INDECOMPOSABLE MODULES 
OVER NAGATA VALUATION DOMAINS

D. ARNOLD AND M. DUGAS

(Communicated by Ronald M. Solomon)

Abstract. For a discrete valuation ring $R$, let $fr(R)$ be the supremum of the ranks of indecomposable finite rank torsion-free $R$-modules. Then $fr(R) = 1, 2, 3,$ or $\infty$. A complete list of indecomposables is given if $fr(R) \leq 3$, in which case $R$ is known to be a Nagata valuation domain.

Let $R$ be a discrete valuation ring with prime $p$ and quotient field $Q$, and let $R^*$ be the $p$-adic completion of $R$ with quotient field $Q^*$. Define $fr(R) = \sup \{\text{rank} X : X \text{ indecomposable torsion-free } R\text{-module of finite rank}\}$. In this paper, we show that $fr(R) = 1, 2, 3,$ or $\infty$. This resolves a conjecture by P. Vamos that $fr(R) = 1, 2,$ or $\infty$.

It is well known that $fr(R) = \infty$ in case $[Q^* : Q] = \infty$ and $fr(R) = 1$ if $[Q^* : Q] = 1$. Call $R$ a Nagata valuation domain if $2 \leq [Q^* : Q]$ is finite [Z]. In this case $\text{char } Q^* = q > 0$; $Q^* = Q(u)$ for some unit $u$ of $R^*$ with $u^n = \lambda$, a unit of $R$; and $[Q^* : Q]$ is a power of $q$ [V, R]. Examples of Nagata valuation domains are given in [N] and [V].

Zanardo [Z] shows that if $[Q^* : Q] = 2$, then $fr(R) = 2$. Moreover, in this case there are, up to isomorphism, only three indecomposables: $R$, $Q$, and $R^*$. His example showing that $fr(R) \geq 6$ for $[Q^* : Q] = 3$ is in error.

Henceforth, assume $[Q^* : Q] = n \geq 2$. Then $Q^*$ is a splitting field for each finite rank $R$-module $X$; i.e., $R^* \otimes X$ is the direct sum of a free $R^*$-module and a $Q^*$-module. Thus, quasi-homomorphism results of Lady [L1, L3] for modules over a discrete valuation ring with a fixed splitting field are applicable.

As summarized in [L1, Theorem 1] and proved in [L3, Theorem 5.1], for:

- $n = 2$, there are three strongly indecomposables: $R$, $Q$, and $R^*$;
- $n = 3$, there are five strongly indecomposables: $R$, $Q$, $R^*$, $C^-R$ (p-rank 1, rank 2), and $C^+ R^*$ (p-rank 2, rank 3);
- $n = 4$, there are strongly indecomposables of arbitrarily large finite rank, but all strongly indecomposable are potentially describable (tame representation type);

- $n \geq 5$, there are strongly indecomposables of arbitrarily large finite rank, but a description is generally regarded as hopeless (wild representation type).

Received by the editors February 25, 1993.

1991 Mathematics Subject Classification. Primary 13C05, 13E05.

Research supported in part by NSF grant DMS-9101000.

©1994 American Mathematical Society 0002-9939/94 $1.00 + .25 per page
Since strongly indecomposables are indecomposable, Lady’s theorem yields $fr(R) = \infty$ for $n \geq 4$. We give an alternate proof by easily constructed examples in §3. This is sufficient for our purposes and avoids the deep arguments used in [L3].

The only unresolved case is $n = 3$. In this case, we show that $fr(R) = 3$ and give a complete list of indecomposables up to isomorphism: $R$, $Q$, $R^*$, $C^-R$, and infinitely many of $p$-rank 2, rank 3 (all quasi-isomorphic to $C^+R^*$). The strongly indecomposable $R$-module $C^+R^*$ is the quasi-homomorphism dual of $R^*$ defined in [A1].

1. Preliminaries

The $p$-rank of an $R$-module $X$ is the $R/pR$-dimension of $X/pX$. Fundamental properties of $p$-rank are given in [A1].

Lemma 1.1 [A1, Proposition 1.3, Lemma 1.5]. Two finite rank $R$-modules $G$ and $H$ are quasi-isomorphic if and only if $p$-rank $G = p$-rank $H$, rank $G = \text{rank} \ H$, and there is a monomorphism $f : G \to H$. Moreover, quasi-isomorphism implies isomorphism for modules of $p$-rank 1.

2. Indecomposables for $[Q^* : Q] = 3$

As noted in the introduction, char $Q = 3$ and $Q^* = Q(u)$ for some unit $u$ of $R^*$ with $u^3 = \lambda$, a unit of $R$. This notation is maintained throughout the rest of this section.

Define $A[u]$ to be the pure $R$-submodule of $R^*$ generated by $\{1, u\}$. Then $A[u] = (Q \oplus Qu) \cap R^*$ is strongly indecomposable with $p$-rank 1 and rank 2 and, hence, is quasi-isomorphic to $C^-R$ by Lady’s theorem. The following lemma is proved in [Z, Proposition 5] using Kurosch matrix-invariant arguments from [A1]. However, it can also be proved directly from the definition of $A[u]$ (a proof is not included).

Lemma 2.1 [Z, Corollary 12, Theorem 8]. The module $A[u]$ is (strongly) indecomposable. Moreover, if $X$ is an indecomposable $R$-module of rank $< 2$, then $X$ is isomorphic to $R$, $Q$, or $A[u]$.

Next let $a, b \in R^* \setminus R$ and define $A[a, b]$ to be the pure $R$-submodule of $R^* \oplus R^*$ generated by $(1, 0), (0, 1),$ and $(a, b)$. In particular, $QA[a, b] = Q(1, 0) \oplus Q(0, 1) \oplus Q(a, b)$ and $A[a, b] = QA[a, b] \cap (R^* \oplus R^*)$. Up to isomorphism, this definition of $A[a, b]$ coincides with that of [Z]. Then $A[a, b]$ has $p$-rank 2 and rank 3. A routine argument shows that $A[a, b]$ is (strongly) indecomposable if and only if $\{1, a, b\}$ is a $Q$-independent set. In this case, $A[a, b]$ is quasi-isomorphic to $C^+R^*$ by Lady’s theorem. Moreover, $A[u, u^2]$ is the quasi-homomorphism dual of $R^*$, noting that $R^*$ has $p$-rank 1 and rank 3.

Lemma 2.2. Suppose that $(a, b)$ and $(c, d)$ are $R^*$-vectors.

(a) If $(c, d) = s(a, b)M + P$ for an invertible $2 \times 2$ $R$-matrix $M$, a $Q$-vector $P$, and $0 \neq s \in Q$, then $A[a, b] \approx A[c, d]$.


(c) If $r$ is a unit of $R$ and $j > i$, then $A[u + pru^2, pu^2] \approx A[u, pu^2]$. 
Proof. (a) Define an $R^*$-automorphism $\phi$ of $R^* \oplus R^*$ by $\phi(x) = xM^{-1}$. Then $\phi$ induces a homomorphism $A[c, d] \rightarrow A[a, b]$ since $(Q \oplus Q)M^{-1}$ is contained in $Q \oplus Q$ and $(c, d)M^{-1} = s(a, b) + PM^{-1}$. In fact, this is an isomorphism since $A[a, b]$ and $\phi(A[c, d])$ are both pure rank-3 submodules of $R^* \oplus R^*$.

(b) Let $A = A[u, p'u^2]$ and $B = A[p'u, u^2]$ with $i \geq 1$. There is an $R^*$-endomorphism $\phi$ of $R^* \oplus R^*$ defined by

$$\phi(1, 0) = (1, 1) = (1, 0) + (0, 1) \in A,$$

$$\phi(0, 1) = (-u^{-2}, -p'uj^{-1} + p^2i) = -\lambda^{-1}(u, p'u^2) + p^{2i}(0, 1) \in A,$$

recalling that $u^3 = \lambda$. Now $\phi$ is an automorphism as

$$\begin{pmatrix} 1 & 1 \\ -u^{-2} & -p'uj^{-1} + p^2i \end{pmatrix}$$

has determinant $d \equiv -u^{-2} \pmod{pR^*}$, a unit of $R^*$. Moreover, $\phi(B)$ is contained in $A$ since

$$\phi(p'u, u^2) = p^i(u(1, 1) + u^2(-u^{-2}, -p'uj^{-1} + p^2i) = (p^i - 1, p^2i u^2)$$

$$= p^i(u, p'u^2) - (1, 0) \in A.$$

It follows that $\phi: B \rightarrow A$ is an isomorphism.

(c) Let $A = A[u, p'u^2]$ and $B = A[u + p'r'u^2, p'u^2]$, and assume that either $i \geq 1$ or else $i = 0$ and $ru$ is not congruent to $1$ modulo $pR^*$.

Define an $R^*$-endomorphism $\phi$ of $R^* \oplus R^*$ by

$$\phi(1, 0) = (1 - p'ru, -p'ru^2 + p^2i + r^2)$$

$$= (1, 0) - p'ru^2 + p^2i + r^2 \in A,$$

$$\phi(0, 1) = (0, 1 - p^3r^3) \in A.$$

Then $\phi$ is an automorphism if $i \geq 1$, since the coefficient determinant $d = (1 - p'r)(1 - p^3r^3) \equiv 1 \pmod{pR^*}$. If $i = 0$, then $d = (1 - ru)(1 - \lambda^3)$. Since $\text{char} Q^* = 3$, $1 - \lambda^3 = 1 - (ru)^3 = (1 - ru)^3$, whence $d = (1 - ru)^4$.

Thus, $\phi$ is an automorphism, as $ru$ is not congruent to $1$ mod $pR^*$.

Now $\phi(B)$ is contained in $A$ since

$$\phi(u + p'r^2, p'u^2)$$

$$= (u + p'r^2)\phi(1, 0) + p'u^2\phi(0, 1)$$

$$= (u + p'r^2)(1 - p'ru, -p'ru^2 + p^2i + r^2) + p^2u^2(0, 1 - p^3r^3)$$

$$= (u - p^3r^3, p'u^2 - p'r^2\lambda)$$

$$= (u, p'u^2 - p^2r^2\lambda(1, 0) - p'r^2\lambda(0, 1) \in A,$$

recalling that $u^3 = \lambda$. As in the proof of (b), $B \approx \phi(B) = A$.

It remains to show that it is sufficient to assume that either $i \geq 1$ or else $i = 0$ and $u$ is not congruent to $1$ modulo $pR^*$. To see this, assume that $i = 0$ and $ru = 1 + ps$ for some $s = s_0 + s_1u + s_2u^2 \in R^*$. Then $u + ru^2 = (2 + ps_0)u + ps_1u^2 + ps_2u^3$. Since $ps_2u \in Q$, it follows from (a) that $B = A[u + ru^2, p'u^2] \approx A[(2 + ps_0)u + ps_1u^2, p'u^2]$. As $\text{char} Q = 3$, $2 + ps_0 = -1 + ps_0$ is a unit of $R^*$. Thus, $B \approx A[u + ptu^2, p'u^2]$ for $t = (2 + ps_0)^{-1}s_1$ by (a). If $i' = p$-height$(pt) \geq j$, an application of (a) shows that $B \approx A$. Otherwise, $j < i'$ and $i' \geq 1$, as desired.
Theorem 2.3. If $X$ is an indecomposable $R$-module of rank 3, then $X$ is isomorphic to $R^*$ or $A[u, p^j u^2]$ for some $j$.

Proof. Note that $p$-rank $X \neq 0$ or 3, as $X$ is reduced with no free summands (see [A1]). If $p$-rank $X = 1$, then $X$ embeds in its completion which is isomorphic to $R^*$. Since $R^*$ also has $p$-rank 1 and rank 3, $X \approx R^*$ by Lemma 1.1.

Now assume that $X$ is indecomposable with $p$-rank 2 and rank 3. Then $X \approx A[a, b]$ with $(a, b) = (u, u^2)M$ for some $2 \times 2$ $R$-matrix $M$ with $\det M \neq 0$ [Z]. We outline another proof that avoids matrix invariants. Let $Rx \oplus Ry$ be a basic submodule of $X$ and extend to a maximal free submodule $Rx \oplus Ry \oplus Rz$ of $X$. Then $X$ embeds as a pure submodule of $R^x \oplus R^y \approx (R^* \otimes X)/d(R^* \otimes X)$, where $d(R^* \otimes X)$ is the maximal divisible submodule. It follows that $X \approx A[a, b]$, where image $z = ax \oplus by$ for $a, b \in R^*$. Since $Q^* = Q(u) = Q \oplus Qu \oplus Qu^2$, we may write $(a, b) = (u, u^2)M + P$ for some $R$-matrix $M$ and $R$-vector $P$. Apply Lemma 2.2(a) to see that, up to isomorphism, $P$ may be chosen to be 0.

In view of Lemma 2.2(a), the isomorphism class of $A[a, b]$ is preserved by invertible $R$-column operations on $M$. In particular, it suffices to assume that $M$ is of the form

$$
\begin{pmatrix}
p^k & 0 \\
p^r & p^j
\end{pmatrix}
$$

with $i < j$ and $r$ either 0 or a unit of $R$. This follows from the observation that if an element in a row has least $p$-height, then the other entry in its row can be set to 0 using an invertible $R$-column operation. Moreover, column interchange and multiplication of a column by a unit are invertible $R$-operations.

We now have $X \approx A[a, b]$ with $(a, b) = (p^ku + p^r u^2, p^j u^2)$, $j > i$ and $r$ either 0 or a unit of $R$.

First, assume $k \leq i$. Then $X \approx A[u + p^{i-k}ru^2, p^{j-k}u^2]$ by Lemma 2.2(a). Moreover, $A[u + p^{i-k}ru^2, p^{j-k}u^2] \approx A[u, p^{j-k}u^2]$ via Lemma 2.2(c). Thus, $X \approx A[u, p^{j-k}u^2]$.

Now assume $k > i$. Factor out $p^i$ and apply Lemma 2.2(a) to assume, up to isomorphism, that $[a, b] = [p^{k-i}u + ru^2, p^{j-i}u^2]$. If $r = 0$, then $X \approx A[a, b] \approx A[u, p^i u^2]$ for some $t$, obtained by factoring out $p^{\min(k-i, j-i)}$ and applying Lemma 2.2(b) in the case $k - i > j - i$.

Finally assume that $r$ is a unit. Then $X \approx A[a, b] = A[ru^2 + p^{k-i}u, p^{j-i}u^2] \approx A[u^2 + p^{i'}r' u, p^{j'}u^2]$ for $i' = k - i$, $j' = j - i$, and $r' = r^{-1}$ (Lemma 2.2(a)). Since $(u^2)^2 = u\lambda$, substituting $v$ for $u^2$ in the latter term and relabeling exponents and units gives $X \approx A[v + p^{i'} rv^2, p^{j'} v]$ for a unit $r = r'/\lambda$ of $R$. Invertible $R$-column operations on

$$
\begin{pmatrix}
1 & p^j \\
p^{i'} & 0
\end{pmatrix}
$$

reduce to the case that $X \approx A(v + p^{i'} rv^2, p^{i'j'}v^2)$. However, $Q^* = Q(u) = Q(v)$ with $v^3 = \lambda^2$, a unit of $R$. Thus, Lemma 2.2, with $u$ replaced by $v$, is true. The argument of the first case then shows that $X \approx A[v, p^{i'} v^2]$ for some $t$. Hence, by Lemma 2.2, $X \approx A[u^2, p^{i'} \lambda u] \approx A[u^2, p^{i'} u] \approx A[p^{i'} u, u^2] \approx A[u, p^{i'} u^2]$, as desired.

For finite rank torsion-free $R$-modules $G$ and $H$, define $S_G(H)$ to be the
image of the evaluation map $\text{Hom}(G, H) \otimes_R G \to H$. Fundamental properties of $S_G(-)$ are given in [A2, Chapter 5] for torsion-free abelian groups of finite rank.

**Proposition 2.4.** (a) If $A[u, p'\cdot u^2] \approx A[u, p'\cdot u^2]$, then $i = j$.

(b) There are embeddings $A[u, p'\cdot u^2] \rightarrow A[u, p^{-i-1}u^2]$ and $A[u, p^{-i-1}u^2] \to A[u, p'\cdot u^2]$. In each case the image has index $p$.

(c) If $G$ and $H$ are indecomposable with $p$-rank 2 and rank 3, then $S_G(H) = H$.

**Proof.** (a) can be proven as in [Z, Proposition 16] for the case $i = 0, j = 1$. We outline an alternate proof that avoids matrix invariants. An $R$-isomorphism $\phi: A = A[u, p'\cdot u^2] \to B = A[u, p'\cdot u^2]$ lifts to an $R^*$-isomorphism of completions $\phi^*: A^* = R^* \oplus R^* \to B^* = R^* \oplus R^*$. Since $\phi(u, p'\cdot u^2) \in B$ and $\phi^{-1}(u, p'\cdot u^2) \in A$, it follows from a computation of $p$-heights that $i = j$.

(b) There is a monomorphism $f: A[u, p^{-i-1}u^2] \to A[u, p'\cdot u^2]$ induced by an $R^*$-endomorphism $\phi$ of $R^* \oplus R^*$ with $\phi(1, 0) = (1, 0)$ and $\phi(0, 1) = (0, p)$. Moreover, there is a monomorphism $f': A[u, p'\cdot u^2] \to A[u, p^{-i-1}u^2]$ induced by $\phi'(1, 0) = (p, 0)$ and $\phi'(0, 1) = (0, 1)$. Note that $ff' = p$ and $f'f = p$. Hence, if $H_i = A[u, p'\cdot u^2]$, then $pH_i$ is contained in image $f$. But $p$-rank $H_i = 2$ and $H_i$ is not isomorphic to $H_i$ by (b). It follows that $H_i/image f \approx R/pR$. Similarly, $H_{i-1}/image f' \approx R/pR$.

(c) For $i \geq 1$ and for $\phi'$ and $\phi$ defined as in the proof of (b), there is $g: A[p^{-i-1}u, u^2] \to A[p'\cdot u, u^2]$ induced by $\phi'$ and $g': A[p'\cdot u, u^2] \to A[p^{-i-1}u, u^2]$ induced by $\phi$ with $gg' = p$ and $g'g = p$. It now follows that if $G_i = A[p^i\cdot u, u^2]$, then $G_i/image g \approx R/pR \approx G_{i-1}/image g'$.

In view of Theorem 2.3, it is sufficient to show that

$$f \oplus \delta_i g \delta_{i-1}^{-1}: H_{i-1} \oplus H_i \rightarrow H_i \text{ and } f' \oplus \delta_i g' \delta_{i-1}^{-1}: H_i \oplus H_{i-1} \rightarrow H_{i-1}$$

are onto, for $\delta_i$ the isomorphism $G_i = A[p^i\cdot u, u^2] \to A[u, p^i\cdot u^2] = H_i$ given in Lemma 2.2(b). Assume that $f \oplus \delta_i g \delta_{i-1}^{-1}$ is not onto. Since $H_i/image pH_i \approx R/pR \oplus R/pR$ and $pH_i$ is properly contained in both the image of $f$ and the image of $\delta_i g \delta_{i-1}^{-1}$, it follows that image $f = image \delta_i g \delta_{i-1}^{-1}$. Hence, $f \delta_{i-1}(G_{i-1}) = \delta_i g(G_{i-1})$. But this is a contradiction, as can be seen by observing that $f$ is a restriction of $\phi$ and $g$ is a restriction of $\phi'$. The proof that $f' \oplus \delta_{i-1} g'$ is onto is analogous.

**Lemma 2.5.** Assume that $X$ is a finite rank $R$-module with submodule $K$ such that $A = X/K \approx A[u] = A[u, p^i\cdot u^2]$ for some $i \geq 0$. If $S_A(X) = X$, then $K$ is a summand of $X$.

**Proof.** It suffices to prove that $\text{End}(A[u])$ and $\text{End}(A[u, p^i\cdot u^2])$ are commutative. This is a consequence of [AR2, Theorems 5.6 and 5.8] as the abelian group proof therein carries over to modules over discrete valuation rings. Recall that $A[u]$ has $p$-rank 1 and is reduced. Hence its completion is isomorphic to $R^*$. In particular, $\text{End}(A[u])$ is isomorphic to a subring of $R^*$. Moreover, $A[u, p^i\cdot u^2]$ is quasi-isomorphic to $A[u, u^2]$ which is the dual of $R^*$, as noted above. Thus, $Q\text{End}(A[u, p^i\cdot u^2]) = Q\text{End}(A[u, u^2]) = Q\text{End}(R^*) = QR^*$. It follows that $\text{End}(A[u, p^i\cdot u^2])$ is commutative.

**Theorem 2.6.** If $X$ is a finite rank $R$-module, then $X$ is the direct sum of modules of rank $\leq 3$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Proof. Choose pure strongly indecomposable submodules $X_i$ of $X$ with $X/(X_1 \oplus \cdots \oplus X_m)$ $p^k$-bounded. Each $X_j$ is isomorphic to $R$, $R^*$, $Q$, $A[u]$, or $A[u, p^i u^2]$ for some $r \geq 0$ by Lady’s theorem, Lemma 2.1, and Theorem 2.3. If $X_i$ is isomorphic to the pure injective module $R^*$ or $Q$, then $X_i$ is a summand of $X$. Moreover, if $X_i \approx R$, then $X$ has a cyclic summand, since $X$ modulo the pure submodule generated by $\{X_j : j \neq i\}$ is isomorphic to $R$.

We may now assume that each $X_j$ is isomorphic to $A[u]$ of some $A[u, p^i u^2]$. By induction on rank $X$ and $|X/(X_1 \oplus \cdots \oplus X_m)|$, it suffices to further assume that $X/(X_1 \oplus \cdots \oplus X_m) \approx R/pR$ and prove that $X$ has a summand of rank $\leq 3$. Write $X = (X_1 \oplus \cdots \oplus X_m) + R(x_1 + \cdots + x_m)/p$. Let $K$ be the pure submodule of $X$ generated by $\{X_j : j \neq i\}$ and $A = X/K$, quasi-isomorphic to $X_1$. Then $A$ has $p$-rank 1, rank 2 or rank 3, and has no free summands, being quasi-isomorphic to a strongly indecomposable $X_1$. Hence, $A$ is indecomposable [A1, Proposition 4.1].

It is now sufficient to prove that $S_A(X) = X$, in which case $X$ has a summand isomorphic to $A$ of rank $\leq 3$ by Lemma 2.5. There is some $Y = X_i$, say $i = 1$, with $S_Y(X_j) = X_j$ for each $j$. This follows from the natural exact sequence $A[u, p^i u^2] \rightarrow A[u] \rightarrow 0$, Proposition 2.4(c), and the assumption that each $X_j \approx A[u]$ or $A[u, p^i u^2]$. Moreover, for $A = X/K \approx X_1 + R(x_1/p)$, $S_A(X_j) = X_j$ for each $j$, again by Proposition 2.4(c) or Lemma 2.1 and the fact that $A$ is indecomposable.

Write $X_i = pX_i + Rx_i$, an indecomposable module for the same reason that $A$ is. For each $i$, there is $y_i \in A$, a unit $r_i$ of $R$, and $f_i : A \rightarrow X_i'$ with $f_i(y_i) \equiv r_ix_i \mod pX_i$. This is because if $X_i' = S_A(X_i')$ is contained in $pX_i$, then $x_i \in pX_i$ and letting $r_i = 1$ will do. Note, for future reference, that we may as well assume that $f_i(y_i) \equiv x_i \mod pX_i$. To see this, choose a unit $s_i$ of $R$ with $r_i = s_i r_i + p t_i$, $t_i \in R$. Then $s_i f_i(y_i) \equiv x_i \mod pX_i$, as desired.

We begin with the case $m = 2$ and find $x \in A$ and $g_i : A \rightarrow X_i'$ with $g_i(x) \equiv x_1 \mod pX_1$ and $g_2(x) \equiv x_2 \mod pX_2$. If either $f_1(y_2) \equiv s_1 x_1 \mod pX_1$ or $f_2(y_1) \equiv s_2 x_2 \mod pX_2$ for units $s_1, s_2$ of $R$, then let $x = y_1$, respectively, $x = y_2$. Otherwise, $f_1(y_2) \in pX_1$ and $f_2(y_1) \in pX_2$. In this case, let $x = y_1 + y_2$. In any case, there are units $t_i$ of $R$ with $f_i(x) \equiv t_i x_i \mod pX_i$. As above, choose $g_i$ to be an appropriate $R$-unit multiple of $f_i$.

Next let $A' = pA + Rx$, an indecomposable submodule of $A$ for the same reason that $A$ is indecomposable. Restriction induces a well-defined $\phi = g_1 \oplus g_2 : A' \rightarrow pX = pX_1 \oplus pX_2 + R(x_1 \oplus x_2)$ with $\phi(x) \in (x_1 \oplus x_2) + pX_1 \oplus pX_2$. Since $S_A(A') = A'$ by Proposition 2.4(c) and $S_A(X_i) = X_i$ for each $i$, it follows that $S_A(pX) = pX$ and so $S_A(X) = X$. This completes the proof for $m = 2$.

We illustrate an induction argument with $m = 3$. From the $m = 2$ case $S_A(X_1 x_2') = X_1 x_2'$ for $X_1 x_2' = pX_1 \oplus pX_2 + R(x_1 \oplus x_2)$. Consequently, there is $x \in A$ and $g_12 : A \rightarrow X_1 x_2'$ with $g_12(x) \equiv (x_1 \oplus x_2) \mod pX_1 \oplus pX_2$. Otherwise, $x_1 \oplus x_2 \in S_A(X_1 x_2') = X_1 x_2'$ is contained in $pX_1 \oplus pX_2$. Recall that there is $y_3 = A$ and $f_3 : A \rightarrow X_1 x_2'$ with $f_3(y_3) \equiv x_3 \mod pX_3$. If $f_3(x) \equiv s_3 x_3 \mod pX_3$ for some unit $s_3$ of $R$, then let $a = x$. If $f_3(y_3) \equiv s_3 x_3 \mod pX_1 \oplus pX_2$ for some unit $s$ of $R$, then let $a = y_3$. Otherwise, let $a = x + y_3$. It follows that $a \in A$ with $g_12(a) \equiv r(x_1 \oplus x_2) \mod pX_1 \oplus pX_2$ and $f_3(a) \equiv r_3 x_3 \mod pX_3$ for units $r$ and $r_3$ of $R$. As in the $m = 2$ case, we may assume that $r = r_3 = 1$ and construct $\phi : A' = pA + Ra \rightarrow pX$ with $\phi(a) \equiv (x_1 \oplus x_2 \oplus x_3)$.  

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
(mod $pX_1 \oplus pX_2 \oplus pX_3$). It follows, as above, that $S_\lambda(X) = X$.

The proof is concluded by an induction on $m$; the argument for passing from $m$ to $m+1$ is analogous to that of the preceding paragraph.

3. INDECOMPOSABLES FOR $[Q^*:Q] = n \geq 4$

The following are examples showing that $fr(R) = \infty$ for $[Q^*:Q] = n \geq 4$. The detailed computations needed to verify that the modules are actually strongly indecomposable are omitted.

**Example 3.1.** Assume $n \geq 4$. Given $m \geq 2$, there is a strongly indecomposable $R$-module with $p$-rank $m$ and rank $2m$.

**Proof.** Case I: $\text{char } Q^* > 5$. Since $Q^*$ is purely inseparable over $Q$, there is $u \in R^*$ such that $1, u, u^2, u^3,$ and $u^4$ are $Q$-independent. Let $M$ be an $m \times m$ simple Jordan block $R$-matrix, i.e., the diagonal elements of $M$ are a fixed unit $\lambda$ of $R$, the super diagonal elements are all $1$'s, and the remaining entries are $0$. Define $X = A[\Gamma] = (Q^m) \cap (Q^m \oplus Q^m \Gamma)$, where $\Gamma = uM + u^2I_m$, and $R$-module with $p$-rank $m$ and rank $2m$.

It can be shown that $\text{End}(X)$ is represented by the set of $2m \times 2m$ $R$-matrices $(\begin{pmatrix} 1 & 0 \\ 0 & \Pi \end{pmatrix})$ with $\Pi M = M\Pi$. This can be seen by equating $Q$-coefficients $1, u, u^2, u^3,$ and $u^4$. Consequently, $Q \text{End}(X) = Q[\Sigma] = Q[x]/(x-\lambda)^m$ is a ring with no nontrivial idempotents, whence $X$ is strongly indecomposable.

Case II: $\text{char } Q^* = 3$. If there is $u \in R^*$ with $1, u, u^2, u^3,$ and $u^4$ $Q$-independent, the construction of Case I suffices. Otherwise, there are $u, v \in R^*$ with $u^3, v^3 \in R$ and $1, u, v, u^2v, v^2u, u^2v^2, u^2, v^2$ are $Q$-independent. Choose $M$ as in Case I, and define $X = A[\Gamma]$ for $\Gamma = uM + vI$. An argument similar to that of Case I shows that $Q \text{End}(X) = Q[X]/(x-\lambda)^m$ and $X$ is strongly indecomposable.

Case III: $\text{char } Q^* = 2$. We are left with two possibilities not covered in Case I: there is $u \in R^*$ with $1, u, u^2,$ and $u^3$ $Q$-independent and $u^4 \in R$, or there is $u, v \in R^*$ with $u^2, v^2 \in R$ and $\{1, u, v, uv\}$ $Q$-independent. In the first case, define $X = A[\Gamma]$ for $\Gamma = uM + u^3I$. An argument similar to that of Case I shows that $X$ is strongly indecomposable.

For the second case, define $X = A[\Gamma]$ for $\Gamma = uM + vI$. Once again, it can be shown that $Q \text{End}(X)$ has no nontrivial idempotents, but the argument is slightly more complicated than the previous cases.

**ACKNOWLEDGMENT**

We thank the referee for helpful suggestions that improved the readability of the paper. Undefined notation and terminology is as in [A1], [A2], or [F].

**REFERENCES**


DEPARTMENT OF MATHEMATICS, BAYLOR UNIVERSITY, WACO, TEXAS 76798-7328
E-mail address: arnoldd@baylor.edu
E-mail address: dugasm@baylor.edu