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AMENABLE ACTIONS AND WEAK CONTAINMENT OF CERTAIN REPRESENTATIONS OF DISCRETE GROUPS

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ABSTRACT. We consider a countable discrete group Γ acting ergodically on a standard Borel space S with quasi-invariant measure μ. Let π be a *unitary* representation of Γ on $L^2(S, d\mu, \mathscr{H})$ "nicely" related with S. We prove that if Γ acts amenably on S then π is weakly contained in the regular representation.

Let Γ be a countable discrete group acting (on the right) on a standard Borel space S. Suppose that the action is ergodic with μ as a quasi-invariant probability measure on S. Let $P(s, g)$ be the Radon-Nikodym cocycle of the action, that is, $P(s, g) = \frac{d\mu(sg)}{d\mu(s)}$. Construct a representation of Γ as follows:

(1) choose any Hilbert space $\mathscr{H}$,
(2) choose any (Borel) cocycle $A(s, g) : S \times G \to \mathscr{U}(\mathscr{H})$ where $\mathscr{U}(\mathscr{H})$ denotes the group of unitary operators on $\mathscr{H}$,
(3) construct $\mathscr{H}_\pi = L^2(S, d\mu, \mathscr{H})$ with the usual inner product given by $\langle F, G\rangle = \int_S \langle F(s), G(s)\rangle_{\mathscr{H}}\, d\mu(s)$,

and finally define

$$(\pi(g)F)(s) = P(s, g)^{\frac{1}{2}} A(s, g) F(sg).$$

Observe now that although the group *is not* amenable it turns out that π is weakly contained in the regular representation for many natural constructions. In particular, this is true when

(a) Γ is a free group and S is its Martin boundary (see [K-S, Appendix]),
(b) Γ is a hyperbolic group and S is its boundary (see [A] and use the techniques of [K-S]), and
(c) Γ is a lattice in a semisimple Lie group G and $S = G/B$ is the maximal Furstenberg boundary of G (see [Q-S, §4]).

Recall that a group is amenable if and only if every unitary representation of Γ is weakly contained in $\infty\pi_{\text{reg}}$ (where π_{reg} denotes the regular representation) [H] and [Z3, Proposition 7.3.6].

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Zimmer introduced a class of Borel spaces which make a nonamenable group G "look like" an amenable one when acting on them. These spaces are called *amenable* G-spaces (G is said to act amenably on them).

To see that cases (a) and (c) deal with amenable actions of Γ on S recall the following

Proposition [Z3]. *Let G be a locally compact group, H a closed subgroup, and Γ a lattice in G. Then Γ acts amenably on G/H if and only if H is amenable.*

Case (c) follows since minimal parabolic subgroups of semisimple Lie groups are amenable. For case (a) the reader should think of the free group acting on a homogeneous tree as a discrete cocompact subgroup of the full automorphism group G of the tree. The Martin boundary of Γ is the set of all "ends" of the tree, and it was proved by Nebbia [N] that the stabilizer (in G) of an end is amenable. Case (b) was proved by Adams in [A].

In this paper we shall prove the following

Theorem. *Suppose that a countable discrete group Γ acts ergodically on (S, μ) with quasi-invariant measure μ. Suppose that the action is amenable. Then every representation π constructed as above is weakly contained in π_{reg}.*

We also believe that the converse is true; namely, we believe that if *for every* Hilbert space $\mathscr{H}$ the above-constructed representation $(L^2(S, d\mu, \mathscr{H}), \pi)$ is weakly contained in π_{reg} then the action is amenable.

Proof of the result. Denote by π_{reg} the right regular representation of Γ. Construct $\pi_{\text{reg}} \otimes \pi$ acting on $\ell^2(\Gamma) \otimes \mathscr{H}_\pi$. Recall that $U : \ell^2(\Gamma) \otimes \mathscr{H}_\pi \to \ell^2(\Gamma, \mathscr{H}_\pi)$ given by

$$(Uf)(x) = \pi(x)f(x)$$

intertwines $\pi_{\text{reg}} \otimes \pi$ to $\pi_{\text{reg}} \otimes I$, where I denotes the trivial representation. Thus $\pi_{\text{reg}} \otimes \pi$ is equivalent to a direct sum of copies of π_{reg}, and it is sufficent to prove that π is weakly contained in $\pi_{\text{reg}} \otimes \pi$.

Fix any finite subset E of Γ. Let $F \in L^2(S, d\mu, \mathscr{H})$. Inside $\ell^2(\Gamma) \otimes L^2(S, d\mu, \mathscr{H})$ we shall construct a sequence of functions $(f_n)_{n=0}^\infty$ such that $\langle \pi_{\text{reg}} \otimes \pi(x) f_n, f_n \rangle$ approaches $\langle \pi(x)F, F \rangle$ for x in E.

Identify $\ell^2(\Gamma) \otimes L^2(S, d\mu, \mathscr{H})$ with $L^2(S, d\mu, \ell^2(\Gamma) \otimes \mathscr{H})$. Set $f_n(s) = \phi_n(s) \otimes F(s)$ where the ϕ_n are elements of

$$\ell^2(\Gamma) \otimes L^2(S, d\mu) = L^2(S, d\mu, \ell^2(\Gamma)).$$

Compute

$$\begin{aligned}
&\langle \pi_{\text{reg}} \otimes \pi(g) f_n, f_n \rangle \\
&\quad = \int_S P(s, g)^{\frac{1}{2}} \langle \pi_{\text{reg}}(g)\phi_n(sg), \phi_n(s) \rangle_{\ell^2(\Gamma)} \langle A(s, g)F(sg), F(s) \rangle_{\mathscr{H}} \, d\mu(s)
\end{aligned}$$

and

$$\langle \pi(g)F, F \rangle = \int_S P(s, g)^{\frac{1}{2}} \langle A(s, g)F(sg), F(s) \rangle_{\mathscr{H}} \, d\mu(s).$$

In order to prove our result we need a sequence of functions $\phi_n(s, x) \in L^2(S, d\mu, \ell^2(\Gamma))$ such that $\sum_{x \in \Gamma} \phi_n(sg, xg)\phi_n(s, x)$ is weakly converging, in $L^2(S, d\mu)$, to the function identically one on S.

For amenable groups the existence of such a sequence is related to existence of an invariant mean in $(L^\infty(G))^*$.

An analogue of the invariant mean property was proved for discrete groups by Zimmer in [Z1, Z2] in the case of an amenable action. The reader may take the following as a definition of an amenable action:

Proposition 1 (Zimmer). *Suppose that Γ is a countable discrete group acting ergodically on (S, μ). Then S is an amenable Γ-space if and only if there is a norm one linear map $\sigma : L^\infty(S \times \Gamma) \to L^\infty(S)$ such that:*

(1) $\sigma(\mathbf{1}) = \mathbf{1}$, $f \geq 0$ *implies* $\sigma(f) \geq 0$;
(2) *if $A \subseteq S$ is measurable, $\sigma(f \cdot \chi_{p^{-1}A}) = \sigma(f)\chi_A$ where $p : S \times \Gamma \to S$ is the projection; and*
(3) $\sigma(f \cdot g) = \sigma(f) \cdot g$, *where* $f \cdot g(s, x) = f(sg, xg)$ *and* $\phi \cdot g(s) = \phi(sg)$.

In [Z1] it is shown how to construct σ once we know that the action of Γ on S is amenable. From these we may deduce other properties of σ which will be needed. For the reader's convenience we briefly recall the construction. Let $\tilde{U}_g : L^2(S \times \Gamma) \to L^2(S \times \Gamma)$ be defined by

$$(\tilde{U}_g f)(s, x) = f(sg, xg)P(s, g)^{\frac{1}{2}} \ ; .$$

For $f \in L^\infty(S)$ define a multiplication operator M_f on $L^2(S \times \Gamma)$ by letting

$$(M_f h)(s, x) = f(s)h(s, x) .$$

Let R denote the von Neumann algebra generated by $\{\tilde{U}_g, M_f\}$ in $\mathscr{B}(L^2(S \times \Gamma))$. It was proved in [Z2] that if Γ acts amenably on S then there is a norm one projection and hence a conditional expectation $P : \mathscr{B}(L^2(S \times \Gamma)) \to R$.

Consider now the following decomposition for $L^2(S \times \Gamma)$. For $f \in L^2(S \times \Gamma)$ let $f_s(x) = f(sx, x)P(s, x)^{\frac{1}{2}}$. Then $f \to \int_S^\oplus f_s \, d\mu$ is a unitary isomorphism of $L^2(S \times \Gamma) \simeq \int_S^\oplus \ell^2(\Gamma) \, d\mu$. Further every element of R is decomposable with respect to this direct integral decomposition. (Namely, $\tilde{U}_g$ corresponds to $\int_S^\oplus \pi_{\text{reg}}(g)$ and M_f to $\int_S^\oplus f^s$ where $(f^s h_s)(x) = f(sx)h_s(x)$.) For every $f \in L^\infty(S \times \Gamma)$ we have the multiplication operator $M_f \in \mathscr{B}(L^2(S \times \Gamma))$. Now $P(M_f)$ is decomposable; hence, we can write $P(M_f) = \int_S^\oplus T_s^f \, d\mu$. Our map σ is now given by

$$(\sigma f)(s) = \langle T_s^f(\delta_e), \delta_e \rangle_{\ell^2(\Gamma)} ,$$

where δ_e denotes the characteristic function of the identity e.

In particular, if F is an element of $L^\infty(S)$ and we let $F(s, g) = F(s)$ for every $g \in \Gamma$ then

$$\text{(i)} \quad (\sigma F)(s) = F(s);$$

furthermore, for every $h \in L^\infty(S \times \Gamma)$

$$\text{(ii)} \quad (\sigma h F)(s) = (\sigma h)(s)F(s) ,$$

since $P(M_h M_F) = P(M_h)M_F$ because F is an element of R.

In order to use our function σ we need the Radon-Nikodym derivative $P(s, g)^{\frac{1}{2}}$ to be a bounded function of s for any fixed g. This, in general, may or may not be true. However, we shall prove that it is always possible to find a measure ν equivalent to μ, for which this is true.

Lemma 1. *Let Γ be a discrete group. There exists a length function $x \to \|x\|$ such that the cardinality of the sets $\{\|x\| \le N\}$ is less than or equal to 3^N. Given such a length function choose $K > 3$ and set*

$$\nu(E) = \sum_{x \in \Gamma} K^{-\|x\|} \mu_x(E) \quad \text{where } \mu_x(E) = \mu(Ex)$$

for every measurable E. Then ν is equivalent to μ and the Radon-Nikodym derivative $\frac{d\nu(sg)}{d\nu(s)}$ is less than or equal to $K^{\|g\|}$.

Proof. Choose any ordering $\{x_n\}_{n=0}^{\infty}$ for the elements of Γ satisfying $x_0 = e$. Set

$$\|x\| = \inf \left\{ \sum_{j=1}^{J} n_j : x = x_{n_1}^{\pm} x_{n_2}^{\pm} \cdots x_{n_J}^{\pm} \right\} .$$

It is immediate that $\|\cdot\|$ is a length function. Set $\Phi(N) = \sharp\{x : \|x\| = N\}$. We observe that the number of strictly positive solutions $(n_1, n_2, \dots, n_k)$ of the equation

$$n_1 + n_2 + \cdots + n_k = N$$

is $\binom{N-1}{k-1}$. Hence

$$\Phi(N) \le \sum_{k=1}^{N} \binom{N-1}{k-1} 2^k = 2 \cdot 3^{N-1} .$$

Finally

$$\sharp\{x : \|x\| \le N\} \le 1 + 2 + 2 \cdot 3 + \cdots + 2 \cdot 3^{N-1} = 3^N .$$

It is obvious from the definition of ν and from the quasi-invariance of μ that ν and μ are equivalent measures. Let E be a measurable set. Compute

$$\begin{aligned} \nu(Eg) &= \sum_{x \in \Gamma} K^{-\|x\|} \mu(Egx) = \sum_{x \in \Gamma} K^{-\|x\| - \|g\| + \|g\|} \mu(Egx) \\ &\le K^{\|g\|} \sum_{x \in \Gamma} K^{-\|gx\|} \mu(Egx) = K^{\|g\|} \nu(E) . \end{aligned}$$

This completes the proof. □

From this point on we shall assume that the Radon-Nikodym derivative $P(s, g)$ is a bounded function of s for any given g.

Proof of the theorem. Let

$$P = \left\{ \phi \in L^1(S \times \Gamma) \,|\, \phi(s, x) \ge 0 \text{ and } \int_S \sum_{x \in \Gamma} \phi(s, x)\, d\mu(s) = 1 \right\}.$$

First of all observe that P is weak-* dense in the unit ball of $(L^\infty(S \times \Gamma))^*$. Construct a linear functional σ^* on $L^\infty(S \times \Gamma)$ by letting

$$\sigma^*(f) = \int_S (\sigma f)(s)\, d\mu(s) .$$

Recall that $(h \cdot g)(s, x) = h(sg, xg)$ and $(\psi \cdot) g(s) = \psi(sg)$.

Choose h in $L^\infty(S \times \Gamma)$, ϵ positive, and g in a finite set E of Γ. First of all we shall show that it is possible to find $\phi \in L^1(S \times \Gamma)$ such that

$$(1.1) \quad \left| \int_S \left(\sum_{x \in \Gamma} (\phi(sg, xg) - \phi(s, x)) h(s, x) \right) d\mu(s) \right| < \epsilon \quad \text{for every } g \text{ in } E .$$

In fact, by property (c) of Proposition 1, we have

$$\begin{aligned} \int_S (\sigma h)(s)\, d\mu(s) &= \int_S (\sigma h \cdot g^{-1})(s) P(s, g^{-1})\, d\mu(s) \quad \text{(because of (ii))} \\ &= \int_S (\sigma h \cdot g^{-1} P(s, g^{-1}))(s)\, d\mu(s). \end{aligned}$$

Hence it is possible to choose $\phi \in L^1(S \times \Gamma)$ so that

$$(1.2) \qquad \left| \int (\sigma h)(s) - \left(\sum_{x \in \Gamma} \phi(s, x) h(s, x) \right) d\mu(s) \right| < \epsilon$$

and

(1.3)

$$\left| \int_S (\sigma h \cdot g^{-1} P(s, g^{-1}))(s) - \left(\sum_{x \in \Gamma} \phi(s, x) h(sg^{-1}, xg^{-1}) P(s, g^{-1}) \right) d\mu(s) \right| < \epsilon.$$

But

$$\begin{aligned} &\int_S \left(\sum_{x \in \Gamma} \phi(s, x) h(sg^{-1}, xg^{-1}) P(s, g^{-1}) \right) d\mu(s) \\ &\quad = \int_S P(s, g^{-1}) \sum_{x \in \Gamma} \phi(s, xg) h(sg^{-1}, x)\, d\mu(s) \end{aligned}$$

and letting $sg^{-1} = s'$

$$= \int_S \sum_{x \in \Gamma} \phi(sg, xg) h(s, x)\, d\mu(s) .$$

Subtracting (1.3) from (1.2) we get (1.1)

Moreover, since $\sigma(\mathbf{1}) = \mathbf{1}$, we may also require that for any given $h' \in L^\infty(S)$

$$(1.4) \qquad \left| \int_S h'(s)(1 - \sum_{x \in \Gamma} \phi(s, x))\, d\mu(s) \right| < \epsilon .$$

We shall use now Namioka's argument to pass from the weak to the strong closure. Let $F = L^1(S) \times \prod_{g \in \Gamma} L^1(S \times \Gamma) = F_0 \times \prod_{g \in \Gamma} F_g$ where $L^1(S)$ and each copy of $L^1(S \times \Gamma)$ has the norm topology. The weak topology on F is the product topology for the weak topology on $L^1(S)$ and on $L^1(S \times \Gamma)$. Define a linear map $T : P \to F$ by letting

$$(T(\phi))_0 = \sum_{x \in \Gamma} \phi(s, x) \quad \text{and} \quad (T(\phi))_g = \phi(sg, xg) - \phi(s, x) .$$

Then (1.1) and (1.4) say that the point $(\mathbf{1}, 0, \dots, 0 \dots)$ is in the weak closure of $T(P)$, and since $T(P)$ is convex, it follows that the same point is in the strong closure of $T(P)$. Hence we can find a net of elements $\psi_j \in P$ such that

$$\sum_{x \in \Gamma} \psi_j(s, x) \to \mathbf{1} \quad \text{and} \quad \psi_j(sg, xg) - \psi_j(s, x) \to 0$$

strongly, respectively, in $L^1(S)$ and in $L^1(S \times \Gamma)$.

Finally let $\phi_j(s, x) = \sqrt{\psi_j(s, x)}$. By standard arguments

$$\|\phi_j \cdot g - \phi_j\|_{L^2(S \times \Gamma)} \leq \|\psi_j \cdot - \psi_j\|^{\frac{1}{2}}_{L^1(S \times \Gamma)}.$$

Choose any F in $L^2(S, d\mu, \mathscr{H})$. Set $f_n(s, x) = \phi_n(s, x) \otimes F(s)$. Let

$$G(s, g) = P(s, g)^{\frac{1}{2}} \langle A(s, g) F(sg), F(s) \rangle_{\mathscr{H}}.$$

Since the set of functions $F : S \to \mathscr{H}$ such that $\langle (\pi(g)F)(s), F(s) \rangle$ is a bounded function of s is dense in $L^2(S, d\mu, \mathscr{H})$, we may assume that $G(s, g)$ is a bounded function of s for any given $g \in \Gamma$. We may also assume that $G(s, g) \geq 0$. Observe that $\|\sqrt{G(s, g)}(\phi_n \cdot g - \phi_n)\|_{L^2(S \times \Gamma)} \to 0$. We have

$$\begin{aligned} &\|\sqrt{G(s, g)}(\phi_n \cdot g - \phi_n)\|_{L^2(S \times \Gamma)} \\ &\quad = \int_S G(s, g) \sum_{x \in \Gamma} \psi_n(sg, xg)\, d\mu(s) + \int_S G(s, g) \sum_{x \in \Gamma} \psi_n(s, x)\, d\mu(s) \\ &\qquad - 2 \int_S G(s, g) \sum_{x \in \Gamma} (\phi_n(sg, xg) \phi_n(s, x)) d\mu(s). \end{aligned}$$

But $\sum_{x \in \Gamma} \psi_n(s, x)$ converges to $\mathbf{1}$ in $L^1(S)$, and so does $\sum_{x \in \Gamma} \psi_n(sg, xg)$ since $\|\psi_n \cdot g - \psi_n\|_{L^1(S \times \Gamma)} \to 0$. Hence,

$$\int_S G(s, g) \sum_{x \in \Gamma} (\phi_n(sg, xg) \phi_n(s, x))\, d\mu(s) \to \int_S G(s, g)\, d\mu(s). \quad \square$$

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