COUNTABLE PARACOMPACTNESS OF Σ-PRODUCTS

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Abstract. It is known that Σ-products of compact spaces always are countably
paracompact but not necessarily normal. In the present paper it is proved that
a Σ-product of paracompact σ-spaces is normal if and only if it is countably
paracompact.

1. Introduction

The equivalence of normality and countable paracompactness in Cartesian
products has been investigated by many authors [5, 8, 9, 17, 22] so that this
topic constitutes a very interesting part in the theory of product spaces [7, 14].
In this paper the equivalence of normality and countable paracompactness will
be considered for Σ-products.

The concept of Σ-products was introduced by Corson [2] who proved that
Σ-products of complete metric spaces are normal. In Gul’ko [4] and Rudin [15]
the following is shown:
(i) A Σ-product of metric spaces is normal.
This answers affirmatively a long outstanding question raised by Corson [2].
Kombarov [8] later generalized (i) by obtaining the following result:
(ii) A Σ-product of paracompact p-spaces is (collectionwise) normal if and
only if it has countable tightness.

In connection with the above results, the following Questions 1 and 2 are
considered by Yajima [18] and Kodama, respectively.

Question 1. Is a Σ-product of paracompact σ-spaces normal if it has countable
tightness?

Question 2. Is a Σ-product of Lašnev spaces normal?

Question 1 has been answered positively. In fact Yajima [18] even proved
(iii) A Σ-product of paracompact Σ-spaces is (collectionwise) normal if it
has countable tightness.

Since paracompact p-spaces are Σ-spaces, (iii) is also a generalization of
the "if" part of (ii). However, the countable tightness is no longer a necessary
condition for a Σ-product of paracompact Σ-spaces to be normal, because there
exists a collectionwise normal $\Sigma$-product of $M_1$-spaces which has no countable tightness [18]. Moreover, since there exists a nonnormal $\Sigma$-product of $M_1$-spaces [18], in Question 1 the assumption of countable tightness cannot be dropped. On the other hand, Rudin [16] proved that any $\Sigma$-product of metric spaces is shrinking and hence countably paracompact. It is also known from Yajima [19] that any normal $\Sigma$-product of $\sigma$-spaces is countably paracompact.

The main purpose of this paper is to establish the equivalence of normality and countable paracompactness of $\Sigma$-products of paracompact $\sigma$-spaces. Namely we prove the following theorem.

**Theorem 1.** A $\Sigma$-product of paracompact $\sigma$-spaces is normal if and only if it is countably paracompact.

In the rest of the paper, we also consider the subshrinking property of $X \times \kappa$, where $X$ is a semistratifiable space and $\kappa$ an uncountable regular cardinal with the usual order topology. The subshrinking property was introduced by Yajima [20] which is important for the study of shrinking property (see Yajima [20] and Hoshina [6]). Yajima [21] recently proved that $X \times \kappa$ is subshrinking for any $\sigma$-space $X$, and he asked whether $X \times \kappa$ is subshrinking for a semistratifiable space $X$. We shall prove

**Theorem 2.** Let $X$ be a semistratifiable space with $\chi(X) < \kappa$. Then $X \times \kappa$ is subshrinking.

All spaces considered here are assumed to be regular $T_1$. The set of natural numbers is denoted by $\mathbb{N}$ and natural numbers are denoted by $i$, $j$, $k$, and $n$. $\kappa$ always denotes an uncountable regular cardinal with the usual order topology.

## 2. Proof of Theorem 1

Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be the Cartesian product of spaces $X_\lambda$, $\lambda \in \Lambda$, and let $s = (s_\lambda)_{\lambda \in \Lambda}$ be a fixed point of $X$. The subspace $\Sigma = \{ x \in X : x_\lambda = s_\lambda \text{ for all but countably many } \lambda \in \Lambda \}$ of $X$ is called a $\Sigma$-product of spaces $X_\lambda$, $\lambda \in \Lambda$. Such a point $s \in \Sigma$ is called the base point of $\Sigma$, which is often omitted.

Let $X$ be a $\Sigma$-product of spaces $X_\lambda$, $\lambda \in \Lambda$, with a base point $(s_\lambda)_{\lambda \in \Lambda}$. For a point $x \in X$, denote by $\text{Supp}(x)$ the set $\{ \lambda \in \Lambda : x_\lambda \neq s_\lambda \}$. Let $\Delta$ be an index set such that for each $\xi \in \Delta$, $R_\xi$ is a subset of $\Lambda$. Then we denote by $X_\xi$ the Cartesian product $\prod_{\lambda \in R_\xi} X_\lambda$ and by $p_\xi$ the projection of $X$ onto $X_\xi$ for each $\xi \in \Delta$.

Let $\xi = (\alpha_{ij})_{i,j \leq n}$ be an $n \times n$ matrix. By $\xi_k$ we denote the $k \times k$ matrix $(\alpha_{ij})_{i,j \leq k}$ for $1 \leq k \leq n$. In particular, $\xi_{n-1}$ is often abbreviated as $\xi_-$ and $\xi_0$ denotes the empty set $\varnothing$.

A space is called a $\sigma$-space if it has a $\sigma$-locally finite net [13]. Note that Lašnev spaces (i.e., closed images of metric spaces) are $M_1$, and $M_1$-spaces are paracompact $\sigma$ [13]. It is well known that the countable product of paracompact $\sigma$-spaces ($\Sigma$-spaces) is paracompact $\sigma$ ($\Sigma$).

The following two lemmas are useful to prove Theorem 1.

**Lemma 1** [5, Lemma 2.1]. Let $X$ be a countably paracompact space and let $E$ and $F$ be a pair of disjoint subsets. Suppose that $F$ is closed and there exists open sets $U_n$, $n \in \mathbb{N}$, such that $E \subset \bigcap_{n \in \mathbb{N}} U_n$ and $(\bigcap_{n \in \mathbb{N}} U_n) \cap F = \varnothing$. Then $E$ and $F$ are separated by open sets.
Lemma 2 [11, Theorem 1]. Let $X$ be a $\sigma$-space. Then there exists a sequence \{${\mathcal F}_n : n \in \mathbb{N}$\} of locally finite closed covers of $X$, satisfying

(a) ${\mathcal F}_n = \{F(\alpha_1 \cdots \alpha_n) : \alpha_1, \ldots, \alpha_n \in \Omega\}$ for each $n \in \mathbb{N}$,

(b) $F(\alpha_1 \cdots \alpha_n) = \bigcup\{F(\alpha_1 \cdots \alpha_n \alpha) : \alpha \in \Omega\}$ for each $\alpha_1, \ldots, \alpha_n \in \Omega$,

(c) For each $x \in X$, there exists a sequence $\alpha_1, \alpha_2, \ldots \in \Omega$ such that $x \in \bigcap_{n \in \mathbb{N}} F(\alpha_1 \cdots \alpha_n)$ and each open nbd of $x$ contains some $F(\alpha_1 \cdots \alpha_n)$.

The above sequence \{${\mathcal F}_n : n \in \mathbb{N}$\} is called a spectral $\sigma$-net of $X$ and the sequence \{${\mathcal F}(\alpha_1 \cdots \alpha_n) : n \in \mathbb{N}$\} in (c) is called a local $\sigma$-net of $X$ at $x$.

Our proof of Theorem 1 is based on the idea in Yajima [18, 20] and we shall use the following fact: a space $X$ is normal if and only if for every pair $A, B$ of disjoint closed subsets of $X$ there exists a $\sigma$-locally finite open cover $\mathcal{U}$ of $X$ such that either $\bigcap \mathcal{U} \cap A = \emptyset$ or $\bigcap \mathcal{U} \cap B = \emptyset$ for every $U \in \mathcal{U}$.

Proof of Theorem 1. Let $X$ be a $\Sigma$-product of paracompact $\sigma$-spaces $X_{\lambda}$, $\lambda \in \Lambda$, with a base point $s = (s_\lambda)_{\lambda \in \Lambda} \in X$, and suppose $X$ is countably paracompact. To prove that $X$ is normal, let $A$ and $B$ be a pair of disjoint closed subsets of $X$; we shall find a $\sigma$-locally finite open cover $\mathcal{G}$ of $X$ such that either $\bigcap \mathcal{G} \cap A = \emptyset$ or $\bigcap \mathcal{G} \cap B = \emptyset$ for every $U \in \mathcal{G}$. Let $\Delta_0 = \{\xi_0\}$, where $\xi_0 = (\emptyset)$, and take an arbitrary nonempty countable subset $R_0 \subset \Lambda$.

Now, for each $n \in \mathbb{N}$ we construct a collection $\mathcal{G}_n$ of open sets in $X$ and an index set $\Delta_n$ of $n \times n$ matrices such that for each $\xi \in \Delta_n$, $\Omega(\xi)$, $E(\xi)$, $H(\xi)$, and $x_\xi$ are given satisfying the following conditions:

1. Each $\mathcal{G}_n$ is locally finite in $X$ such that for each $G \in \mathcal{G}_n$, $G$ is disjoint from $A$ or $B$.
2. For each $\xi \in \Delta_n$, $\{F(\alpha_1 \cdots \alpha_k) : \alpha_1, \ldots, \alpha_k \in \Omega(\xi)\}$, $k \in \mathbb{N}$, is a spectral $\sigma$-net of $X_\xi$.
3. For each $\xi = (\alpha_{ij}), \sum \leq n \in \Delta_n$,
   (a) $\xi_\gamma \in \Delta_{\gamma-1}$, $\alpha_{in} \in \Omega(\xi_-)$ for $1 \leq i \leq n - 1$, and $\alpha_{nj} \in \Omega(\xi_-)$ for $1 \leq j \leq n$, where for $n = 1$, $\alpha_{11} \in \Omega(\xi_0)$;
   (b) $E(\xi) = \bigcap_{\sum \leq n \in \Delta_n} \bigcap_{1 \leq j \leq n} (F(\alpha_{ij} \cdots \alpha_{i1}))$.
4. \{H(\xi) : $\xi \in \Delta_n$\} is a locally finite collection of open sets of $X$ with $H(\xi) \subseteq E(\xi)$ for each $\xi \in \Delta_n$.
5. For each $\xi \in \Delta_{n-1}$, $E(\xi)$ is covered by $\mathcal{G}_n \cup \{E(\eta) : \eta \in \Delta_n \text{ with } \eta_- = \xi\}$, and $X$ is covered by $\mathcal{G}_1 \cup \{E(\eta) : \eta \in \Delta_1\}$.
6. For each $\xi \in \Delta_n$,
   (a) $x_\xi \in A \cap E(\xi)$ if $n$ is odd and $x_\xi \in B \cap E(\xi)$ if $n$ is even;
   (b) $R_\xi = R_{\xi_-} \cup \text{Supp}(x_\xi)$.

Assume that the above construction has already been performed for no greater than $n$, where, without loss of generality, we may assume that $n$ is odd. Take a $\xi \in \Delta_n$. Put

$$M_\xi = \{\eta = (\alpha_{ij}), \sum \leq n+1 : \eta_- = \xi, \alpha_{i,j+1} \in \Omega(\xi_{i,j+1})$$

for $1 \leq i \leq n$ and $\alpha_{n+1,j} \in \Omega(\xi)$ for $1 \leq j \leq n + 1\}.$

For each $\eta = (\alpha_{ij}), \sum \leq n+1 \in M_\xi$, we define

$$E(\eta) = \bigcap_{i=1}^{n+1} p_{\xi_{i,j}}^{-1}(F(\alpha_{i,j} \cdots \alpha_{i,n+1})).$$
Moreover, we put
\[ \Delta_\xi = \{ \eta \in M_\xi : B \cap E(\eta) \neq \emptyset \}. \]
It is easily seen that \( \{ p_\xi(E(\eta)) : \eta \in M_\xi \} \) is a locally finite collection of closed sets of \( X_\xi \) with \( p_\xi^{-1}(E(\eta)) = E(\eta) \) for each \( \eta \in \Delta_\xi \). And so if we define \( S(\xi) \) by
\[ S(\xi) = \bigcup \{ E(\eta) : \eta \in M_\xi \setminus \Delta_\xi \}, \]
\( p_\xi(S(\xi)) \) is closed in \( X_\xi \) with \( S(\xi) = p_\xi^{-1}(S(\xi)) \). Note that \( S(\xi) \subset E(\xi) \cap (X \setminus B) \). It follows from the perfect normality of \( X_\xi \) that
\[ S(\xi) = \bigcap_{n=1}^{\infty} p_\xi^{-1}(V_n) \subset \bigcap_{n=1}^{\infty} p_\xi^{-1}(V_n) \subset X \setminus B \]
for some countably many open sets \( V_n, n \in \mathbb{N}, \) in \( X_\xi \). Since \( X \) is countably paracompact, Lemma 1 implies that there exists an open set \( G_\xi \) in \( X_\xi \) contained in \( H(\xi) \) such that
\[ S(\xi) \subset G_\xi \subset \overline{G}_\xi \subset X \setminus B. \]
Here, keeping \( \xi \in \Delta_n \), we let
\[ G_{n+1} = \{ G_\xi : \xi \in \Delta_n \} \quad \text{and} \quad \Delta_{n+1} = \bigcup_{\xi \in \Delta_n} \Delta_\xi. \]
To define \( H(\eta) \) for \( \eta \in \Delta_n \), note, as mentioned above, that \( \{ p_\xi(E(\eta)) : \eta \in M_\xi \} \) is a locally finite collection of closed sets of \( X_\xi \) with \( p_\xi^{-1}(E(\eta)) = E(\eta) \subset E(\xi) \) for \( \eta \in M_\xi \). By the paracompactness of \( X_\xi \), there exists a locally finite collection \( \{ W(\eta) : \eta \in M_\xi \} \) of open sets of \( X_\xi \) such that \( p_\xi(E(\eta)) \subset W(\eta) \) and thus \( E(\eta) \subset p_\xi^{-1}(W(\eta)) \). Let \( H(\eta) = p_\xi^{-1}(W(\eta)) \cap H(\xi) \). It follows from the inductive assumption (4) that \( \{ H(\eta) : \eta \in \Delta_{n+1} \} \) is locally finite with \( H(\eta) \supset E(\eta) \). For each \( \eta \in \Delta_{n+1} \), we can choose some \( x_\eta \in B \cap E(\eta) \). Let \( R_\eta = R_{\eta-} \cup \text{Supp}(x_\eta) \). Since \( X_\eta \) is a \( \sigma \)-space it follows from Lemma 2 that there exists a spectral \( \sigma \)-net
\[ \{ F(\alpha_1 \cdots \alpha_k) : \alpha_1, \ldots, \alpha_k \in \Omega(\eta) \}, \quad k \in \mathbb{N}, \]
of \( X_\eta \) for each \( \eta \in \Delta_{n+1} \). Then the conditions (1)-(6) are satisfied for \( n + 1 \). Here we check only (5). Pick any \( \xi = (\alpha_{ij})_{i,j \leq n} \in \Delta_n \). Then
\[ F(\alpha_{i1} \cdots \alpha_{in}) = \bigcup \{ F(\alpha_{i1} \cdots \alpha_{i,n+1}) : \alpha_{i,n+1} \in \Omega(\xi_{i-1}) \} \]
for \( i = 1, \ldots, n \), and
\[ X = \bigcup \{ p_\xi^{-1}(F(\alpha_{n+1,1} \cdots \alpha_{n+1,n+1})) : \alpha_{n+1,j} \in \Omega(\xi) \text{ for } j = 1, \ldots, n + 1 \}. \]
It follows that
\[ E(\xi) = \bigcup \{ E(\eta) : \eta \in M_\xi \} \subset \bigcup \mathcal{G}_{n+1} \cup (\bigcup \{ E(\eta) : \eta \in \Delta_\xi \}). \]
We now set \( \mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{G}_n \). By (1), \( \mathcal{F} \) is a \( \sigma \)-locally finite collection of open sets of \( X \) such that for \( G \in \mathcal{F}, \overline{G} \) misses \( A \) or \( B \). To complete the proof, it suffices to show that \( \mathcal{F} \) covers \( X \). Assume the contrary and pick some \( x \in X \setminus \bigcup \mathcal{F} \). By (2) and (5), we can inductively choose a sequence \( \{ \alpha_{ij} : i, j \geq 1 \} \) such that for each \( n \geq 1 \), \( \xi^{(n)} = (\alpha_{ij})_{i,j \leq n} \in \Delta_n \) and \( \{ F(\alpha_{n1} \cdots \alpha_{nk}) : k \in \mathbb{N} \} \)
is a local $\sigma$-net of $X_{(\xi^{(n-1)})}$ at point $p_{(\xi^{(n-1)})}(x)$, where $\alpha_{nk} \in \Omega(\xi^{(n-1)})$ and $\xi^{(0)} = \xi_0$. Now fix $m \geq 1$. If $n > m$, then

$$x_{\xi^{(n)}} \in E(\xi^{(n)}) \subset p_{\xi^{(m)}}^{-1}(F(\alpha_{m+1,1} \cdots \alpha_{m+1,n})).$$

We thus have $p_{\xi^{(m)}}(x_{\xi^{(n)}}) \in F(\alpha_{m+1,1} \cdots \alpha_{m+1,n})$ for each $n > m$. Since

$\{F(\alpha_{m+1,1} \cdots \alpha_{m+1,k}) : k \in \mathbb{N}\}$

is a local $\sigma$-net of $X_{(\xi^{(m)})}$ at point $p_{\xi^{(m)}}(x)$, the sequence $\{p_{\xi^{(m)}}(x_{\xi^{(n)}})\}_{n>m}$ converges to $p_{\xi^{(m)}}(x)$. Define a point $y = (y_\lambda)_{\lambda \in \Lambda}$ in $X$ by letting $y_\lambda = x_\lambda$ if $\lambda \in \bigcup_{n=1}^{\infty} R_{(\xi^{(n)})}$ and $y_\lambda = s_\lambda$ otherwise. Then one can prove that the sequence $\{x_{\xi^{(n)}}\}_{n \in \mathbb{N}}$ converges to $y$, and thus $y \in A \cap B$. This is a contradiction. The proof of Theorem 1 is complete.

Question 2 is still open. By Theorem 1 we now have

**Corollary 1.** A $\Sigma$-product of Lašnev $(M_1)$ spaces is normal if and only if it is countably paracompact.

By Theorem 1 and [20, Corollary 1] we also have

**Corollary 2.** The following are equivalent for a $\Sigma$-product $X$ of paracompact $\sigma$-spaces.

1. $X$ is collectionwise normal.
2. $X$ is shrinking.
3. $X$ is normal.
4. $X$ is countably paracompact.

It is not possible to replace $\sigma$-spaces by $\Sigma$-spaces in Theorem 1. Since, as pointed out in the abstract, $\Sigma$-products of compact spaces always are countably paracompact but not necessarily normal [4].

### 3. Proof of Theorem 2

A space is said to be semistratifiable [3] if there exists a function $g$ of $X \times \mathbb{N}$ into the topology of $X$ satisfying

1. $\bigcap_{n \in \mathbb{N}} g(x, n) = \{x\}$ for each $x \in X$;
2. if $\{x_n\}$ is a sequence of points in $X$ with $x \in \bigcap_{n \in \mathbb{N}} g(x_n, n)$ for some $x \in X$, then $\{x_n\}$ converges to $x$.

A space $X$ is said to be shrinking if for every open cover $\{G_\gamma : \gamma \in \Gamma\}$ of $X$ there exists a closed cover $\{F_\gamma : \gamma \in \Gamma\}$ of $X$ such that $F_\gamma \subseteq G_\gamma$ for each $\gamma \in \Gamma$. If the closed cover can be weakly chosen as a closed cover $\mathcal{F} = \{F_\gamma \times n : \gamma \in \Gamma\}$ and $n \in \mathbb{N}$ with $F_\gamma \times n \subseteq G_\gamma$ for each $\gamma \in \Gamma$ and $n \in \mathbb{N}$, then the space is said to be subshrinking. Such a cover $\mathcal{F}$ is called a subshrinking of $\{G_\gamma : \gamma \in \Gamma\}$.

It follows from Bešlagić [1] that a space is shrinking if and only if it is normal and subshrinking. Note that subparacompact spaces are subshrinking.

**Proof of Theorem 2.** Let $X$ be a semistratifiable space and $\mathcal{G} = \{G_\gamma : \gamma \in \Gamma\}$ an open cover of $X \times \kappa$. We shall find a subshrinking for $\mathcal{G}$.

For a set $F \subset X$, set

$\mathcal{W}(F) = \{W : W \text{ is open in } \kappa \text{ such that } F \times W \subseteq G_\gamma \text{ for some } \gamma \in \Gamma\}$

and

$\mathcal{Y} = \{V : V \text{ is open in } X \text{ such that } \kappa = \bigcup \mathcal{W}(V)\}.$

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Now for each $n \in \mathbb{N}$ we construct inductively two $\sigma$-locally finite collections $\mathcal{G}_n$ and $\mathcal{F}_n$ of closed sets of $X$ satisfying the following conditions (1)-(3):

1. $\mathcal{F}_{n+1}$ can be expressed as $\mathcal{F}_{n+1} = \bigcup\{\mathcal{F}_F : F \in \mathcal{F}_n\}$.
2. For each $C \in \mathcal{G}_n$, $\kappa = \bigcup \mathcal{W}(C)$.
3. For each $F \in \mathcal{F}_n$,
   a. $F \subset g(x_F, n)$ for some $x_F \in X \setminus \bigcup \mathcal{V}$;
   b. $F$ is covered by $\mathcal{G}_{n+1} \cup \mathcal{F}_F$; and $X$ is covered by $\mathcal{G}_1 \cup \mathcal{F}_1$.

Assume $n \in \mathbb{N}$, and $\mathcal{G}_i$ and $\mathcal{F}_i$ for $i \leq n$ have already been defined satisfying the conditions. Take an $F \in \mathcal{F}_n$ and fix it. Put

$$\mathcal{W} = \{F \cap V : V \in \mathcal{V}\} \cup \{F \cap g(x, n+1) : x \in F \setminus \bigcup \mathcal{V}\}.$$ 

By the subparacompactness of $X$, there exists a $\sigma$-locally finite closed cover $\mathcal{F}$ of $F$ refining $\mathcal{W}$. Let $\mathcal{E}_F = \{F \in \mathcal{F} : F \subset V \text{ for some } V \in \mathcal{V}\}$ and $\mathcal{F}_F = \mathcal{F} \setminus \mathcal{E}_F$. Here running $F \in \mathcal{F}_n$ we put

$$\mathcal{E}_{n+1} = \bigcup\{\mathcal{E}_F : F \in \mathcal{F}_n\} \quad \text{and} \quad \mathcal{F}_{n+1} = \bigcup\{\mathcal{F}_F : F \in \mathcal{F}_n\}.$$ 

Then both $\mathcal{E}_{n+1}$ and $\mathcal{F}_{n+1}$ are locally finite satisfying conditions (1)-(3).

Claim. $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ covers $X$.

Assume the contrary and pick some $x \in X \setminus \bigcup \mathcal{C}$. Then one can easily find a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X \setminus \bigcup \mathcal{V}$ such that $x \in g(x_n, n)$ for each $n \in \mathbb{N}$. It follows from the definition of semistratifiable spaces above Definition 3 that the sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to $x$. Since $\chi(X) < \kappa$, we may take a $\lambda < \kappa$ and find a nbd base $\{V(x_n, \alpha) : \alpha < \lambda\}$ for $x_n$, $n \in \mathbb{N}$. By the definition of $x_n$, for each $n \in \mathbb{N}$ there exists a point

$$\xi(n, \alpha) \in \kappa \setminus \bigcup \mathcal{W}(V(x_n, \alpha))$$

for each $\alpha < \lambda$. Let $\xi(n)$ be a cluster point of the net $\{\xi(n, \alpha) : \alpha < \lambda\}$, $n \in \mathbb{N}$, and let $\xi$ be a cluster point of the sequence $\{\xi(n)\}_{n \in \mathbb{N}}$. Then $(x, \xi) \in G_{\gamma}$ for some $\gamma \in \Gamma$. It is not hard to find an $n$, an $\alpha < \lambda$, and a nbd of $O_\xi$ of $\xi$ such that

$$V(x_n, \alpha) \times O_\xi \subset G_{\gamma}.$$ 

It follows that $\xi(n, \alpha) \in \bigcup \mathcal{W}(V(x_n, \alpha))$.

Now decompose $\mathcal{C}$ as $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n'$ so that $\mathcal{C}_n'$ is locally finite. Pick a $C \in \mathcal{C}$ and fix it. For each $\alpha \in \kappa$, there exists an $f(\alpha) < \alpha$ such that $C \times (f(\alpha), \alpha] \subset G_{\gamma}$ for some $\gamma \in \Gamma$; let $\gamma(\alpha, C)$ denote this $\gamma$. By the Pressing Down Lemma, there exists a $\beta \in \kappa$ and a stationary set $S \subset \kappa$ such that $f(\alpha) = \beta$ for all $\alpha \in S$. Therefore we have $C \times (\beta, \alpha] \subset G_{\gamma(\alpha, C)}$ for all $\alpha \in S$. Without loss of generality we can assume that either all $\gamma(\alpha, C)$, $\alpha \in S$, are the same or different. If all $\gamma(\alpha, C)$, $\alpha \in S$, are the same, we may put $\gamma(C) = \gamma(\alpha, C)$ for all $\alpha \in S$. We let $\beta_C$ denote the chosen $\beta$ and index the chosen stationary set $S$ as $S = \{\alpha(C, \mu) : \mu \in \kappa\}$. Here keeping $C \in \mathcal{C}$, decompose $\mathcal{C}_n'$ for each $n \in \mathbb{N}$ as

$$\mathcal{C}_n'(1) = \{C \in \mathcal{C}_n' : \text{ all } \gamma(\alpha(C, \mu), C), \mu < \kappa, \text{ are the same}\}$$
and $s_n'(2) = \mathcal{E}_n' \setminus \mathcal{E}_n'(1)$. We now put

$$H_{n\gamma} = \left( \bigcup \{ C \times (\beta_C, \kappa) : C \in \mathcal{E}_n'(1) \text{ with } \gamma(C) = \gamma \} \right)$$

$$\cup \left( \bigcup \{ C \times (\beta_C, \alpha(C, \mu)) : C \in \mathcal{E}_n'(2) \text{ and } \mu < \kappa \right.$$ with $\gamma(\alpha(C, \mu), C) = \gamma \}$$

for each $\gamma \in \Gamma$ and $n \geq 1$. Then $H_{n\gamma}$ is a closed set in $X \times \kappa$ with $H_{n\gamma} \subset G_\gamma$ for each $\gamma \in \Gamma$ and $n \geq 1$.

Moreover, for each $C \in \mathcal{E}$, since the subspace $C \times [0, \beta_C]$ is subparacompact, there exists a closed cover $\{ Z_{n,C,\gamma} : n \in \mathbb{N} \text{ and } \gamma \in \Gamma \}$ of it such that $Z_{n,C,\gamma} \subset G_\gamma$ for each $n \in \mathbb{N}$ and $\gamma \in \Gamma$. Let us set

$$H_{n,m,\gamma} = \bigcup \{ Z_{n,C,\gamma} : C \in \mathcal{E}_m'(1) \}$$

for each $n, m \in \mathbb{N}$ and $\gamma \in \Gamma$. It is easy to see that $H_{n,m,\gamma}$ is closed with $H_{n,m,\gamma} \subset G_\gamma$ for each $n, m \in \mathbb{N}$ and $\gamma \in \Gamma$. So we find a subshrinking

$$\{ H_{n,m,\gamma} : n, m \in \mathbb{N} \text{ and } \gamma \in \Gamma \} \cup \{ H_{n\gamma} : n \in \mathbb{N} \text{ and } \gamma \in \Gamma \}$$

for the open cover $\mathcal{G}$ which completes the proof.

Notice that for any subparacompact space $X$ Yajima [21] gives a sufficient condition for $X \times \kappa$ to be subshrinking.

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