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A COUNTER-EXAMPLE IN JACOBSON RADICALS

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ABSTRACT. A plausible conjecture states that an element of the Jacobson radical of the endomorphism ring of an abelian p-group increases the height of any non-zero element of the socle. I construct a counter-example.

1. INTRODUCTION

Throughout this paper I use the standard notation of abelian group theory, as found for example in the books of Fuchs [F1, F2].

In 1963, R.S. Pierce [P] defined a special ideal in the endomorphism ring $\mathscr{E}(G)$ of an abelian p-group G as follows:

$$\mathbf{H} = \{\eta \in \mathscr{E}(G) : h(g\eta) > h(g) \text{ for all } g \in G[p] \text{ of finite height}\}.$$

He showed that $\mathbf{H}$ is an upper bound for the Jacobson radical $\mathbf{J}(G)$ of the endomorphism ring $\mathscr{E}(G)$ of G and is equal to it if G is separable and has no direct summand which is an unbounded direct sum of cyclic groups. This was the beginning of a series of papers which described $\mathbf{J}(G)$ in terms of its action on $G[p]$ for various classes of abelian groups, including torsion-complete groups [P], direct sums of cyclic groups [L], valued vector spaces [FS], totally projective groups [H], and sufficiently projective groups [HJ]. The problem of finding a ring-theoretic characterisation of $\mathbf{H}$ was raised in discussion at the Oberwolfach conference on abelian groups in 1993, but answered already by Liebert in 1974 [L]; namely, $\mathbf{H}$ is the unique maximal idempotent-free ideal of $\mathscr{E}(G)$.

In many of the papers mentioned above, the authors strengthened the definition of Pierce's radical $\mathbf{H}$ to: if G is a reduced abelian p-group of length λ, then

$$\mathbf{H} = \{\eta \in \mathscr{E}(G) : h(g\eta) > h(g) \text{ for all } 0 \neq g \in G[p]\}.$$

For the sake of clarity, let us denote this version of Pierce's radical by $\mathbf{H}_\lambda$ and the original one by $\mathbf{H}_\omega$. Each of the papers cited above contains a proof that $\mathbf{J}(G) \subseteq \mathbf{H}_\lambda$, so it is plausible to conjecture that this is true without any restriction on G. In this note, I show that the conjecture is false in general and present a sufficient condition for it to be true.

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2. The counter–example

Let M be the Prüfer group $\langle a, a_1, a_2, \ldots \rangle$ where $pa = 0$ and for $i \in \mathbf{Z}^+$, $p^i a_i = a$ [F1, p. 150]. M has length $\omega+1$, $p^\omega M = \langle a \rangle$, and $M/p^\omega M \cong B$, the standard basic group.

Let $\overline{B}$ be the torsion completion of B, and let $\overline{M}$ be an extension of $\langle a \rangle$ by $\overline{B}$ such that the following diagram is exact and commutative:

$$\begin{tikzcd} 0 \arrow[r] & \langle a \rangle \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \langle a \rangle \arrow[r] & \overline{M} \arrow[r] & \overline{B} \arrow[r] & 0 \end{tikzcd}$$

where the last vertical arrow is the inclusion of B in $\overline{B}$, and hence f is monic. To clarify the following calculations, let b denote the copy of a in $\overline{M}$ and for all $i \in \mathbf{Z}^+$ let $b_i = a_i f$. Thus $\overline{M}$ has length $\omega + 1$ and $p^\omega \overline{M} = \langle b \rangle$. Let $G = M \oplus \overline{M}$.

In [M, Theorems 2.4 and 3.4], Megibben showed that if a p–group G satisfies $G = C \oplus D$ where $p^\omega C \cong p^\omega D$ is cyclic of order p, $C/p^\omega C$ is a direct sum of cyclic groups, and $D/p^\omega D$ is torsion complete, then $p^\omega D$ is fully invariant in G.

I shall use this property to show that $\mathscr{E}(G)$ has a quasi-regular ideal containing an element which does not increase the height of every non–zero element of $G[p]$. Let

$$N = H_\omega \cap \{\nu \in \mathscr{E}(G) : p^k G[p]\nu = 0 \text{ for some } k \in \mathbf{Z}^+\}$$

and

$$K = H_\omega \cap \{\kappa \in \mathscr{E}(G) : M\kappa \leq \overline{M} \text{ and } \overline{M}\kappa = 0\}.$$

Lemma. *$N + K$ is a quasi–regular ideal of $\mathscr{E}(G)$.*

Proof. Since N is an ideal and $N+K$ is additively closed, to show that $N+K$ is an ideal it suffices to show that $\mathscr{E}(G) \cdot K \subseteq N + K$ and $K \cdot \mathscr{E}(G) \subseteq N + K$.

Let $\alpha \in \mathscr{E}(G)$, $\kappa \in K$; clearly $\alpha\kappa$ and $\kappa\alpha \in H_\omega$. Let π and ρ be the canonical projections of G onto M and $\overline{M}$ respectively, considered as elements of $\mathscr{E}(G)$.

We may write α as $\pi\alpha\pi + \rho\alpha\pi + \alpha\rho$. Since $\pi\alpha\pi\kappa + \alpha\rho\kappa$ maps M into $\overline{M}$ and $\overline{M}$ to zero, $\pi\alpha\pi\kappa + \alpha\rho\kappa \in K$. On the other hand, $\rho\alpha\pi$ maps b to zero, by Megibben's result above, so it induces a homomorphism from the torsion-complete group $\overline{B}$ to M. By [R, Theorem 1], every homomorphism from a torsion-complete group to a countable group is small, which means that the kernel of $\rho\alpha\pi$ contains $p^k\overline{M}[p]$ for some $k \in \mathbf{Z}^+$. Hence there exists $k \in \mathbf{Z}^+$ such that $p^k G[p]\rho\alpha\pi = 0$, so $\rho\alpha\pi\kappa \in N$. Hence $\alpha\kappa \in N + K$, so $\mathscr{E}(G) \cdot K \subseteq N + K$.

Similarly, $\kappa\alpha = \kappa\alpha\pi + \kappa\alpha\rho$. By an argument similar to that in the previous paragraph, $\kappa\alpha\pi \in N$, while $\kappa\alpha\rho \in K$. Hence $\kappa\alpha \in N + K$, so $K \cdot \mathscr{E}(G) \subseteq N + K$.

To see that $N + K$ is quasi–regular, let $\nu \in N$ and $\kappa \in K$. Let k be such that $p^k G[p]\nu = 0$. Then $(\nu + \kappa)^k$ is a sum of products of k factors, each a multiple of ν or κ. Since ν, $\kappa \in H_\omega$, each such factor increases the height of socle elements of finite height, so $G[p](\nu+\kappa)^k \subseteq p^k G[p]$. Since $\kappa^2 = 0$ and $p^k G[p]\nu = 0$, we have $G[p](\nu+\kappa)^{k+2} = 0$. Hence $[1-(\nu+\kappa)][\sum_{i=0}^{k+2}(\nu+\kappa)^i]$ fixes $G[p]$, so by [P, Lemma 13.1], $1-(\nu+\kappa)$ is an automorphism of G. Hence $N+K$ is a quasi-regular ideal of $\mathscr{E}(G)$.

It remains only to identify an element η of $N+K$ and a non-zero element g of $G[p]$ such that $h(g\eta) = h(g)$. Define η by $\overline{M}\eta = 0$, $a\eta = b$, and for all $i \in \mathbf{Z}^+$, $a_i\eta = pb_{i+1}$. Then η is well defined, in the sense that the relations on M and $\overline{M}$ are preserved, $\eta \in K$, and $h(a\eta) = \omega + 1 = h(a)$.

3. A weak sufficient condition for $\mathbf{J} \subseteq \mathbf{H}_\lambda$

The reason that the counter-example described above is valid is clearly that the subgroup $p^\omega G[p]$ contains a proper fully invariant subgroup. The object of this section is to describe a condition on a group G of arbitrary length which prevents this possibility by ensuring that the only fully invariant subgroups of $G[p]$ are the functorial ones $p^\nu G[p]$.

Call an abelian p-group G **socle transitive** if whenever a, $b \in G[p]$ with $h(a) = h(b)$, there exists an automorphism of G mapping a to b; call G **socle fully transitive** if whenever a, $b \in G[p]$ with $h(a) \leq h(b)$, there exists an endomorphism of G mapping a to b. Corner [C] constructed examples which show that these two properties are independent. They are apparently weaker than transitivity and full transitivity respectively, but it is still unknown whether they are in fact equivalent to them. Certainly all sufficiently projective groups and all separable groups are socle transitive and socle fully transitive. I now prove a sufficient condition for $\mathbf{J}(G) \subseteq \mathbf{H}_\lambda$.

Proposition. *Let G be a reduced abelian p-group of length λ. If G is socle transitive or socle fully transitive, then $\mathbf{J}(G) \subseteq H_\lambda$.*

Proof. Suppose there exists $\eta \in \mathbf{J}(G)$ and $0 \neq a \in G[p]$ such that $h(a\eta) = h(a)$. Then there exists $\mu \in \mathscr{E}(G)$ such that $a\eta\mu = a$. But $\eta\mu \in \mathbf{J}(G)$, so $1-\eta\mu$ is monic, a contradiction.

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