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CONTINUITY OF DERIVATIONS ON H^*-ALGEBRAS

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ABSTRACT. We prove that the separating subspace for a derivation on a nonassociative H^*-algebra is contained in the annihilator of the algebra. In particular, derivations on nonassociative H^*-algebras with zero annihilator are continuous.

I. INTRODUCTION

For a complete normed algebra A, a basic continuity problem is to find algebraic conditions on A which ensure that *derivations* on A are continuous, the most celebrated result in this direction being the theorem by Johnson and Sinclair [15] that asserts derivations on semisimple (associative) Banach algebras are continuous. In the nonassociative context only very particular results about this problem are known, namely, *derivations on A are continuous whenever A is a JB^*-algebra* [25], *a classical Banach Lie algebra of operators on a Hilbert space* [14], *or a semisimple complete normed alternative algebra* [22]. Of particular relevance to this paper is the fact, recently proved by Zalar [26], that *derivations on Mal'cev H^*-algebras with zero annihilator are continuous.*

The purpose of this paper is to show that, for a derivation on a general nonassociative H^*-algebra, the separating subspace is contained in the annihilator of the algebra, so we prove the automatic continuity of derivations on H^*-algebras with zero annihilator. As an application we obtain a generalization of a theorem, by Rodríguez [18], on Jordan characterization of H^*-algebras. H^*-algebras, introduced and studied by Ambrose [1] in the associative case, have been considered also in the case of the most familiar classes of nonassociative algebras [5, 7, 9, 10, 17, 19, 20, 23, 24] and even in the general nonassociative context [8, 10, 11]. Actually three crucial results in the theory of nonassociative H^*-algebras will be used in our proof:

Theorem 1. *Every H^*-algebra is the orthogonal sum of its annihilator and a H^*-algebra with zero annihilator.*

Theorem 2 [10]. *Every H^*-algebra with zero annihilator is the closure of the orthogonal sum of its minimal closed ideals, and these are topologically simple H^*-algebras.*

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Theorem 3 [8]. *Every topologically simple H^*-algebra is a centrally closed prime algebra.*

We recall that a prime algebra A over a field $\mathbf{K}$ is said to be *centrally closed* if, for every nonzero ideal I of A and for every linear mapping $f: I \to A$ with

$$f(ax) = af(x) \quad \text{and} \quad f(xa) = f(x)a$$

for all a in A and x in I, there exists λ in $\mathbf{K}$ such that $f(x) = \lambda x$ for all x in I (see [12]). Also we note that the introduction of techniques of central closeability in the treatment of problems of automatic continuity is perhaps the most interesting novelty in this paper.

II. The theorem

Our first result will be an easy consequence of Theorem 3.1 in [12] which asserts that, given linearly independent elements $x_1, \dots, x_p$ in a centrally closed prime algebra A, there exists T in the usual multiplication algebra of A (from now on denoted by $M(A)$) such that $T(x_1) = \cdots = T(x_{p-1}) = 0$ and $T(x_p) \neq 0$. In fact, we will need only the case $p = 2$ of this statement, whose proof is then almost straightforward: if $T(x_2) = 0$ whenever T lies in $M(A)$ with $T(x_1) = 0$, then $S(x_1) \to S(x_2)$ is a well-defined linear mapping (say f) from the nonzero ideal $I = M(A)(x_1)$ into A satisfying

$$f(ax) = af(x) \quad \text{and} \quad f(xa) = f(x)a$$

for all a in A and x in I, so x_1 and x_2 are linearly dependent because $f(x_1) = x_2$ and A is centrally closed.

Proposition 1. *Let A be a centrally closed prime algebra such that* $\dim(T(A)) > 1$ *for all nonzero T in the multiplication algebra $M(A)$ of A. Then there exist sequences $\{b_n\}$ in A and $\{T_n\}$ in $M(A)$ such that $T_n \cdots T_1 b_n \neq 0$ and $T_{n+1} T_n \cdots T_1 b_n = 0$ for all n in $\mathbf{N}$.*

Proof. Let b_1 in A and T_1 in $M(A)$ be such that $T_1 b_1 \neq 0$, and suppose inductively that $b_1, \dots, b_k$ and $T_1, \dots, T_k$ have been chosen so that $T_j \cdots T_1 b_{j-1} = 0$ and $T_j \cdots T_1 b_j \neq 0$ for $j = 2, \dots, k$. Since $\dim(T_k \cdots T_1)(A) > 1$, there exists b_{k+1} in A such that $T_k \cdots T_1 b_k$ and $T_k \cdots T_1 b_{k+1}$ are linearly independent, so there exists T_{k+1} in $M(A)$ such that

$$T_{k+1} T_k \cdots T_1 b_k = 0 \quad \text{and} \quad T_{k+1} T_k \cdots T_1 b_{k+1} \neq 0.$$

The sequences $\{b_n\}$ and $\{T_n\}$ constructed in this way satisfy the requirements in the proposition. □

Now we will see that the possibility $\dim(T(A)) = 1$ for some T in the multiplication algebra of a topologically simple complete normed algebra A implies the continuity of derivations on A. The proof is very simple, involving only the well-known easy fact that the separating subspace for a derivation on a normed algebra is a closed ideal and the following purely algebraic observation. If, for an element a in an algebra A, we denote by L_a and R_a the operators of left and right multiplication by a on A, respectively, and D is a derivation on A, then we have

$$DL_a - L_a D = L_{D(a)} \quad \text{and} \quad DR_a - R_a D = R_{D(a)};$$

hence, for every T in $M(A)$ the operator $d(T) = DT - TD$ lies in $M(A)$. Recall that a normed algebra is said to be topologically simple if it has nonzero product and has no nonzero proper closed ideals.

Proposition 2. *Let D be a derivation on a topologically simple complete normed algebra A, and suppose that there exists a nonzero T in $M(A)$ with finite-dimensional range. Then D is continuous.*

Proof. Since T is continuous with finite-dimensional range, DT is continuous, so also TD is continuous because $TD = DT - d(T)$ and $d(T)$ lies in $M(A)$. Thus the separating subspace $s(D)$ for D is contained in the kernel of T. Since $s(D)$ is a closed ideal of A and T is nonzero, we obtain that $s(D) = 0$, and the result follows from the closed graph theorem. □

Now we can conclude the proof of our main result. Recall that an *H^*-algebra* is a complex algebra A with a (conjugate-linear) algebra involution $*$, whose underlying vector space is a Hilbert space satisfying

$$(ab \mid c) = (a \mid cb^*) = (b \mid a^*c)$$

for all a, b, c in A. Note that, since for an element a in an H^*-algebra A the adjoint L_a^* of the operator L_a is L_{a^*} and the adjoint of R_a is R_a^*, for every T in $M(A)$ we have that T^* lies in $M(A)$. Recall finally that the *annihilator* of an algebra A is defined as the set of those a in A satisfying $ab = ba = 0$ for every b in A.

Theorem 4. *Every derivation on an H^*-algebra A with zero annihilator is continuous.*

Proof. We assume first that A is topologically simple, and we argue by contradiction. If D is not continuous, then on the one hand we can use Proposition 2, Theorem 3, and Proposition 1 to obtain the existence of sequences $\{b_n\}$ in A and $\{T_n\}$ in $M(A)$ such that $T_n \cdots T_1 b_n \neq 0$ and $T_{n+1} \cdots T_1 b_n = 0$ for all n in $\mathbf{N}$, and clearly we may assume $\|b_n\| = \|T_n\| = 1$ for all n in $\mathbf{N}$. On the other hand the discontinuity of D together with the closed graph theorem implies the existence of an element b in A such that the linear functional $x \mapsto (D(x) \mid b)$ from A into $\mathbf{C}$ is not continuous. But it is straightforward to show that the set $I = \{y \in A\colon x \mapsto (D(x) \mid y) \text{ is continuous}\}$ is an ideal of A which is closed thanks to the classical Banach-Steinhaus theorem. It follows from the topological simplicity of A that $I = 0$, *i.e.*, the functional $x \mapsto (D(x) \mid y)$ is discontinuous for every nonzero element y in A. Now, using this fact, we can construct inductively a sequence $\{a_n\}$ in A with the property that for all n in $\mathbf{N}$ we have $\|a_n\| \leq 2^{-n}$, and

$$\begin{aligned}|(D(a_n) \mid T_n \cdots T_1 b_n)| \geq n &+ \left|\sum_{j=1}^{n-1} (D(T_1^* \cdots T_j^* a_j) \mid b_n)\right| \\ &+ \|d(T_1^* \cdots T_n^*)\| + \|d(T_1^* \cdots T_{n+1}^*)\|.\end{aligned}$$

Now we consider the element a in A defined by $a = \sum_{j=1}^{\infty} T_1^* \cdots T_j^* a_j$, and

for n in N write $c_n = a_{n+1} + \sum_{j=n+2}^{\infty} T_{n+2}^* \cdots T_j^* a_j$. Then we have

$$\begin{aligned}
(D(a) \mid b_n) &= \sum_{j=1}^{n-1} (D(T_1^* \cdots T_j^* a_j) \mid b_n) + (D(T_1^* \cdots T_n^* a_n) \mid b_n) \\
&\quad + \left(D\left(\sum_{j=n+1}^{\infty} T_1^* \cdots T_j^* a_j \right) \mid b_n \right) \\
&= \sum_{j=1}^{n-1} (D(T_1^* \cdots T_j^* a_j) \mid b_n) + (d(T_1^* \cdots T_n^*)(a_n) + T_1^* \cdots T_n^* D(a_n) \mid b_n) \\
&\quad + (D(T_1^* \cdots T_{n+1}^* c_n) \mid b_n) \\
&= \sum_{j=1}^{n-1} (D(T_1^* \cdots T_j^* a_j) \mid b_n) + (d(T_1^* \cdots T_n^*)(a_n) \mid b_n) + (D(a_n) \mid T_n \cdots T_1 b_n) \\
&\quad + (d(T_1^* \cdots T_{n+1}^*)(c_n) \mid b_n) + (D(c_n) \mid T_{n+1} \cdots T_1 b_n) \\
&= (D(a_n) \mid T_n \cdots T_1 b_n) + \sum_{j=1}^{n-1} (D(T_1^* \cdots T_j^* a_j) \mid b_n) \\
&\quad + (d(T_1^* \cdots T_n^*)(a_n) \mid b_n) + (d(T_1^* \cdots T_{n+1}^*)(c_n) \mid b_n),
\end{aligned}$$

where for the last equality we have used that $T_{n+1} \cdots T_1 b_n = 0$. Therefore, since $\|c_n\| \leq 1$, we obtain

$$\begin{aligned}
\|D(a)\| &\geq |(D(a) \mid b_n)| \\
&\geq |(D(a_n) \mid T_n \cdots T_1 b_n)| - \left| \sum_{j=1}^{n-1} (D(T_1^* \cdots T_j^* a_j) \mid b_n) \right| \\
&\quad - |(d(T_1^* \cdots T_n^*)(a_n) \mid b_n)| - |(d(T_1^* \cdots T_{n+1}^*)(c_n) \mid b_n)| \\
&\geq |(D(a_n) \mid T_n \cdots T_1 b_n)| - \left| \sum_{j=1}^{n-1} (D(T_1^* \cdots T_j^* a_j) \mid b_n) \right| \\
&\quad - \|d(T_1^* \cdots T_n^*)\| - \|d(T_1^* \cdots T_{n+1}^*)\| \\
&\geq n.
\end{aligned}$$

The contradiction $\|D(a)\| \geq n$ for all n in $\mathbf{N}$ shows that D is continuous if A is topologically simple.

The general case will be reduced to that proved above by means of Theorem 2, as follows. If M is any minimal closed ideal of A, M is a direct summand of A, so $D(M)$ is contained in M because A is semiprime. Since M is a topologically simple H^*-algebra, it follows from the first part of the proof that D is continuous on M. Now let $\{x_n\}$ be a sequence in A such that $\lim x_n = 0$ and $\lim D(x_n) = x$ for some x in A. For all y in M we have that

$$0 = \lim D(x_n y) = \lim(D(x_n)y + x_n D(y)) = xy,$$

and in the same way $yx = 0$. Thus $s(D)M = Ms(D) = 0$ for every minimal closed ideal M of A, so $s(D)A = As(D) = 0$ and thus $s(D) = 0$. By the closed graph theorem D is continuous. □

Derivations on (nonassociative) H^*-algebras appears naturally in [18], and our theorem can be immediately applied in order to obtain a generalization, of [18, Theorem 4].

Corollary 1. *Let A be a flexible complex algebra with algebra involution, and assume that the algebra A^+, obtained by symmetrization of the product of A, is an H^*-algebra with zero annihilator for some inner product and given involution. Then A with the same inner product and involution is an H^*-algebra.*

Proof. The flexibility of A assures us that, for any a in A, the mapping $D_a\colon b \mapsto ab - ba$ is a derivation on A^+ which is continuous by the above theorem. Now [11, Theorem 2.1] shows us that $D_a^*(b) = -D(b^*)^*$ for all a and b in A. This means that $(ab - ba \mid c) = (b \mid a^*c - ca^*)$ for all a, b, and c in A. Since A^+ is an H^*-algebra, we have that $(ab + ba|c) = (b|a^*c + ca^*)$ and so $(ab|c) = (b|a^*c)$ and $(ab|c) = (a|cb^*)$ for all a, b and c in A. □

One of the important questions in the theory of Banach algebras is whether the separating subspace $s(D)$ for a derivation D is contained in the radical. We prove that for an H^*-algebra A and a derivation D on A we have that

$$s(D) \subset \mathrm{Ann}(A).$$

Theorem 5. *The separating subspace $s(D)$ for a derivation D on an H^*-algebra A is contained in the annihilator,* $\mathrm{Ann}(A)$, *of A.*

Proof. By Theorem 1, A is the orthogonal sum of $\mathrm{Ann}(A)$ and an H^*-algebra B with zero annihilator. If we denote by π_B the orthogonal projection from A onto B we have that $\pi_B \circ D_{|B}$ is a derivation on B which is continuous by Theorem 4. Also it is clear that $D(\mathrm{Ann}(A)) \subset \mathrm{Ann}(A)$. Now let $\{x_n\}$ be a sequence in A such that $\lim x_n = 0$ and $\lim D(x_n) = x$ for some x in A. Then, since $\pi_B D(x_n) = \pi_B D(\pi_B(x_n)) = \pi_B D_{|B}(\pi_B(x_n))$ for all n in $\mathbf{N}$, we have that $\pi_{|B}(x) = 0$, so $x \in \mathrm{Ann}(A)$. □

Remark. H^*-algebras over the field of real numbers have been considered also in the literature (see [3, 6, 17] for the associative case, and [2, 4, 13, 21] for the nonassociative case). Since the complexification of a real H^*-algebra is in a natural way an H^*-algebra and derivations on the given algebra extend canonically to derivations on the complexification, it follows, from Theorem 5, that the separating subspace for derivations on real H^*-algebras are contained in the annihilator of the algebra, so derivations are continuous when the real H^*-algebra has zero annihilator.

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