PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 3, November 1994

L-SERIES AND MODULAR FORMS OF HALF-INTEGRAL WEIGHT

RHONDA L. HATCHER

(Communicated by Dennis A. Hejhal)

ABSTRACT. Let f be a normalized Hecke eigenform of weight $2k$, with k odd. The main result of this paper is an equation representing the value of $L(f, s)L(f \otimes \varepsilon, s)$ at $s = k$ in terms of the Fourier coefficients of a modular form of half-integral weight.

1. INTRODUCTION

Let N be a prime, $f \in S_{2k}^{\text{new}}(\Gamma_0)$. In this paper, we will derive an equation for the value of the product of L-series, $L(f, k)L(f \otimes \varepsilon, k)$ at $s = k$ in terms of the squares of the Fourier coefficients of a modular form of half-integral weight. The main identity is a geometric proof of a formula due to Waldspurger [13]. The method of the proof is a generalization of the approach used by Gross [3] in proving the result for $k = 1$.

2. THE DIRICHLET L-SERIES

Let K be an imaginary quadratic field of discriminant $-D$ where N is inert. Let $\mathcal{O}$ be the ring of integers in K and $A =$ ideal class in K. Set $\varepsilon =$ quadratic character of $(\mathbb{Z}/D\mathbb{Z})^*$ determined by $\varepsilon(p) = \left(\frac{-D}{p}\right)$ for $p \nmid D$ and $2u =$ number of units in K, and set

$$r_A(m) = \begin{cases} \text{number of ideals of norm } m \text{ in } A & \text{for } m \geq 1, \\ 1/(2u) & \text{for } m = 0. \end{cases}$$

We now define a Dirichlet series associated with f and the ideal class A. Let $\sum_{m \geq 1} a_m q^m$, with $q = e^{2\pi i z}$, be the Fourier expansion of f. Define the Dirichlet L-series associated with f and the ideal class A by

$$L(f, A, s) = \sum_{\substack{m \geq 1 \\ (m, N)=1}} \left(\frac{\varepsilon(m)}{m^{2s-2k+1}}\right) \sum_{m \geq 1} \left(\frac{a_m r_A(m)}{m^s}\right).$$

$L(f, A, s)$ has an analytic continuation to an entire function of s. For proof of this fact see [4].

Received by the editors February 14, 1993.
1991 *Mathematics Subject Classification*. Primary 11F66; Secondary 11G40.

Let χ be a complex character of the group $\mathrm{Pic}(\mathscr{O})$, and suppose $f = \sum_{m\geq 1} a_m q^m$ is a normalized eigenform of weight $2k$ for the Hecke algebra $\mathbb{T}$. Define

$$L(f, \chi, s) = \sum_A \chi(A) L(f, A, s),$$

where the sum is over all A in $\mathrm{Pic}(\mathscr{O})$.

It can be shown that if $\varepsilon(N) = 1$, then $L(f, \chi, s)$ vanishes at $s = k$. The values of $L(f, \chi, s)$ at $s = k$ in the case $\varepsilon(N) = -1$ were studied in [6]. In particular, an equation representing the value of $L(f, \chi, k)$ in terms of height pairings of special points on a vector bundle V is derived. In the next section we will describe the vector bundle V and the equation for $L(f, \chi, k)$.

3. The representation of $L(f, \chi, k)$ in terms of a height pairing

Let B be the quaternion algebra over $\mathbb{Q}$ ramified at the rational prime N and at ∞. Let R be a maximal order of B. Let $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$ be the profinite completion of $\mathbb{Z}$, and let $\widehat{\mathbb{Q}} = \widehat{\mathbb{Z}} \otimes \mathbb{Q} = \coprod_p \mathbb{Q}_p$ be the ring of finite adèles of $\mathbb{Q}$. Set $B_p = B \otimes \mathbb{Q}_p$. Then $R_p = R \otimes \mathbb{Z}_p$ is the local component of R in B_p. Let $\widehat{B} = B \otimes \widehat{\mathbb{Q}} = \coprod_p B_p$ and $\widehat{R} = R \otimes \widehat{\mathbb{Z}} = \prod_p R_p$.

Let $B_0 = \{b \in B : \mathrm{Tr}(b) = 0\}$, and let U be the representation of B^* on the elements $b_0 \in B_0$ with action by B^* defined by $b_0\gamma = (1/\mathbb{N}(\gamma))\gamma^{-1}b_0\gamma$. The center $\mathbb{Q}^*$ of B^* acts by $b_0 x = x^{-2} b_0$ for $x \in \mathbb{Q}^*$. An inner product on U is given by $[u_1, u_2] = \frac{1}{2}\mathrm{Tr}(u_1\overline{u_2})$. Let W_{2k-1} be the inner product space $\mathrm{Sym}^{2k-2}(W)$, where $W = \mathbb{C}x \oplus \mathbb{C}y$ is the two-dimensional representation of $\mathrm{SU}(2)$ with $[x, x] = [y, y] = 1$ and $[x, y] = 0$. Since $W_3 = \mathrm{Sym}^2(W) \cong \{b \in B : \mathrm{Tr}(b) = 0\}$ and $W_{2k-1} = \mathrm{Sym}^{2k-2}(W) \subset \mathrm{Sym}^{k-1}(\mathrm{Sym}^2(W))$, we can write the inner product space $\mathrm{Sym}^{k-1}(U)$ as the orthogonal direct sum $\mathrm{Sym}^{k-1}(U) = W_{2k-1} \oplus M$. Note that W_{2k-1} is the unique irreducible summand of highest weight $2k-2$, and it is a representation of B^* of dimension $2k-1$.

The quaternion algebra B over $\mathbb{Q}$ corresponds to an algebraic curve Y of genus zero over $\mathbb{Q}$ as described in [6, p. 541]. The vector bundle V is defined by

$$V = \widehat{R}^* \backslash \widehat{B}^* \times Y \times W_{2k-1}/B^*.$$

Let $\{I_1, I_2, \ldots, I_n\}$ be a set of left ideals of R which represent the distinct ideal classes. The set $\{I_1, I_2, \ldots, I_n\}$ corresponds to a choice of elements $g_1, g_2, \ldots, g_n$ in $\widehat{R}^* \backslash \widehat{B}^*$ such that $\widehat{B}^* = \bigcup_{i=1}^n \widehat{R}^* g_i B^*$. Let R_i denote the right order of I_i. For each $\Gamma_i = R_i/\langle \pm 1\rangle$, define $W^{\Gamma_i} = \{w \in W : \gamma(w) = w \text{ for all } \gamma \in \Gamma_i\}$. Define $\mathrm{Pic}(V) = \bigoplus_{i=1}^n W^{\Gamma_i}$. Any $v \in V$ is equivalent to the double coset (g_i, y, w) for some i. Define the class of the point v to be the projection of w in W^{Γ_i} given by $\mathrm{class}(v) = \frac{1}{|\Gamma_i|}\sum_{\gamma\in\Gamma_i} \gamma(w)$.

Define a mapping $\langle\ ,\ \rangle$ from $V \times V$ into $\mathbb{Q}$ as follows. If $v_1 = g \times \widetilde{y_1} \times \widetilde{w_1} \cong g_i \times y_1 \times w_1$ and $v_2 = h \times \widetilde{y_2} \times \widetilde{w_2} \cong g_j \times y_2 \times w_2$, then

$$\langle v_1, v_2\rangle = \begin{cases} 0 & \text{if } i \neq j, \\ \sum_{\gamma\in\Gamma_i} [w_1, \gamma(w_2)]_{W_{2k-1}} & \text{if } i = j. \end{cases}$$

This pairing induces a height pairing $\langle\ ,\ \rangle$ on $\mathrm{Pic}(V)$.

We now define a special element w_0 of $\mathrm{Sym}^{2k-2}(W)$. This w_0 will be used in the definition of a special point of V of discriminant $-D$. For the remainder of the paper, we fix K an imaginary quadratic field of discriminant $-D$, where the prime N is inert, and an embedding $f : K \to B$. Let $v_0 = \sqrt{-D} \in U$. Then v_0^{k-1} lies in $\mathrm{Sym}^{k-1}(U)$. Let $w_0 \in \mathrm{Sym}^{2k-2}(W)$ be the component of v_0^{k-1} in $\mathrm{Sym}^{2k-2}(W)$.

Let $X = \widehat{R}^*\backslash\widehat{B}^* \times Y/B^*$. There exists a canonical identification $Y(K) \simeq \mathrm{Hom}(K, B)$ as described in [6, p. 541]. The special points of V of discriminant $-D$ are defined to be points of the form $v = g \times y \times w_0$, where $g \times y$ lies in the image of $\widehat{R}^*\backslash\widehat{B}^* \times Y(K)$ in $X(K)$, and the embedding f corresponding to y satisfies $f(K) \cap g^{-1}\widehat{R}g = f(\mathscr{O}_{-D})$, where $\mathscr{O}_{-D}$ is the order of discriminant $-D$. An action of $\mathrm{Pic}(\mathscr{O})$ on the special points of V of discriminant $-D$ given by $v \to v_A$ for $A \in \mathrm{Pic}(\mathscr{O})$ is defined in [6, p. 545].

Define $e_\chi = \sum_A \chi^{-1}(A)v_A$, where the sum is over all A in $\mathrm{Pic}(\mathscr{O})$. Let $e_{f,\chi}$ be the projection of the divisor e_χ in $\mathrm{Pic}(V)\otimes\mathbb{C}$ to the f-isotypical eigenspace for the Hecke algebra $\mathbb{T}$. The following result is proved in [6, Proposition 8.2, p. 558] for the case D prime, and it follows from [7, Proposition 4.1, p. 340] that the result can be extended to the case D composite.

Proposition 1. *If* $\varepsilon(N) = -1$, *then*

$$L(f, \chi, k) = \frac{(f, f)}{u^2 D^{k-\frac{1}{2}}(k-1)!^2}\langle e_{f,\chi}, e_{f,\chi}\rangle,$$

where $(\ ,\ )$ *is the Petersson product as defined in* [6, p. 547].

In the special case $\chi = 1$, $L(f, \chi, k)$ can be written in the form

$$L(f, \chi, k) = L(f, k)L(f \otimes \varepsilon, k),$$

where $f \otimes \varepsilon = \sum_{m\geq 1} a_m\varepsilon(m)q^m$ is the twist of f, $L(f, s) = \sum_{m\geq 1} a_m m^{-s}$, and $L(f \otimes \varepsilon, s) = \sum_{m\geq 1} a_m\varepsilon(m)m^{-s}$.

Define e_D to be the class in $\mathrm{Pic}(V)$ of the rational divisor

$$\frac{1}{2u}\sum_{\mathrm{disc}(v)=-D}(v).$$

Then, as explained in [6, p. 560], in the special case $\chi = 1$ Proposition 1 becomes

Corollary 1. *If* $\varepsilon(N) = -1$, *then*

$$L(f, k)L(f \otimes \varepsilon, k) = \frac{(f, f)}{D^{k-\frac{1}{2}}(k-1)!^2}\langle e_{f,D}, e_{f,D}\rangle.$$

In the next section we will use the result of Corollary 1 to arrive at an equation representing the value of $L(f, k)L(f\otimes\varepsilon, k)$ in terms of the Fourier coefficients of a modular form of half-integral weight.

4. The main identity

Define the formal series $g = \sum_{D>1} e_D q^D$, with the e_D as defined above. We will need the following result.

Lemma 1. *For all* $e \in \mathrm{Pic}(V)$ *the series*

$$g(e) = \sum_{D>1} \langle e, e_D \rangle q^D$$

is a modular form of weight $k + \frac{1}{2}$ *on* $\Gamma_0(4N)$ *with integral coefficients.*

Proof. Let $\{\alpha_j\}$ be the basis for $\mathrm{Pic}(V)$ that you get by tensoring the basis $\{e_1, e_2, \dots, e_n\}$ of $\mathrm{Pic}(X)$ described in [3] with the basis of $\mathrm{Sym}^{2k-2}(W)$ given by $\{x^{2k-2}, x^{2k-3}y, \dots, y^{2k-2}\}$. Recall that $\mathrm{Sym}^{2k-2}(W)$ embeds into $\mathrm{Sym}^{k-1}(U)$ as described earlier. It will suffice to show that for every α_j, $g(\alpha_j)$ is a modular form of weight $k + \frac{1}{2}$ on $\Gamma_0(4N)$.

Suppose $\alpha_j = e_i \otimes x^l y^m$. Then

$$g(\alpha_j) = g(e_i \otimes x^l y^m) = \sum_{D>1} \langle e_i \otimes x^l y^m, e_D \rangle.$$

Using the fact that

$$e_D = \sum_{i=1}^{n} a_i(D)(e_i^\vee \otimes w_0),$$

where w_0 is the component of $(\sqrt{-D})^{k-1}$ in $\mathrm{Sym}^{2k-2}(W)$, $e_i^\vee = e_i/|\Gamma_i|$, and $a_i(D)$ is the number of embeddings of $\mathscr{O}_D \to R_i$ modulo R_i^*, it follows that

$$g(\alpha_j) = \sum_{D>1} \left\langle e_i \otimes x^l y^m, \sum_{i=1}^{n} a_i(D)(e_i^\vee \otimes w_0) \right\rangle q^D = \sum_{D>1} a_i(D)[x^l y^m, w_0] q^D.$$

Applying a result of Pizer [11, Proposition 2.10], we see that the inner product $[x^l y^m, w_0]$ is equal to $p(\sqrt{-D})$, where p is a spherical polynomial of degree $k-1$ in three variables x_1, x_2, x_3, with $\sqrt{-D} = x_1\mu_1 + x_2\mu_2 + x_3\mu_3$, where $\{\mu_1, \mu_2, \mu_3\}$ is a basis of U. Following the work of Gross in [3], we let S_i be the suborder of index 8 in R_i which is defined by $S_i = \mathbb{Z} + 2R_i$, and let S_i^0 be the subgroup of elements of trace zero in S_i. Then S_i^0 has rank three over $\mathbb{Z}$. From [3], we know that $a_i(D)$ is one half of the number of elements $b \in R_i$ with $b \equiv 0, 1 \bmod 2R_i$, $\mathrm{Tr}\, b = 0$, and $\mathbb{N}b = D = -b^2$. It follows that

$$g(\alpha_j) = \sum_{D>1} a_i(D) p(\sqrt{-D}) q^D = \frac{1}{2} \sum_{b \in S_i^o} p(b) q^{\mathbb{N}b}.$$

Since $\mathbb{N}b$ is a positive definite ternary quadratic form and $p(b)$ is a spherical polynomial of degree $k-1$ in three variables, it follows from a result of Schoeneberg [12] generalized by Pfetzer [10, p. 452] that $g(\alpha_i)$ is a modular form of weight $k + \frac{1}{2}$ on $\Gamma_0(4N)$. The lemma now follows.

Define e_f to be a nonzero element in the f-isotypical component of $\mathrm{Pic}(V) \otimes \mathbb{R}$, where f is an eigenform for the Hecke algebra $\mathbb{T}$. Then from Lemma 1, we know that

$$g(e_f) = \sum_D \langle e_f, e_D \rangle q^D \equiv \sum_D m_D q^D$$

is a modular form of weight $k + \frac{1}{2}$ on $\Gamma_0(4N)$ with integral coefficients. Furthermore, $g(e_f)$ is an eigenform for the Hecke algebra on the space of modular

forms of weight $k+\frac{1}{2}$ on $\Gamma_0(4N)$ with eigenvalues corresponding to the eigenvalues of f. This fact follows from an argument analogous to that given by Gross in [3] for the case $k=1$.

The main result, which we prove next, gives a representation of the value of $L(f,k)L(f\otimes\varepsilon,k)$ in terms of the Fourier coefficients m_D. The formula in Proposition 2 is due to Waldspurger [13].

Proposition 2. *If* $\varepsilon(N)=-1$, *then*

$$L(f,k)L(f\otimes\varepsilon,k)=\frac{(f,f)}{D^{k-\frac{1}{2}}(k-1)!^2}\frac{m_D^2}{\langle e_f,e_f\rangle}.$$

Proof. From Proposition 1, we know that

$$L(f,k)L(f\otimes\varepsilon,k)=\frac{(f,f)}{D^{k-\frac{1}{2}}(k-1)!^2}\langle e_{f,D},e_{f,D}\rangle.$$

Hence, it will suffice to show that

$$\langle e_{f,D},e_{f,D}\rangle=\frac{m_D^2}{\langle e_f,e_f\rangle}.$$

To see that this is true, first note that $m_D=\langle e_f,e_D\rangle=\langle e_f,e_{f,D}\rangle$. Therefore,

$$e_{f,D}=\frac{m_D}{\langle e_f,e_f\rangle}e_f$$

in $(\mathrm{Pic}(V)\otimes\mathbb{R})^f$. It follows that

$$\begin{aligned}\langle e_{f,D},e_{f,D}\rangle&=\left\langle\frac{m_D}{\langle e_f,e_f\rangle}e_f,\frac{m_D}{\langle e_f,e_f\rangle}e_f\right\rangle\\&=\frac{m_D^2}{\langle e_f,e_f\rangle^2}\langle e_f,e_f\rangle=\frac{m_D^2}{\langle e_f,e_f\rangle}.\end{aligned}$$

This completes the proof.

References

1. S. Böcherer and R. Schulze-Pillot, *On a theorem of Waldspurger and on Eisenstein series of Klingen type*, Math. Ann. **288** (1990), 361–388.
2. ——, *The Dirichlet series of Koecher and Maas and modular forms of weight* $3/2$, preprint.
3. B. Gross, *Heights and the special values of L-series*, CMS Conf. Proc., vol. 7, Amer. Math. Soc., Providence, RI, 1987, pp. 115–187.
4. B. Gross and D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), 225–320.
5. B. Gross, W. Kohnen, and D. Zagier, *Heegner points and derivatives of L-series.* II, Math. Ann. **278** (1987), 497–562.
6. R. Hatcher, *Heights and L-series*, Canad. J. Math **42** (1990), 533–560.
7. ——, *Special values of L-series*, Proc. Amer. Math. Soc. **114** (1992), 337–343.
8. W. Kohnen, *Newforms of half-integral weight*, J. Reine Angew. Math. **333** (1982), 32–72.
9. ——, *Fourier coefficients of modular forms of half-integral weight*, Math. Ann. **271** (1985), 237–268.
10. V. Pfetzer, *Die Wirkung der Modulsubstitutionen auf mehrfache Thetareihen zu quadratischen Formen ungerader Variablenzahl*, Arch. Math. **4** (1953), 448–454.

11. A. Pizer, *An algorithm for computing modular forms on* $\Gamma_0(N)$, J. Algebra **64** (1980), 340–390.

12. B. Schoeneberg, *Das Verhalten von mehrfachen Thetareihen bei Modulsubstitutionen*, Math. Ann. **116** (1939), 511–523.

13. J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*, J. Math. Pures Appl. (9) **60** (1981), 375–484.

DEPARTMENT OF MATHEMATICS, TEXAS CHRISTIAN UNIVERSITY, FORT WORTH, TEXAS 76129
E-mail address: `hatcher@gamma.is.tcu.edu`