalo in [CFG], gave a new, nonprobabilistic proof of Harnack's inequality and obtained local continuity of weak solutions of equation (1) (see also Hinz and Kalf [HK] and Simader [Si] for analogous results with $\mathscr{L} = \Delta$). These solutions are in general not Hölder continuous, but they have this regularity when V belongs to a particular subset of $S(\Omega)$, the classical

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Morrey space $L^{1,\lambda}(\Omega)$ for $\lambda > 1$ (see [D]). V belongs to this space when

$$\sup_{\substack{x\in\Omega \\ r>0}} r^{-\mu} \int_{|x-y|<r} |V(y)|\, dy < +\infty, \qquad \mu = \lambda(n-2). \tag{2}$$

More recently the same question was investigated for operators $\mathscr{L}$ which are the sum of squares of vector fields. More precisely let $\mathscr{L}$ be an operator of the form

$$\mathscr{L} \equiv \sum_{j=1}^{p} X_j^2 \tag{3}$$

where $X_1, \dots, X_p$ are C^∞ first-order vector fields on an open set $\Omega \subset R^n$ satisfying Hörmander's condition for hypoellipticity

$$\operatorname{rank} \operatorname{Lie}[X_1, \dots, X_p](x) = n \tag{4}$$

at every point $x \in \Omega$ and such that

$$X_j = X_j^* \quad \text{for every} \quad j = 1, \dots, p \tag{5}$$

(X_j^* is the formal adjoint of X_j).

In [CGL], assuming V to be in an appropriate version of the Stummel-Kato class defined below, a Harnack inequality and local continuity of weak solutions of (1) were established.

In this paper, assuming again that $\mathscr{L}$ satisfies (3)–(5), we introduce a new space $L^{1,\lambda}_{\mathscr{L}}(\Omega)$, which is the natural generalization of the classical Morrey space to the framework of these operators and which contains the $\mathscr{L}$-Stummel-Kato class introduced in [CGL]. In §3 we list some examples for the reader's convenience. Then, in §4 we extend the results in [D] to weak solutions of $\mathscr{L}u + Vu = 0$, proving their local Hölder continuity when the potential V belongs to $L^{1,\lambda}_{\mathscr{L}}(\Omega)$.

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2. Definition of Morrey spaces

In the sequel we assume that the operator $\mathscr{L}$ satisfies (3)–(5) and coincides with the Laplacian Δ in the complement of some compact subset of Ω. This condition does not imply any restriction because of the local character of our result. On the other side it ensures the existence of a fundamental solution Γ of class C^∞ off the diagonal of $R^n \times R^n$ (see [S]). We will call Ω_r its level sets, which are the natural analogues of the Euclidean balls for the Laplace operator:

$$\Omega_r(x) \equiv \left\{ y \in \Omega : \frac{1}{\Gamma(x, y)} < r \right\} \quad \text{with} \quad r > 0 \text{ and } x \in \Omega.$$

Let $\eta: R^+ \to R^+$ be an increasing function such that $\lim_{r\to 0} \eta(r) = 0$.

Recall that a function $V \in L^1_{\mathrm{loc}}(\Omega)$ belongs to the *$\mathscr{L}$-Stummel-Kato* class $S(\Omega)$ when there exists a constant $C > 0$ such that

$$\sup_{x\in\Omega} \int_{\Omega_r(x)} |V(y)| \Gamma(x, y)\, dy \le C\eta(r).$$

In particular, when $\eta(r) = r^\lambda$, we will simply denote $S_\lambda(\Omega) = S_{r^{\lambda-1}}(\Omega)$.

Now we give our definition of $\mathscr{L}$-Morrey spaces.

Definition 2.1. If $\lambda > 1$, we will say that V belongs to the Morrey space $L^{1,\lambda}_{\mathscr{L}}(\Omega)$, if $V \in L^1_{\text{loc}}(\Omega)$ and

$$\|V\|_{L^{1,\lambda}_{\mathscr{L}}(\Omega)} \equiv \sup_{\substack{r>0\\ x\in\Omega}} r^{-\lambda} \int_{\Omega_r(x)} |V(y)|dy < +\infty.$$

Remark 2.1. We explicitly note that in the case $\mathscr{L} = \Delta$, $L^{1,\lambda}_{\Delta}$ is the classical Morrey space $L^{1,\lambda(n-2)}$.

The following relation holds between these spaces.

Proposition 2.1. *If* $\lambda > 1$ *then* $L^{1,\lambda}_{\mathscr{L}}(\Omega) = S_\lambda(\Omega)$.

Proof. Let $V \in L^{1,\lambda}_{\mathscr{L}}(\Omega)$. Then for every $x \in \Omega$ and $r > 0$, we have

$$\begin{aligned}\int_{\Omega_r(x)} |V(y)|\Gamma(x, y)dy &= \sum_{k=0}^{\infty} \int_{\Omega_{r(x)/2^k} \setminus \Omega_{r(x)/2^{k+1}}} |V(y)|\Gamma(x, y)dy \\ &\leq \sum_{k=0}^{\infty} \frac{r}{2^{k+1}} \int_{\Omega_{r(x)/2^k}} |V(y)|\,dy \leq 2\sum_{k=0}^{\infty} 2^{k(1-\lambda)} \|V\|_{L^1_{\mathscr{L}}(\Omega)} r^{\lambda-1} \leq \|V\|_{L^{1,\lambda}_{\mathscr{L}}(\Omega)} r^{\lambda-1}.\end{aligned}$$

Hence we get $L^{1,\lambda}_{\mathscr{L}}(\Omega) \subseteq S_\lambda(\Omega)$. The other inclusion is obvious, since

$$r^{-\lambda} \int_{\Omega_r(x)} |V(y)|dy \leq r^{-\lambda+1} \int_{\Omega_r(x)} \Gamma(x, y)|V(y)|dy.$$

3. Examples

Now we give some examples to clarify our definition.

Example 3.1. *Sublaplacian operators on homogeneous spaces.* Let $\mathscr{L}$ be an operator satisfying (3)–(5). If the Lie algebra $g = \text{Lie}(X_1, \dots, X_p)$ is nilpotent and has dimension n, it is possible to define a group law on R^n, which makes R^n into a Lie group, G. Its Lie algebra is g, and the exponential map is merely the identity. We say that g is stratified and $\mathscr{L}$ is a sublaplacian for G if there exists m such that

$$g = \bigoplus_{i=1}^{m} V_j \tag{6}$$

where $V_1 = \langle X_1, \dots, X_p\rangle$, $V_{j+1} = [V_j, V_1]$ for every $j = 1, \dots, m-1$, and $[V_m, V_1] = 0$. In this case a natural family of algebra automorphisms for the operator $\mathscr{L}$ is

$$\delta_r\left(\sum_{j=1}^{m} Y_j\right) = \sum_{j=1}^{m} r^j Y_j \qquad (Y_j \in V_j).$$

In other words it is possible to choose a basis in R^n such that every element x can be represented in the form $x = (x_1, \dots, x_m)$, with $x_1 \in R^{\dim V_1}, \dots, x_m \in R^{\dim V_m}$ and δ_r is simply

$$\delta_r(x_1, \dots, x_m) = (rx_1, \dots, r^m x_m). \tag{7}$$

On R^n we can also define the function

$$d_0(x_1, \dots, x_m) = \left(\sum_{j=1}^{m} |x_j|^{2m/j}\right)^{1/2m}. \tag{8}$$

This is a norm, which is homogeneous of degree one with respect to the dilations δ_r and satisfies

$$d_0(x) = 0 \quad \text{if and only if} \quad x = 0;\ d_0(x^{-1}) = d_0(x).$$

(Here x^{-1} denotes the inverse element in the group law defined on R^n, while the group law itself will be denoted $\circ$.) Thus the function

$$d(x, y) = d_0(x^{-1} \circ y)$$

is a distance on R^n, called the control distance associated to $\mathscr{L}$. The real number $Q = \sum_{j=1}^{m} j \dim(V_j)$ is called the *homogeneous dimension* of G on analogy with the Euclidean dimension. Indeed, since δ_r sends $B(x, 1)$ to $B(x, r)$ for every $x \in R^n$ and $r > 0$, then a simple change of variable gives

$$|B(x, r)| = r^Q |B(x, 1)|,$$

where $|\cdot|$ is Lebesgue measure. Moreover, a precise estimate of Γ is known (see [F]):

$$\Gamma(x, y) \cong d(x, y)^{-Q+2}.$$

Hence Definition 2.1 can be rewritten in the following way: the function V belongs to $L^{1,\lambda}_{\mathscr{L}}(\Omega)$ if

$$\sup_{\substack{x\in\Omega \\ r>0}} r^{-\mu} \int_{d(x,y)<r} |V(y)|\,dy < +\infty, \qquad Q - 2 < \mu, \tag{9}$$

where $\mu = \lambda(Q - 2)$.

Examples of sublaplacians follow.

Example 3.2. *The classical Laplacian.* If $\mathscr{L} = \Delta$ then by (6)

$$g = \operatorname{Lie}(X_1, \dots, X_n) = \langle X_1, \dots, X_n \rangle.$$

The dilations in (7) simply reduce to $\delta_r(x) = rx$, and the homogeneous dimension is $Q = n$. Similarly from (8) we deduce that the natural distance is the Euclidean one and the definition of $L^{1,\lambda}_{\mathscr{L}}$ is simply the classical definition of Morrey space $L^{1,\lambda(n-2)}(\Omega)$.

Example 3.3. *The sublaplacian operator on the Heisenberg group.* Let us denote by $\xi = (x, y, t)$ the elements of R^{2n+1}, with $x \in R^n$, $y \in R^n$, $t \in R$, and set

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},$$

and

$$\mathscr{L} = \sum_{j=1}^{n} X_j^2 + Y_j^2.$$

The Lie algebra associated has rank $Q = 2n+2$, and the group G is R^{2n+1} with the group law

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(xy' - x'y)).$$

Moreover, $\mathscr{L}$ is a sublaplacian operator on G. Indeed $g = V_1 \oplus V_2$, where

$$V_1 = \langle X_1, \dots, X_n, Y_1, \dots, Y_n\rangle, \qquad V_2 = \left\langle \frac{\partial}{\partial t} \right\rangle.$$

Thus $Q = \dim V_1 + 2\dim V_2 = 2n+2$, the dilations are

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t),$$

and the homogeneous norm is

$$d_0(x, y, t) = ((x^2 + y^2)^2 + t^2)^{1/4}.$$

Remark 3.1. For a general operator satisfying (3)–(5) the homogeneous dimension is not defined, but it is possible to define a local homogeneous dimension, which, however, cannot be used to give a definition of Morrey space analogous to (9). For the reader's convenience we recall the definition.

Let U be a bounded open set such that $U \subset\subset \Omega$. If $X_0, \dots, X_p$ together with their commutators of length at most m span R^n at every point of U, we say that $X_0, \dots, X_p$ are of type m, and we set

$$X^{(1)} = \{X_0, \dots, X_p\},\ X^{(2)} = \{[X_0, X_1], \dots, [X_{p-1}, X_p]\},\ \text{etc.}$$

Let $Y_1, \dots, Y_q$ be some enumeration of the components of $X^{(1)}, \dots, X^{(m)}$. If $Y_i \in X^{(j)}$, we say that Y_i has formal degree $\deg_i = \deg(Y_i) = j$. If $I = (i_1, \dots, i_n) \in N^n$, with $1 \le i_j \le q$ and $x \in U$, we set

$$\deg(I) = \deg_{i_1} + \cdots + \deg_{i_n}, \qquad \lambda_I(x) = \det(Y_{i_1}, \dots, Y_{i_n}).$$

We also call, following [NSW]

$$\Lambda(x, r) = \sum_I |\lambda_I(x)| r^{\deg(I)} \quad \forall x \in U, r > 0,$$

where the index I in the summation varies in the set of all n-uple previously defined. For $\delta > 0$, let $C(\delta)$ be the class of absolutely continuous mappings φ: $[0, 1] \to \Omega$ such that $\varphi'(t) = \sum_{j=1}^q a_j(t) Y_j(\varphi(t))$ a.e. $t \in [0, 1]$ with $|a_j(t)| < \delta^{d_j}$. Then define

$$d(x, y) = \inf\{\delta > 0 : \exists \varphi \in C(\delta) \text{ with } \varphi(0) = x, \varphi(1) = y\}.$$

The function d defines a metric, and Λ gives an estimate of the Lebesgue measure of the ball of this metric. Indeed in [NSW] and [SC] the following results are proved.

For every bounded set $U \subset\subset \Omega$ there exist $C_1, C_2, r_0 > 0$ such that, for every $x \in U$, $r < r_0$,

$$C_1 \le \frac{|B(x, r)|}{\Lambda(x, r)} \le C_2 \tag{10}$$

where $|\cdot|$ stands for Lebesgue measure.

For every bounded set $U \subset\subset \Omega$ there exist $C_1, C_2, r_0 > 0$ such that for every $x \in U$ and every $y \in U \backslash \{x\}$ with $d(x, y) \le r_0$

$$C_1 \frac{d(x, y)^2}{|B(x, d(x, y))|} \le \Gamma(x, y) \le C_2 \frac{d(x, y)^2}{|B(x, d(x, y))|}. \tag{11}$$

From (11) it follows that the following limit exists:

$$Q(x) = \lim_{r \to 0^+} \frac{\log(|B(x, r)|)}{\log r}. \tag{12}$$

Here $Q(x)$ is called the *local homogeneous dimension*, and when $\mathscr{L}$ is a sublaplacian operator, $Q(x)$ is the homogeneous dimension.

The following example will clarify the meaning by Definition 2.1 when the local homogeneous dimension is not a constant.

Example 3.4. The Grushin operator. Let us denote by $\xi = (x, y)$ the elements of R^n with $x \in R^M$, $y \in R^N$ and $n = M + N$. Then the Grushin operator is defined as

$$\mathscr{L} \equiv \sum_{j=1}^{M} X_j^2 + \sum_{j=1}^{N} Y_j^2$$

where

$$X_j = \frac{\partial}{\partial x_j} \quad \text{and} \quad Y_j = x_j \frac{\partial}{\partial y_j}.$$

Since the rank of the Lie algebra g generated by the vector fields X_j and Y_j is $M + 2N$ and the rank of the image of the same Lie algebra at every point is $M + N$, $\mathscr{L}$ is not a sublaplacian operator. Its local homogeneous dimension is

$$Q(x, y) = \begin{cases} M + N, & x \neq 0, y \in R^N, \\ M + 2N, & x = 0, y \in R^N. \end{cases}$$

Let us now note that the class defined by (9) contains only the null function. Indeed, if Ω is an open set containing the origin and μ is a real number greater than $Q(x, y) - 2$ for all $(x, y) \in \Omega$, then μ is larger than $M + 2N - 2$. Thus if $N \ge 2$, μ is larger than $M + N = n$, which is the homogeneous dimension of R^n in those points $(x, y) \in R^M \times R^N$ in which $x \neq 0$. Then we obtain $V = 0$ a.e. in Ω. However, $\mathscr{L}$ satisfies (3)–(5); hence, Definition 2.1 can be used, and our result applies.

4. Regularity of the solutions of $\mathscr{L}u + Vu = 0$

In the following we will denote

$$D_{\mathscr{L}} u \equiv (X_1 u, \ldots, X_p u)$$

for every $u \in C_0^\infty(\Omega)$. We will call $H_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$||u||_{H^1(\Omega)} = \left(\int_\Omega |u|^2 + |D_L u|^2 dy \right)^{1/2},$$

and we say that $u \in H^1_{\text{loc}}(\Omega)$ if $u\varphi \in H_0^1(\Omega)$ for every $\varphi \in C_0^\infty(\Omega)$. Let us now define what we mean by a solution.

Definition 4.1. If V is an $\mathscr{L}$-Morrey potential (or, more generally, an $\mathscr{L}$-Stummel-Kato potential), we say that $u \in H^1_{\mathrm{loc}}(\Omega)$ is a local weak solution of equation

$$\mathscr{L}u + Vu = 0 \tag{13}$$

if and only if

$$\int_\Omega D_{\mathscr{L}}u D_{\mathscr{L}}\varphi \, dx = \int_\Omega V(x)u(x)\varphi(x)\, dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Let us explicitly note that the integral in the right-hand side exists, by Remark 3.2 in [CGL] and Lemma 2.1 here. In [CGL] it is proved that any solution of (13) is continuous. We present here a slightly different proof, working on the level sets of Γ, instead of the balls of the metric. In this way we will obtain an estimate of the continuity modulus of u in terms of Γ, from which the Hölder continuity of u will follow when the potential $V \in L^{1,\lambda}_{\mathscr{L}}(\Omega)$. Indeed our results can be stated as follows.

Theorem 4.1. *Assume that $V \in S(\Omega)$ and $U \subset\subset \Omega$. Then there exist constants α, C_1, $r_0 > 0$ such that for any positive solution of $\mathscr{L}u + Vu \geq 0$ on U and any level set $\Omega_r(x_0)$ with $\Omega_{4r}(x_0) \subset\subset U$ we have*

$$|u(x) - u(x_0)| \leq C_1 \left[\Gamma(x, x_0)^{-\alpha} r^{-\alpha} + \frac{\eta(r)}{(\Gamma(x, x_0)r)^{\alpha/2}} + \eta\left(\frac{r^{1/2}}{\Gamma(x, x_0)}\right)\right] \sup_{\Omega_r(x_0)} |u|$$

for almost all $x \in \Omega_r(x_0)$.

Corollary 4.1. *Assume that $V \in L^{1,\lambda}_{\mathscr{L}}(\Omega)$ and $U \subset\subset \Omega$. Then there exist constants α, C_1, $r_0 > 0$ such that for any positive solution u of $\mathscr{L}u + Vu \geq 0$ on U and any level set $\Omega_r(x_0)$ with $\Omega_r(x_0) \subset\subset \Omega$ we have*

$$\begin{aligned} |u(x) - u(x_0)| \leq C_1[\Gamma(x, x_0)^{-\alpha} r^{-\alpha} + r^{\lambda-1}(\Gamma(x, x_0)^{\alpha/2} r^{-\alpha/2} \\ + \Gamma(x, x_0)^{-(\lambda-1)/2} r^{-(\lambda-1)/2})] \sup |u| \end{aligned}$$

for almost all $x \in \Omega_r(x_0)$.

Remark 4.1. From the estimates (11) and (12) we get

$$\Gamma(x, x_0)^{-1} \leq C \frac{|B(x, d(x, y))|}{d(x, y)^2} \leq C r^{Q(x_0)-2}.$$

Then Corollary 4.1 implies that u is Hölder continuous.

Remark 4.2. When $\mathscr{L}$ is a subelliptic laplacian and has constant homogeneous dimension Q, Corollary 4.1 simply reduces to: For any positive solution u of $\mathscr{L}u + Vu \geq 0$ on U and any ball $\overline{B(x_0, 4r)} \subset\subset U$ we have

$$\begin{aligned} |u(x) - u(x_0)| \leq C_1 \Bigg\{ \left(\frac{d(x, x_0)}{r}\right)^{\alpha(Q-1)/2} \\ + r^{\lambda-1}\left[\left(\frac{d(x, x_0)}{r}\right)^{\alpha(Q-1)/2} + \left(\frac{d(x, x_0)}{r}\right)^{(\lambda-1)(Q-1)/2}\right]\Bigg\} \sup |u| \end{aligned}$$

for every $x \in B(x_0, r)$.

Let us begin with some lemmas. If Λ is the function defined in §3, we define

$$E(x, r) = \Lambda(x, r)/r^2, \qquad x \in R^n, r > 0.$$

For $x_0 \in \Omega$, $E(x_0, \cdot)$ is a polynomial with positive coefficients; hence, it is increasing, and we will denote its inverse by $F(x_0, \cdot)$. Now, from (10) it is possible to deduce that

Proposition 4.1. *For every bounded set $U \subset\subset \Omega$ there exist constants C_0, C_1, C_2, r_0 such that for every $x \in U$, $0 < r < r_0$ we have*

$$B(x, C_1F(x, r)) \subseteq \Omega_r(x) \subseteq B(x, C_2F(x, r)),$$
$$|\Omega_R(x_0)| \leq C_0 RF^2(x_0, R)$$

(*see* [CGL, *Appendix* A1 *and* A2] *for a detailed proof*).

Lemma 4.1 (existence of a cut-off function). *For every bounded set $U \subset\subset \Omega$ for every $x_0 \in U$, for every $r, \varrho > 0$ with $r < \varrho$ and $\Omega_\varrho(x_0) \subseteq U$, there exists a cut-off function $\psi \in C_0^\infty(\Omega_\varrho(x_0))$ such that $0 \leq \psi \leq 1$ in $\Omega_\varrho(x_0)$, $\psi \equiv 1$ in $\Omega_r(x_0)$, and $|D_{\mathscr{L}}\psi| \leq 2\varrho/(\varrho - r)d(y, x_0)$.*

Proof. Let $\xi \in C_0^\infty([0, \varrho])$ be such that $\xi \equiv 1$ on $[0, r]$, $0 \leq \xi \leq 1$ on $[0, t]$, and $|\xi'| \leq 2/\varrho - r$. The function $\psi(y) = \xi(1/\Gamma(y, x_0))$ satisfies all the previous conditions.

Lemma 4.2 (Caccioppoli type inequality). *Let us suppose that $V \in S(\Omega)$. Suppose that $u \in H^1_{\mathrm{loc}}(\Omega)$, $u \geq 0$ and satisfies $\mathscr{L}u + Vu = 0$ in Ω. Then for every bounded set $U \subset\subset \Omega$ there exists C_1, $\varrho_0 > 0$ such that for every $\varrho < \varrho_0$, with $\overline{\Omega_{4\varrho}(x)} \subset U$, we have*

$$\int_{\Omega_\varrho(x)} |D_{\mathscr{L}}u|^2\,dy \leq C_1 \frac{\varrho}{|\Omega_{2\varrho}(x)|} \int_{\Omega_{2\varrho}(x)} u^2\,dy.$$

Proof. In Theorem 3.7 in [CGL] the same result is proved on the balls of the metric. Hence, using Proposition 4.2 and the existence of a cut-off function we easily get the estimate.

Proposition 4.2 (see [CGL]). *For every $u \in C_0^\infty(\Omega)$ a.e. $x \in \Omega$ and every $r > 0$ such that* $\operatorname{supp}(u) \subset\subset \Omega_r(x)$

$$u(x) = -\int_{\Omega_r(x)} Lu(y)\Gamma(x, y)dy.$$

Proof of Theorem 4.1. Let us denote by φ a cut-off function of class $C_0^\infty(\Omega_{2r}(x_0))$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $\Omega_{3/2r}(x_0)$, and

$$|D_H\varphi(x_0)| \leq C_1 \frac{1}{d(x, x_0)}.$$

Then

$$\begin{aligned} u(x) - u(x_0) = & -\int (\Gamma(x, y) - \Gamma(x_0, y))V(y)\varphi(y)u(y)\,dy \\ & + \int (\Gamma(x, y) - \Gamma(x_0, y))D_H\varphi(y)D_Hu(y)\,dy \\ & - \int (D_H\Gamma(x, y) - D_H\Gamma(x_0, y))D_H\varphi(y)u(y)\,dy \\ = & \ \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Now,

$$|\Gamma(x, y) - \Gamma(x_0, y)| \le C\left(\frac{1}{\Gamma(x_0, x)r}\right)^\alpha \Gamma(x_0, y);$$

hence, II can be estimated as

$$\begin{aligned}
\mathrm{II} &\le C\left(\frac{1}{\Gamma(x_0, x)r}\right)^\alpha \int_{\Omega_{2r}(x_0)\setminus\Omega_{3r/2}(x_0)} \Gamma(x_0, y)|D_H u|\,|D_H\varphi|\,dx \\
&\qquad \text{(by Cauchy-Schwartz inequality and the properties of the function } \varphi) \\
&\le \frac{1}{r}\left(\frac{1}{\Gamma(x_0, x)r}\right)^\alpha \left(\int_{\Omega_{2r}(x_0)\setminus\Omega_{3r/2}(x_0)} |D_H u|^2\,dx\right)^{1/2} \\
&\quad \times \left(\int_{\Omega_{2r}(x_0)\setminus\Omega_{3r/2}(x_0)} \frac{1}{d^2(x, x_0)}\,dx\right)^{1/2} \\
&\qquad \text{(by the Caccioppoli inequality and Proposition 4.2)} \\
&\le \frac{1}{r}\left(\frac{1}{\Gamma(x_0, x)r}\right)^\alpha \left(\frac{r}{\Omega_{2r}(x_0)}\int_{\Omega_{2r}(x_0)} u^2 dx\right)^{1/2} \left(\frac{|\Omega_{2r}(x_0)|}{F^2(x_0, r)}\right)^{1/2} \\
&\le \left(\frac{1}{\Gamma(x_0, x)r}\right)^\alpha \sup|u|.
\end{aligned}$$

Analogously

$$\mathrm{III} \le \left(\frac{1}{\Gamma(x_0, x)r}\right)^\alpha \sup|u|.$$

Let us finally estimate I:

$$\begin{aligned}
I &= \int_{1/\Gamma(y, x_0) > N/\Gamma(y, x_0)} (\Gamma(x, y) - \Gamma(x_0, y))V(y)\varphi(y)u(y)\,dy \\
&\quad + \int_{1/\Gamma(y, x_0) \le N/\Gamma(y, x_0)} (\Gamma(x, y) - \Gamma(x_0, y))V(y)\varphi(y)u(y)\,dy \\
&= A + B.
\end{aligned}$$

Now,

$$\begin{aligned}
A &\le \frac{1}{N^\alpha}\int_{1/\Gamma(x_0, y) \le 2r} \Gamma(x_0, y)V(y)\sup|u| \le \frac{1}{N^\alpha}\eta(r)\sup|u| \\
&\le \frac{\eta(r)}{(r\Gamma(x, x_0))^{\alpha/2}}\sup|u|
\end{aligned}$$

if we choose $N = (r\Gamma(x, x_0))^{1/2}$.

Since

$$\frac{1}{\Gamma(x, y)} \le C\left(\frac{1}{\Gamma(x, x_0)} + \frac{1}{\Gamma(x_0, y)}\right) \le C(N+1)\frac{1}{\Gamma(x, x_0)},$$

B can be estimated as

$$\begin{aligned} B &\leq \int_{1/\Gamma(y,x)\leq C(N+1)/\Gamma(y,x_0)} \Gamma(x,y)V(y)u(y)dy \\ &\leq \int_{1/\Gamma(y,x_0)\leq N/\Gamma(y,x_0)} \Gamma(x_0,y)V(y)u(y)dy \\ &\leq \sup|u|\eta\left((N+1)\frac{1}{\Gamma(y,x_0)}\right) = C_1\sup|u|\eta\left(2\left[\frac{r}{\Gamma(y,x_0)}\right]\right). \end{aligned}$$

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