AMENABLE ACTIONS AND WEAK CONTAINMENT OF CERTAIN REPRESENTATIONS OF DISCRETE GROUPS

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Abstract. We consider a countable discrete group $\Gamma$ acting ergodically on a standard Borel space $S$ with quasi-invariant measure $\mu$. Let $\pi$ be a unitary representation of $\Gamma$ on $L^2(S, d\mu, H)$ "nicely" related with $S$. We prove that if $\Gamma$ acts amenably on $S$ then $\pi$ is weakly contained in the regular representation.

Let $\Gamma$ be a countable discrete group acting (on the right) on a standard Borel space $S$. Suppose that the action is ergodic with $\mu$ as a quasi-invariant probability measure on $S$. Let $P(s, g)$ be the Radon-Nikodym cocycle of the action, that is, $P(s, g) = \frac{d\mu(sg)}{d\mu(s)}$. Construct a representation of $\Gamma$ as follows:

1. Choose any Hilbert space $H$,
2. Choose any (Borel) cocycle $A(s, g) : S \times G \to \mathcal{U}(H)$ where $\mathcal{U}(H)$ denotes the group of unitary operators on $H$,
3. Construct $\mathcal{H}_\pi = L^2(S, d\mu, H)$ with the usual inner product given by $\langle F, G \rangle = \int_S (F(s), G(s))_H d\mu(s)$,

and finally define

$$ (\pi(g)F)(s) = P(s, g)^{\frac{1}{2}}A(s, g)F(sg). $$

Observe now that although the group is not amenable it turns out that $\pi$ is weakly contained in the regular representation for many natural constructions. In particular, this is true when

(a) $\Gamma$ is a free group and $S$ is its Martin boundary (see [K-S, Appendix]),
(b) $\Gamma$ is a hyperbolic group and $S$ is its boundary (see [A] and use the techniques of [K-S]),
(c) $\Gamma$ is a lattice in a semisimple Lie group $G$ and $S = G/B$ is the maximal Furstenberg boundary of $G$ (see [Q-S, §4]).

Recall that a group is amenable if and only if every unitary representation of $\Gamma$ is weakly contained in $\pi_{reg}$ (where $\pi_{reg}$ denotes the regular representation) [H] and [Z3, Proposition 7.3.6].

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Zimmer introduced a class of Borel spaces which make a nonamenable group $G$ "look like" an amenable one when acting on them. These spaces are called amenable $G$-spaces ($G$ is said to act amenably on them).

To see that cases (a) and (c) deal with amenable actions of $\Gamma$ on $S$ recall the following

**Proposition [Z3].** Let $G$ be a locally compact group, $H$ a closed subgroup, and $\Gamma$ a lattice in $G$. Then $\Gamma$ acts amenably on $G/H$ if and only if $H$ is amenable.

Case (c) follows since minimal parabolic subgroups of semisimple Lie groups are amenable. For case (a) the reader should think of the free group acting on a homogeneous tree as a discrete cocompact subgroup of the full automorphism group $G$ of the tree. The Martin boundary of $\Gamma$ is the set of all "ends" of the tree, and it was proved by Nebbia [N] that the stabilizer (in $G$) of an end is amenable. Case (b) was proved by Adams in [A].

In this paper we shall prove the following

**Theorem.** Suppose that a countable discrete group $\Gamma$ acts ergodically on $(S, \mu)$ with quasi-invariant measure $\mu$. Suppose that the action is amenable. Then every representation $\pi$ constructed as above is weakly contained in $\pi_{\text{reg}}$.

We also believe that the converse is true; namely, we believe that if for every Hilbert space $\mathcal{H}$ the above-constructed representation $(L^2(S, d\mu, \mathcal{H}), \pi)$ is weakly contained in $\pi_{\text{reg}}$ then the action is amenable.

**Proof of the result.** Denote by $\pi_{\text{reg}}$ the right regular representation of $\Gamma$. Construct $\pi_{\text{reg}} \otimes \pi$ acting on $\ell^2(\Gamma) \otimes \mathcal{H}_\pi$. Recall that $U : \ell^2(\Gamma) \otimes \mathcal{H}_\pi \to \ell^2(\Gamma, \mathcal{H}_\pi)$ given by

$$(Uf)(x) = \pi(x)f(x)$$

intertwines $\pi_{\text{reg}} \otimes \pi$ to $\pi_{\text{reg}} \otimes I$, where $I$ denotes the trivial representation. Thus $\pi_{\text{reg}} \otimes \pi$ is equivalent to a direct sum of copies of $\pi_{\text{reg}}$, and it is sufficient to prove that $\pi$ is weakly contained in $\pi_{\text{reg}} \otimes \pi$.

Fix any finite subset $E$ of $\Gamma$. Let $F \in L^2(S, d\mu, \mathcal{H})$. Inside $\ell^2(\Gamma) \otimes L^2(S, d\mu, \mathcal{H})$ we shall construct a sequence of functions $(f_n)_{n=0}^\infty$ such that

$$\langle \pi_{\text{reg}} \otimes \pi(x)f_n, f_n \rangle_{\ell^2(\Gamma)} \to \langle \pi(x)F, F \rangle$$

for $x$ in $E$.

Identify $\ell^2(\Gamma) \otimes L^2(S, d\mu, \mathcal{H})$ with $L^2(S, d\mu, \ell^2(\Gamma) \otimes \mathcal{H})$. Set $f_n(s) = \phi_n(s) \otimes F(s)$ where the $\phi_n$ are elements of

$$\ell^2(\Gamma) \otimes L^2(S, d\mu) = L^2(S, d\mu, \ell^2(\Gamma)).$$

Compute

$$\langle \pi_{\text{reg}} \otimes \pi(g)f_n, f_n \rangle = \int_S P(s, g)^{\frac{1}{2}} \langle \pi_{\text{reg}}(g)\phi_n(sg), \phi_n(s) \rangle \langle A(s, g)F(sg), F(s) \rangle d\mu(s)$$

and

$$\langle \pi(g)F, F \rangle = \int_S P(s, g)^{\frac{1}{2}} \langle A(s, g)F(sg), F(s) \rangle d\mu(s).$$

In order to prove our result we need a sequence of functions $\phi_n(s, x) \in L^2(S, d\mu, \ell^2(\Gamma))$ such that $\sum_{x \in \Gamma} \phi_n(sg, xg)\phi_n(s, x)$ is weakly converging, in $L^2(S, d\mu)$, to the function identically one on $S$. 

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For amenable groups the existence of such a sequence is related to existence of an invariant mean in \((L^\infty(G))^*\).

An analogue of the invariant mean property was proved for discrete groups by Zimmer in [Z1, Z2] in the case of an amenable action. The reader may take the following as a definition of an amenable action:

**Proposition 1 (Zimmer).** Suppose that \(\Gamma\) is a countable discrete group acting ergodically on \((S, \mu)\). Then \(S\) is an amenable \(\Gamma\)-space if and only if there is a norm one linear map \(\sigma : L^\infty(S \times \Gamma) \rightarrow L^\infty(S)\) such that:

1. \(\sigma(1) = 1, f \geq 0\) implies \(\sigma(f) \geq 0\);
2. if \(A \subseteq S\) is measurable, \(\sigma(f \cdot \chi_{p^{-1}A}) = \sigma(f)\chi_A\) where \(p : S \times \Gamma \rightarrow S\) is the projection; and
3. \(\sigma(f \cdot g) = \sigma(f) \cdot g\), where \(f \cdot g(s, x) = f(sg, xg)\) and \(\phi \cdot g(s) = \phi(sg)\).

In [Z1] it is shown how to construct \(\sigma\) once we know that the action of \(\Gamma\) on \(S\) is amenable. From these we may deduce other properties of \(\sigma\) which will be needed. For the reader's convenience we briefly recall the construction. Let \(\hat{U}_g : L^2(S \times \Gamma) \rightarrow L^2(S \times \Gamma)\) be defined by

\[
(\hat{U}_gf)(s, x) = f(sg, xg)\mu(s, g) \cdot f.
\]

For \(f \in L^\infty(S)\) define a multiplication operator \(M_f\) on \(L^2(S \times \Gamma)\) by letting

\[
(M_fh)(s, x) = f(s)h(s, x).
\]

Let \(R\) denote the von Neumann algebra generated by \(\{\hat{U}_g, M_f\}\) in \(\mathcal{B}(L^2(S \times \Gamma))\). It was proved in [Z2] that if \(\Gamma\) acts amenably on \(S\) then there is a norm one projection and hence a conditional expectation \(P : \mathcal{B}(L^2(S \times \Gamma)) \rightarrow R\).

Consider now the following decomposition for \(L^2(S \times \Gamma)\). For \(f \in L^2(S \times \Gamma)\) let \(f_e(x) = f(sx, x)\mu(s, x)\)\(\frac{1}{2}\). Then \(f \rightarrow \int_S f_e d\mu\) is a unitary isomorphism of \(L^2(S \times \Gamma) \simeq \int_S^\oplus \ell^2(\Gamma) d\mu\). Further every element of \(R\) is decomposable with respect to this direct integral decomposition. (Namely, \(\hat{U}_g\) corresponds to \(\int_S^\oplus \pi_{reg}(g)\) and \(M_f\) to \(\int_S^\oplus f^s\) where \((f^s h_s)(x) = f(sx)h_s(x)\).)

For every \(f \in L^\infty(S \times \Gamma)\) we have the multiplication operator \(M_f \in \mathcal{B}(L^2(S \times \Gamma))\). Now \(P(M_f)\) is decomposable; hence, we can write \(P(M_f) = \int_S^\oplus T_s f d\mu\). Our map \(\sigma\) is now given by

\[
(\sigma f)(s) = \langle T_s f(\delta_e), \delta_e \rangle_{\ell^2(\Gamma)},
\]

where \(\delta_e\) denotes the characteristic function of the identity \(e\).

In particular, if \(F\) is an element of \(L^\infty(S)\) and we let \(F(s, g) = F(s)\) for every \(g \in \Gamma\) then

\[
(i) \quad (\sigma F)(s) = F(s);
\]

furthermore, for every \(h \in L^\infty(S \times \Gamma)\)

\[
(ii) \quad (\sigma h F)(s) = (\sigma h)(s)F(s),
\]

since \(P(M_h M_F) = P(M_h)M_F\) because \(F\) is an element of \(R\).

In order to use our function \(\sigma\) we need the Radon-Nikodym derivative \(P(s, g)\)\(\frac{1}{2}\) to be a bounded function of \(s\) for any fixed \(g\). This, in general, may or may not be true. However, we shall prove that it is always possible to find a measure \(\nu\) equivalent to \(\mu\), for which this is true.
Lemma 1. Let $\Gamma$ be a discrete group. There exists a length function $x \to \|x\|$ such that the cardinality of the sets $\{|\|x\| \leq N\}$ is less than or equal to $3^N$. Given such a length function choose $K > 3$ and set

$$\nu(E) = \sum_{x \in \Gamma} K^{-\|x\|} \mu_x(E) \quad \text{where} \quad \mu_x(E) = \mu(Ex)$$

for every measurable $E$. Then $\nu$ is equivalent to $\mu$ and the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ is less than or equal to $K\|g\|.$

Proof. Choose any ordering $\{x_n\}_{n=0}^{\infty}$ for the elements of $\Gamma$ satisfying $x_0 = e$. Set

$$\|x\| = \inf \left\{ \sum_{j=1}^{J} n_j : x = x_{n_1}^+ x_{n_2}^- \cdots x_{n_J}^- \right\}.$$

It is immediate that $\|\cdot\|$ is a length function. Set $\Phi(N) = \#\{x : \|x\| = N\}$. We observe that the number of strictly positive solutions $(n_1, n_2, \ldots, n_k)$ of the equation

$$n_1 + n_2 + \cdots + n_k = N$$

is $\binom{N-1}{k-1}$. Hence

$$\Phi(N) \leq \sum_{k=1}^{N} \binom{N-1}{k-1} 2^k = 2 \cdot 3^{N-1}.$$

Finally

$$\#\{x : \|x\| \leq N\} \leq 1 + 2 + 2 \cdot 3 + \cdots + 2 \cdot 3^{N-1} = 3^N.$$

It is obvious from the definition of $\nu$ and from the quasi-invariance of $\mu$ that $\nu$ and $\mu$ are equivalent measures. Let $E$ be a measurable set. Compute

$$\nu(E) = \sum_{x \in \Gamma} K^{-\|x\|} \mu(Ex) = \sum_{x \in \Gamma} K^{-\|x\|} \|s\| \|g\| \mu(Ex)$$

$$\leq K \|s\| \sum_{x \in \Gamma} K^{-\|x\|} \mu(Ex) = K \|s\| \nu(E).$$

This completes the proof. $\Box$

From this point on we shall assume that the Radon-Nikodym derivative $P(s, g)$ is a bounded function of $s$ for any given $g$.

Proof of the theorem. Let

$$P = \left\{ \phi \in L^1(S \times \Gamma) \mid \phi(s, x) \geq 0 \text{ and } \int_S \sum_{x \in \Gamma} \phi(s, x) \, d\mu(s) = 1 \right\}.$$

First of all observe that $P$ is weak-* dense in the unit ball of $(L^\infty(S \times \Gamma))^*$. Construct a linear functional $\sigma^*$ on $L^\infty(S \times \Gamma)$ by letting

$$\sigma^*(f) = \int_S (\sigma f)(s) \, d\mu(s).$$

Recall that $(h \cdot g)(s, x) = h(sg, xg)$ and $(\psi \cdot)g(s) = \psi(sg)$. 

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Choose $h$ in $L^\infty(S \times \Gamma)$, $\epsilon$ positive, and $g$ in a finite set $E$ of $\Gamma$. First of all we shall show that it is possible to find $\phi \in L^1(S \times \Gamma)$ such that

\begin{equation}
\left| \int_S \left( \sum_{x \in \Gamma} (\phi(sg, xg) - \phi(s, x))h(s, x) \right) d\mu(s) \right| < \epsilon \quad \text{for every } g \in E .
\end{equation}

In fact, by property (c) of Proposition 1, we have

\[
\int_S (\sigma h)(s) d\mu(s) = \int_S (\sigma h \cdot g^{-1})(s)P(s, g^{-1}) d\mu(s) \quad \text{(because of (ii))}
\]

\[
= \int_S (\sigma h \cdot g^{-1}P(s, g^{-1}))(s) d\mu(s).
\]

Hence it is possible to choose $\phi \in L^1(S \times \Gamma)$ so that

\begin{equation}
\int (\sigma h)(s) - \left( \sum_{x \in \Gamma} \phi(s, x)h(s, x) \right) d\mu(s) < \epsilon
\end{equation}

and

\begin{equation}
\int (\sigma h \cdot g^{-1}P(s, g^{-1}))(s) - \left( \sum_{x \in \Gamma} \phi(s, x)h(sg^{-1}, xg^{-1})P(s, g^{-1}) \right) d\mu(s) < \epsilon .
\end{equation}

But

\[
\int_S \left( \sum_{x \in \Gamma} \phi(s, x)h(sg^{-1}, xg^{-1})P(s, g^{-1}) \right) d\mu(s)
\]

\[
= \int_S P(s, g^{-1}) \sum_{x \in \Gamma} \phi(s, xg)h(sg^{-1}, x) d\mu(s)
\]

and letting $sg^{-1} = s'$

\[
= \int_S \sum_{x \in \Gamma} \phi(sg, xg)h(s, x) d\mu(s).
\]

Subtracting (1.3) from (1.2) we get (1.1)

Moreover, since $\sigma(1) = 1$, we may also require that for any given $h' \in L^\infty(S)$

\begin{equation}
\int_S h'(s)(1 - \sum_{x \in \Gamma} \phi(s, x)) d\mu(s) < \epsilon .
\end{equation}

We shall use now Namioka's argument to pass from the weak to the strong closure. Let $F = L^1(S) \times \prod_{g \in \Gamma} L^1(S \times \Gamma) = F_0 \times \prod_{g \in \Gamma} F_g$ where $L^1(S)$ and each copy of $L^1(S \times \Gamma)$ has the norm topology. The weak topology on $F$ is the product topology for the weak topology on $L^1(S)$ and on $L^1(S \times \Gamma)$. Define a linear map $T : P \to F$ by letting

\[
(T(\phi))_0 = \sum_{x \in \Gamma} \phi(s, x) \quad \text{and} \quad (T(\phi))_g = \phi(sg, xg) - \phi(s, x) .
\]
Then (1.1) and (1.4) say that the point \( (1, 0, \ldots, 0, \ldots) \) is in the weak closure of \( T(P) \), and since \( T(P) \) is convex, it follows that the same point is in the strong closure of \( T(P) \). Hence we can find a net of elements \( \psi_j \in P \) such that
\[
\sum_{x \in \Gamma} \psi_j(s, x) \to 1 \quad \text{and} \quad \psi_j(sg, xg) - \psi_j(s, x) \to 0
\]
strongly, respectively, in \( L^1(S) \) and in \( L^1(S \times \Gamma) \).

Finally let \( \phi_j(s, x) = \sqrt{\psi_j(s, x)} \). By standard arguments
\[
\|\phi_j \cdot g - \phi_j\|_{L^2(S \times \Gamma)} \leq \|\psi_j \cdot g - \psi_j\|_{L^1(S \times \Gamma)}^{1/2}.
\]
Choose any \( F \) in \( L^2(S, d\mu, \mathcal{M}) \). Set \( f_n(s, x) = \phi_n(s, x) \otimes F(s) \). Let
\[
G(s, g) = P(s, g)^\frac{1}{2} \langle A(s, g)F(sg), F(s) \rangle \mathcal{M}.
\]
Since the set of functions \( F : S \to \mathcal{M} \) such that \( \langle (\pi(g)F)(s), F(s) \rangle \) is a bounded function of \( s \) is dense in \( L^2(S, d\mu, \mathcal{M}) \), we may assume that \( G(s, g) \) is a bounded function of \( s \) for any given \( g \in \Gamma \). We may also assume that \( G(s, g) \geq 0 \). Observe that \( \|\sqrt{G(s, g)}(\phi_n \cdot g - \phi_n)\|_{L^2(S \times \Gamma)} \to 0 \). We have
\[
\|\sqrt{G(s, g)}(\phi_n \cdot g - \phi_n)\|_{L^1(S \times \Gamma)}
= \int_S G(s, g) \sum_{x \in \Gamma} \psi_n(sg, xg) d\mu(s) + \int_S G(s, g) \sum_{x \in \Gamma} \psi_n(s, x) d\mu(s)
- 2 \int_S G(s, g) \sum_{x \in \Gamma} (\phi_n(sg, xg) \phi_n(s, x)) d\mu(s).
\]
But \( \sum_{x \in \Gamma} \psi_n(s, x) \) converges to \( 1 \) in \( L^1(S) \), and so does \( \sum_{x \in \Gamma} \psi_n(sg, xg) \) since \( \|\psi_n \cdot g - \psi_n\|_{L^1(S \times \Gamma)} \to 0 \). Hence,
\[
\int_S G(s, g) \sum_{x \in \Gamma} (\phi_n(sg, xg) \phi_n(s, x)) d\mu(s) \to \int_S G(s, g) d\mu(s). \quad \square
\]

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