

THE HANKEL TRANSFORMATION ON M'_μ AND ITS REPRESENTATION

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ABSTRACT. The Hankel transformation was extended by Zemanian to certain generalized functions of slow growth through a generalization of Parseval's equation as

$$(1) \quad \langle h_\mu f, \varphi \rangle = \langle f, h_\mu \varphi \rangle$$

where $\varphi, h_\mu \varphi \in H_\mu, f \in H'_\mu$.

Later, Koh and Zemanian defined the generalized complex Hankel transformation on $J_\mu = \bigcup_{\nu=1}^{\infty} J_{a_\nu, \mu}$, where $J_{a_\nu, \mu}$ is the testing function space which contains the kernel function, $\sqrt{xy}J_\mu(xy)$. A transformation was defined directly as the application of a generalized function to the kernel function, i.e., for $f \in J'_\mu$,

$$(2) \quad (h_\mu f)(y) = \langle f(x), \sqrt{xy}J_\mu(xy) \rangle.$$

In this paper, we extend definition (2) to a larger space of generalized functions. We first introduce the test function space $M_{a, \mu}$ which contains the kernel function and show that $H_\mu \subset M_{a, \mu} \subset J_{a, \mu}$. We then form the countable union space $M_\mu = \bigcup_{\nu=1}^{\infty} M_{a_\nu, \mu}$ whose dual M'_μ has J'_μ as a subspace. Our main result is an inversion theorem stated as follows.

Let $F(y) = (h_\mu f)(y) = \langle f(x), \sqrt{xy}J_\mu(xy) \rangle, f \in M'_\mu$, where y is restricted to the positive real axis. Let $\mu \geq -\frac{1}{2}$. Then, in the sense of convergence in H'_μ ,

$$f(x) = \lim_{r \rightarrow \infty} \int_0^r F(y) \sqrt{xy}J_\mu(xy) dy.$$

This convergence gives a stronger result than the one obtained by Koh and Zemanian (1968).

Secondly, we prove that every generalized function belonging to $M'_{a, \mu}$ can be represented by a finite sum of derivatives of measurable functions. This proof is analogous to the method employed in structure theorems for Schwartz distributions (Edwards, 1965), and similar to one by Koh (1970).

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1. INTRODUCTION

The conventional Hankel transformation is defined by

$$F(y) = h_\mu f = \int_0^\infty f(x) \sqrt{xy} J_\mu(xy) dx,$$

where $0 < y < \infty$, μ is a real number, and J_μ is the Bessel function of the first kind and order μ . In 1966, Zemanian (see [4]) constructed a testing function space H_μ in order to extend this transformation to certain generalized functions.

For each real number μ , a function $\varphi(x)$ is in H_μ if and only if it is defined on $0 < x < \infty$, it is complex-valued and smooth, and for each pair of nonnegative integers m and k ,

$$\gamma_{m,k}^\mu(\varphi) \triangleq \sup_{0 < x < \infty} |x^m (x^{-1} D)^k [x^{-\mu-1/2} \varphi(x)]|$$

exists (i.e., is finite). H_μ is a linear space. Also, each $\gamma_{m,k}^\mu$ is a seminorm on H_μ . The topology of H_μ is that generated by $\{\gamma_{m,k}^\mu\}_{m,k=0}^\infty$.

The Hankel transformation h_μ is an automorphism on H_μ whenever $\mu \geq -\frac{1}{2}$. The generalized functions in the dual H'_μ of H_μ act like distributions of slow growth as $x \rightarrow \infty$. Moreover, H'_μ is the domain of the generalized Hankel transformation h_μ , which is defined via (1). It follows that h_μ is an automorphism on H'_μ . This procedure is reminiscent of Schwartz's method of extending the Fourier transformation to distributions of slow growth.

In 1968, Koh and Zemanian [1] developed an alternative approach to the generalized Hankel transformation. For a real number μ and a positive real number a , they constructed a testing function space $J_{a,\mu}$ as follows.

Let $J_{a,\mu}$ be a testing function space containing all $\varphi(x)$ which are defined and smooth on $I = (0, \infty)$ and for which

$$\begin{aligned} \tau_k^{\mu,a}(\varphi) &= \sup_{x \in I} |e^{-ax} x^{-\mu-1/2} (x^{-\mu-1/2} D x^{2x+1} D x^{-\mu-1/2})^k(\varphi)| \\ &< \infty, \quad k = 0, 1, 2, \dots. \end{aligned}$$

They assign to $J_{a,\mu}$ the topology generated by the countable multinorm $\{\tau_k^{\mu,a}\}_{k=0}^\infty$.

$J_{a,\mu}$ contains the kernel $\sqrt{xy} J_\mu(xy)$ as a function on $0 < x < \infty$ for each fixed complex y in the strip $\Omega = \{y : |\operatorname{Im} y| < a, y \neq 0 \text{ or a negative number}\}$. The Hankel transformation h_μ is now defined on the dual space $J'_{a,\mu}$ via (2).

It is proved in [1] that any generalized function that has a Hankel transform according to (2) will also have a Hankel transform according to (1), and the two transforms will agree.

The definition (2), while not as general as that of (1), is a more natural extension of the classical transformation because the kernel appears explicitly as a testing function. This leads to simpler manipulation for computational purposes. This could not be done in (1) because the kernel is not a member of H_μ , whatever be the value of y .

In this paper, we define a new testing function space $M_{a,\mu}$ between H_μ and $J_{a,\mu}$, namely $H_\mu \subset M_{a,\mu} \subset J_{a,\mu}$, whereby $M_{a,\mu}$ still contains the kernel function. Since $J'_{a,\mu} \subset M'_{a,\mu}$, definition (2) is thus extended to a larger class of

generalized functions. We obtain many properties of $M_{a,\mu}$ and the countable union $M_\mu = \bigcup_{i=1}^{\infty} M_{a_i,\mu}$. An inversion theorem and a representation for $M'_{a,\mu}$ are our main results.

By a smooth function, we mean a function that possesses continuous ordinary derivatives of all orders at all points of its domain. The notation $\langle f, \varphi \rangle$ denotes the number assigned to a member φ of some testing function space by a member f of the dual space.

$D(I)$ denotes the space of smooth functions that have compact support on I . We equip $D(I)$ with the usual topology. Its dual $D'(I)$ is the space of Schwartz distributions on I .

$E(I)$ denotes the space of smooth functions on I . Its dual $E'(I)$ is the space of distributions with compact support on I .

The following theorem given in [4] will be used subsequently.

Theorem 1.1. *If $f(x) \in L_1(0, \infty)$, if $f(x)$ is of bounded variation in a neighbourhood of the point $x = x_0 > 0$, if $\mu \geq -\frac{1}{2}$, and if $F(y)$ is defined as the conventional Hankel transformation, then*

$$\frac{1}{2}[f(x_0 + 0) + f(x_0 - 0)] = h_\mu^{-1}F = \int_0^\infty F(y)\sqrt{x_0y}J_\mu(x_0y)dy.$$

Note that when $\mu \geq -\frac{1}{2}$, the conventional inverse Hankel transformation h_μ^{-1} is defined by precisely the same formula as is the direct Hankel transformation h_μ ; in symbols, $h_\mu = h_\mu^{-1}$.

2. THE TESTING FUNCTION SPACES $M_{a,\mu}$ AND M_μ

Let $a \in I$ and $\mu \in R$. We define $M_{a,\mu}$ as the space of testing functions $\varphi(x)$ which are defined and smooth on $0 < x < \infty$, taking its value in C , and for which

$$\gamma_{m,k}^{a,\mu}(\varphi) = \sup_{x \in I} |e^{-ax}x^m(x^{-1}D)^kx^{-\mu-1/2}\varphi(x)| < \infty, \quad m, k = 0, 1, 2, \dots$$

We assign to $M_{a,\mu}$ the topology generated by the countable multinorm $\{\gamma_{m,k}^{a,\mu}\}$. $M_{a,\mu}$ is a Hausdorff space since $\gamma_{m,0}^{a,\mu}$ is a norm.

The following properties will be inferred.

(i) Let $\mu \geq -\frac{1}{2}$. For a fixed complex number y belonging to the strip $\Omega = \{y : |\operatorname{Im} y| < a, y \neq 0 \text{ or a negative number}\}$,

$$\frac{\partial^m}{\partial y^m}(\sqrt{xy}J_\mu(xy)) \in M_{a,\mu}.$$

Indeed, it is easily verified that (see [1])

$$\frac{\partial^m}{\partial y^m}(\sqrt{xy}J_\mu(xy)) = \sum_{j=0}^m a_j(\mu)y^{j-m}x^j\sqrt{xy}J_{\mu-j}(xy),$$

where the $a_j(\mu)$ are constants depending on μ only.

Considering

$$\begin{aligned} (x^{-1}D)^k x^{-\mu-1/2} x^j \sqrt{xy} J_{\mu-j}(xy) &= \sqrt{y}(x^{-1}D)^k x^{-(\mu-j)} J_{\mu-j}(xy) \\ &= (-1)^k y^{k+1/2} x^{-(\mu-j+k)} J_{\mu-j+k}(xy) \end{aligned}$$

and

$$\begin{aligned}(xy)^{-(\mu-j+k)} J_{\mu-j+k}(xy) &\sim \frac{1}{2^{\mu-j+k} \Gamma(\mu - j + k + 1)} \quad \text{as } x \rightarrow 0^+ \\ &= O[(xy)^{-(\mu-j+k)-1/2} e^{x|\operatorname{Im} y|}] \quad \text{as } x \rightarrow \infty,\end{aligned}$$

it follows that

$$\gamma_{m,k}^{a,\mu}(x^j \sqrt{xy} J_{\mu-j}(xy)) < \infty.$$

Therefore

$$\gamma_{m,k}^{a,\mu} \left[\frac{\partial^m}{\partial y^m} (\sqrt{xy} J_\mu(xy)) \right] \leq \sum_{j=1}^m |a_j(\mu)| \cdot |y|^{j-m} \gamma_{m,k}^{a,\mu}(x^j \sqrt{xy} J_{\mu-j}(xy)) < \infty$$

for a fixed $y \in \Omega$.

(ii) The differential operator $N_\mu \stackrel{\Delta}{=} x^{\mu+1/2} D x^{-\mu-1/2}$ is continuous from $M_{a,\mu}$ into $M_{a,\mu+1}$. Indeed

$$\gamma_{m,k}^{a,\mu+1}(N_\mu \varphi) = \gamma_{m,k+1}^{a,\mu}(\varphi).$$

Note : It is impossible for us to define N_μ^{-1} on $M_{a,\mu+1}$.

The differential operator $\overline{M}_\mu \stackrel{\Delta}{=} x^{-\mu-1/2} D x^{\mu+1/2}$ is continuous from $M_{a,\mu+1}$ into $M_{a,\mu}$. Indeed,

$$\gamma_{m,k}^{a,\mu}(\overline{M}_\mu \varphi) \leq 2|\mu+k+1| \gamma_{m,k}^{a,\mu+1}(\varphi) + \gamma_{m+2,k+1}^{a,\mu+1}(\varphi).$$

(iii) Multipliers in $M_{a,\mu}$. Define

$$O = \left\{ \theta(x) \in C_I^\infty \mid \exists C_\nu \in I \text{ for each integer } \nu \geq 0, \exists \text{ integer } n_\nu, \text{ such that} \right. \\ \left. \left| \frac{(x^{-1}D)^\nu \theta(x)}{1+x^{n_\nu}} \right| \leq C_\nu \right\}.$$

For arbitrary $\theta \in O$ and $\varphi \in M_{a,\mu}$, we have

$$(x^{-1}D)^k x^{-\mu-1/2} \theta \varphi = \sum_{\nu=0}^k \binom{k}{\nu} \frac{(x^{-1}D)^\nu \theta}{1+x^{n_\nu}} (1+x^{n_\nu}) (x^{-1}D)^{k-\nu} x^{-\mu-1/2} \varphi$$

so that

$$\gamma_{m,k}^{a,\mu}(\theta \varphi) \leq \sum_{\nu=0}^k \binom{k}{\nu} C_\nu [\gamma_{m,k-\nu}^{a,\mu}(\varphi) + \gamma_{m+n_\nu,k-\nu}^{a,\mu}(\varphi)].$$

(iv) $M_{a,\mu}$ is complete. The proof is very similar to Lemma 5.2.2 on page 131 in [4].

(v) $H_\mu \subset M_{a,\mu}$ for all $\mu \in R$, $a \in I$. And the topology of H_μ is stronger than that induced on it by $M_{a,\mu}$. Indeed, $e^{-ax} \leq 1$ on $(0, \infty)$ and $\gamma_{m,k}^{a,\mu}(\varphi) \leq \gamma_{m,k}^\mu(\varphi)$.

Our space $M_{a,\mu}$ is a subspace of $J_{a,\mu}$ and has a stronger topology than that induced on it by $J_{a,\mu}$.

To see $M_{a,\mu} \subset J_{a,\mu}$, we note that

$$(3) \quad x^{-\mu-1/2} S_\mu^k \varphi = \sum_{j=0}^k b_j x^{2j} (x^{-1}D)^{k+j} x^{-\mu-1/2} \varphi,$$

where $S_\mu = x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2}$, and the constants b_j depend on μ only. It follows that

$$\tau_k^{a,\mu}(\varphi) \leq \sum_{j=0}^k |b_j| \gamma_{2j,k+j}^{a,\mu}(\varphi)$$

for any $\varphi \in M_{a,\mu}$. This implies our assertion.

Note that H_μ is a proper subset of $M_{a,\mu}$. Indeed, $\sqrt{xy} J_\mu(xy) \in M_{a,\mu}$ as was already shown in (i), but $\sqrt{xy} J_\mu(xy) \notin H_\mu$ since it is not of rapid descent. Also, $M_{a,\mu}$ is a proper subset of $J_{a,\mu}$. In fact, we define

$$\varphi_1(x) = \begin{cases} e^{ax}, & x \geq 1, \\ \text{smooth} & 1/2 < x < 1, \\ 0, & 0 < x \leq 1/2. \end{cases}$$

Let $\varphi = x^{\mu+1/2} \varphi_1(x)$, then $\varphi(x) \in J_{a,\mu}$. By equation (3) we have

$$e^{-ax} x^{-\mu-1/2} S_\mu^k \varphi = \sum_{j=0}^k b_j e^{-ax} x^{2j} (x^{-1} D)^{k+j} x^{-\mu-1/2} \varphi.$$

When $x \geq 1$

$$e^{-ax} x^{-\mu-1/2} S_\mu^k \varphi = \sum_{j=0}^k b_j e^{-ax} x^{2j} (x^{-1} D)^{k+j} e^{ax}.$$

It is easily verified that

$$(x^{-1} D)^{k+j} e^{ax} = C(k, j) x^{-(2k+2j)+1} a e^{ax} + \dots + a^{k+j} x^{-k-j} e^{ax},$$

where $C(k, j)$ is a constant depending on k, j .

It follows that $\tau_k^{a,\mu}(\varphi) < \infty$, but $\gamma_{1,0}^{a,\mu}(\varphi) \geq \sup_{x \in [1, \infty)} |e^{-ax} x e^{ax}| = \infty$.

(vi) If $a > b > 0$, then $M_{b,\mu} \subset M_{a,\mu}$ and the topology of $M_{b,\mu}$ is stronger than that induced on it by $M_{a,\mu}$. This follows immediately from the inequality

$$\gamma_{m,k}^{a,\mu}(\varphi) \leq \gamma_{m,k}^{b,\mu}(\varphi) \quad \text{for } \varphi \in M_{b,\mu}.$$

(vii) $D(I) \subset M_{a,\mu}$ and the topology of $D(I)$ is stronger than that induced on it by $M_{a,\mu}$. By the way, we point out that $D(I)$ is not dense in $M_{a,\mu}$.

(viii) For every choice of μ and a , $M_{a,\mu} \subset E(I)$. Moreover, it is dense in $E(I)$ because $D(I)$ is dense in $E(I)$.

(ix) For each $f \in M'_{a,\mu}$, there exist a pair of nonnegative integers m_0, k_0 and a positive constant C such that for $\varphi \in M_{a,\mu}$

$$|\langle f, \varphi \rangle| \leq C \max_{\substack{0 \leq k \leq k_0 \\ 0 \leq m \leq m_0}} \gamma_{m,k}^{a,\mu}(\varphi).$$

We turn now to the definition of a certain countable-union space $M_{\sigma,\mu}$ (for short M_μ) that arises from the $M_{a,\mu}$ spaces. Our subsequent discussion takes on a simpler form when the space M_μ is used in place of the $M_{a,\mu}$ spaces. Let $\{a_\nu\}_{\nu=1}^\infty$ be a monotonically increasing sequence of positive numbers tending to σ . Here $\sigma = \infty$ is allowed. By virtue of note (vi), $\{M_{a_\nu,\mu}\}_{\nu=1}^\infty$ is a sequence of Fréchet spaces such that $M_{a_1,\mu} \subset M_{a_2,\mu} \subset \dots$, and such that the topology of $M_{a_\nu,\mu}$ is stronger than that induced on it by $M_{a_{\nu+1},\mu}$. Let $M_{\sigma,\mu}$ ($= M_\mu$) =

$\bigcup_{\nu=1}^{\infty} M_{a_\nu, \mu}$ denote the countable-union space generated by the above sequence of spaces. Thus, a sequence $\{\varphi_n\}_{n=1}^{\infty}$ converges in M_μ to φ iff for some fixed a_ν , $\varphi_n, \varphi \in M_{a_\nu, \mu}$ and $\varphi_n \rightarrow \varphi$ in $M_{a_\nu, \mu}$.

We assign to M'_μ the usual weak convergence concept. Accordingly, a sequence $\{f_\nu\}_{\nu=1}^{\infty}$ converges in M'_μ if there exists an $f \in M'_\mu$ such that, for every $\varphi \in M_\mu$

$$|\langle f_\nu, \varphi \rangle - \langle f, \varphi \rangle| \rightarrow 0 \quad \text{as } \nu \rightarrow \infty.$$

The following lemmas are immediate.

Lemma 2.1. *For any fixed complex number y belonging to $\Omega = \{y : |\operatorname{Im} y| < \sigma, y \neq 0 \text{ or a negative number}\}$*

$$\frac{\partial^m}{\partial y^m}(\sqrt{xy}J_\mu(xy)) \in M_\mu, \quad m = 0, 1, 2, \dots$$

Lemma 2.2. *For every choice of $\sigma > 0$, $H_\mu \subset M_\mu$ and convergence in H_μ implies convergence in M_μ .*

3. THE GENERALIZED HANKEL TRANSFORMATION ON M'_μ

Let $\mu \geq -\frac{1}{2}$. In view of (vi), to every $f \in M'_{a, \mu}$ there exists a unique real number σ_f possibly $\sigma_f = \infty$ such that $f \in M'_{b, \mu}$ if $b < \sigma_f$ and $f \notin M'_{b, \mu}$ if $b > \sigma_f$. Therefore, $f \in M'_{\sigma_f, \mu}$. We define the μ th order Hankel transform $h_\mu f$ as the application of f to the kernel $\sqrt{xy}J_\mu(xy)$; i.e.,

$$F(y) = (h_\mu f)(y) = \langle f(x), \sqrt{xy}J_\mu(xy) \rangle,$$

where $y \in \Omega_f = \{y : |\operatorname{Im} y| < \sigma_f, y \neq 0 \text{ or a negative number}\}$. The strip Ω_f will be called the region of definition for $F(y)$.

The following results given in [5] will be needed.

Lemma 3.1. *Let a be a fixed positive number. For all y in the strip $\Omega = \{y : |\operatorname{Im} y| < a, y \neq 0 \text{ or a negative number}\}$, for $0 < x < \infty$, and for $\mu \geq -\frac{1}{2}$,*

$$|e^{-ax}(xy)^{-\mu}x^\mu J_\mu(xy)| \leq A_\mu,$$

where A_μ is a constant with respect to x and y .

Lemma 3.2 (Boundedness of $F(y)$). *$F(y)$ is bounded on any cut strip $\{y : |\operatorname{Im} y| \leq a_1 < a < \sigma_f, y \neq 0 \text{ or a negative number}\}$ according to*

$$|F(y)| \leq |y|^{\mu+1/2}P_a(|y|),$$

where $P_a(|y|)$ is a polynomial depending only on a .

Lemma 3.3 (Analyticity of $F(y)$). *$F(y)$ is an analytic function in $\Omega = \{y : |\operatorname{Im} y| \leq a < \sigma_f, y \neq 0 \text{ or a negative number}\}$ and*

$$D_y F(y) = \left\langle f(x), \frac{\partial}{\partial y} \sqrt{xy}J_\mu(xy) \right\rangle.$$

Theorem 3.1. *Let $f \in M'_\mu$, $\varphi \in H_\mu$, and $\mu \geq -\frac{1}{2}$. Then*

$$\langle \langle f(x), \sqrt{xy}J_\mu(xy) \rangle, \varphi(y) \rangle = \left\langle f(x), \int_0^\infty \sqrt{xy}J_\mu(xy)\varphi(y)dy \right\rangle, \quad y \in I.$$

Proof. Since $\langle f(x), \sqrt{xy}J_\mu(xy) \rangle$ is of slow growth as $y \rightarrow \infty$ by Lemma 3.2, and is Lebesgue integrable on $0 < y < Y$ for $Y \in (0, \infty)$ by Lemma 3.3, we can write

$$(4) \quad \langle \langle f(x), \sqrt{xy}J_\mu(xy) \rangle, \varphi(y) \rangle = \int_0^\infty \langle f(x), \sqrt{xy}, J_\mu(xy) \rangle \varphi(y) dy$$

since $\varphi \in H_\mu$.

Our theorem will be proven when we show that

$$\int_0^\infty \langle f(x), \sqrt{xy}J_\mu(xy) \rangle \varphi(y) dy = \left\langle f(x), \int_0^\infty \varphi(y) \sqrt{xy}J_\mu(xy) dy \right\rangle.$$

Consider the Riemann sum

$$(5) \quad \frac{Y}{m} \sum_{\nu=1}^m \varphi\left(\nu \frac{Y}{m}\right) \left\langle f(x), \sqrt{x\nu \frac{Y}{m}} J_\mu\left(x\nu \frac{Y}{m}\right) \right\rangle.$$

This sum converges to the integral $\int_0^Y \varphi(y) \langle f(x), \sqrt{xy}J_\mu(xy) \rangle dy$ for the integrand is continuous on $0 < y < Y$. Moreover, since $\varphi(y)$ is of rapid descent while $\langle f(x), \sqrt{xy}J_\mu(xy) \rangle$ is bounded by a polynomial in $|y|$, the last integral converges to the right-hand side of (4) as $Y \rightarrow \infty$.

On the other hand, we are able to write (5) as

$$(6) \quad \left\langle f(x), \frac{Y}{m} \sum_{\nu=1}^m \varphi\left(\nu \frac{Y}{m}\right) \sqrt{x\nu \frac{Y}{m}} J_\mu\left(x\nu \frac{Y}{m}\right) \right\rangle$$

and show that, as m and then $Y \rightarrow \infty$, (6) converges to

$$\left\langle f(x), \int_0^\infty \varphi(y) \sqrt{xy}J_\mu(xy) dy \right\rangle.$$

Indeed, by taking the operator $(x^{-1}D)^k x^{-\mu-1/2}$ under the integral and the summation signs, we have

$$(7) \quad \begin{aligned} & \gamma_{m,k}^{a,\mu} \left\{ \int_0^Y \varphi(y) \sqrt{xy}J_\mu(xy) dy - \frac{Y}{m} \sum_{\nu=1}^m \varphi\left(\nu \frac{Y}{m}\right) \sqrt{x\nu \frac{Y}{m}} J_\mu\left(x\nu \frac{Y}{m}\right) \right\} \\ &= \sup_{x \in I} \left| e^{-ax} x^m \left\{ \int_0^Y \varphi(y) (-1)^k y^{k+1/2} x^{-(\mu+k)} J_{\mu+k}(xy) dy \right. \right. \\ & \quad \left. \left. - \frac{Y}{m} \sum_{\nu=1}^m \varphi\left(\nu \frac{Y}{m}\right) (-1)^k \left(\nu \frac{Y}{m}\right)^{\mu+2k+1/2} \left(x\nu \frac{Y}{m}\right)^{-(\mu+k)} J_{\mu+k}\left(x\nu \frac{Y}{m}\right) \right\} \right|. \end{aligned}$$

Because of the factor $e^{-ax} x^m$ and the boundedness of $(xy)^{-(\mu+k)} J_{\mu+k}(xy)$ on $0 < xy < \infty$, given an $\varepsilon > 0$, there exists an X such that for all $x > X$, the quantity under the supremum sign is less than ε for every m . Now, on $0 < x < X$, $0 < y < Y$, the expression $\varphi(y) y^{\mu+2k+1/2} (xy)^{-\mu-k} J_{\mu+k}(xy)$ is uniformly continuous, hence the Riemann sum on the right-hand side of (7) converges to the integral uniformly on $0 < x < X$ as $m \rightarrow \infty$. Thus, (6) converges to $\langle f(x), \int_0^Y \varphi(y) \sqrt{xy}J_\mu(xy) dy \rangle$ as $m \rightarrow \infty$.

Finally, we show that $\int_Y^\infty \varphi(y)\sqrt{xy}J_\mu(xy)dy \rightarrow 0$ as $Y \rightarrow \infty$ in M_μ . This is because of the following inequalities:

$$\begin{aligned} & \left| e^{-ax}x^m(x^{-1}D)^kx^{-\mu-1/2}\int_Y^\infty \varphi(y)\sqrt{xy}J_\mu(xy)dy \right| \\ &= \left| \int_Y^\infty \varphi(y)e^{-ax}x^m(-1)^ky^{\mu+2k+1/2}(xy)^{-(\mu+k)}J_{\mu+k}(xy)dy \right| \\ &\leq A_\mu \int_Y^\infty |\varphi(y)y^{\mu+2k+1/2}| dy \rightarrow 0. \end{aligned}$$

The last inequality is due to Lemma 3.1.

Inversion and uniqueness. We now state an inversion theorem for our generalized Hankel transformation.

Theorem 3.2. Let $F(y) = \langle f(x), \sqrt{xy}J_\mu(xy) \rangle$, $f \in M'_\mu$, $y \in I$. Let $\mu \geq -\frac{1}{2}$. Then, in the sense of convergence in H_μ ,

$$f(x) = \lim_{r \rightarrow \infty} \int_0^r F(y)\sqrt{xy}J_\mu(xy)dy.$$

Proof. Let $\varphi(x) \in H_\mu$, we wish to show that

$$\left\langle \int_0^r F(y)\sqrt{xy}J_\mu(xy)dy, \varphi(x) \right\rangle$$

tends to $\langle f(x), \varphi(x) \rangle$ as $r \rightarrow \infty$. Since $F(y)$ is smooth and $\sqrt{xy}J_\mu(xy)$ is bounded on $0 < xy < \infty$, it follows that $\int_0^r F(y)\sqrt{xy}J_\mu(xy)dy$ is continuous and bounded with respect to x . Hence we have

$$\left\langle \int_0^r F(y)\sqrt{xy}J_\mu(xy)dy, \varphi(x) \right\rangle = \int_0^\infty \int_0^r F(y)\sqrt{xy}J_\mu(xy)dy\varphi(x)dx.$$

By Fubini's theorem we can change the order of integration and obtain

$$\begin{aligned} (8) \quad & \int_0^\infty \varphi(x) \int_0^r F(y)\sqrt{xy}J_\mu(xy)dy dx \\ &= \int_0^r \langle f(x), \sqrt{xy}J_\mu(xy) \rangle \int_0^\infty \varphi(x)\sqrt{xy}J_\mu(xy)dx dy. \end{aligned}$$

Set $\Phi(y) = \int_0^\infty \varphi(x)\sqrt{xy}J_\mu(xy)dx$. Then $\Phi(y) \in H_\mu$ since $\mu \geq -\frac{1}{2}$.

By Theorem 3.1, the right-hand side of (8) can be written as

$$\left\langle f(x), \int_0^r \sqrt{xy}J_\mu(xy) \int_0^\infty \varphi(x)\sqrt{xy}J_\mu(xy)dxdy \right\rangle.$$

Now, we wish to show

$$L_r(x) = \int_0^r \sqrt{ty}J_\mu(ty) \int_0^\infty \varphi(x)\sqrt{xy}J_\mu(xy)dxdy$$

converges in M_μ to $\varphi(t)$ as $r \rightarrow \infty$.

By the last part of the proof of Theorem 3.1, we get

$$\lim_{r \rightarrow \infty} \int_0^r \sqrt{xy}J_\mu(xy)\Phi(y)dy = \int_0^\infty \sqrt{xy}J_\mu(xy)\Phi(y)dy = \varphi(x).$$

The last equality is due to Theorem 1.1.

As a result of the inversion theorem, we have the following theorem.

Theorem 3.3. Let $F(y) = h_\mu f = G(y) = h_\mu g$ for $y \in I$, then $f = g$ in the sense of equality in H'_μ .

Now, we come to prove a characterization theorem for the generalized function in $M'_{a,\mu}$.

Theorem 3.4. A functional f is in $M'_{a,\mu}$ if and only if there exist bounded measurable functions $g_{m,k}(x)$ defined on I , for $m = 0, 1, 2, \dots, m_0$ and $k = 0, 1, 2, \dots, k_0$, where m_0 and k_0 are nonnegative integers depending on f , and such that

$$(9) \quad \langle f, \varphi \rangle = \left\langle \sum_{m=0, k=0}^{m_0, k_0} x^{-\mu-1/2} \left(-D \frac{1}{x}\right)^k \{e^{-ax} x^m (-D) g_{m,k}(x)\}, \varphi(x) \right\rangle$$

for every $\varphi \in M_{a,\mu}$.

Proof. Let $f \in M'_{a,\mu}$. By using the property (ix), there exist a pair of nonnegative integers m_0, k_0 , and a positive constant C such that for $\varphi \in M_{a,\mu}$

$$|\langle f, \varphi \rangle| \leq C \max_{\substack{0 \leq k \leq k_0 \\ 0 \leq m \leq m_0}} \gamma_{m,k}^{a,\mu}(\varphi) = C \max_{\substack{0 \leq k \leq k_0 \\ 0 \leq m \leq m_0}} \sup_{x \in I} |e^{-ax} x^m (x^{-1} D)^k x^{-\mu-1/2} \varphi|.$$

Since $\varphi \in M_{a,\mu}$, there exists $C_{m,k} > 0$ such that

$$\sup_{x \in I} |e^{-ax} x^{m+2} (x^{-1} D)^k x^{-\mu-1/2} \varphi| \leq C_{m,k} \text{ for fixed } a \text{ and } \mu.$$

It follows that

$$\lim_{x \rightarrow \infty} e^{-ax} x^m (x^{-1} D)^k x^{-\mu-1/2} \varphi = 0;$$

hence

$$e^{-ax} x^m (x^{-1} D)^k x^{-\mu-1/2} \varphi = \int_{\infty}^x D_t \{e^{-at} t^m (t^{-1} D)^k t^{-\mu-1/2} \varphi(t)\} dt.$$

From

$$\begin{aligned} D_t \{e^{-at} t^m (t^{-1} D)^k t^{-\mu-1/2} \varphi(t)\} \\ = D_t \{e^{-at} t^m\} (t^{-1} D)^k t^{-\mu-1/2} \varphi(t) + e^{-at} t^{m+1} (t^{-1} D)^{k+1} t^{-\mu-1/2} \varphi(t) \end{aligned}$$

and the fact that $\varphi \in M_{a,\mu}$, it follows that

$$\begin{aligned} \int_0^\infty |D_t \{e^{-at} t^m (t^{-1} D)^k t^{-\mu-1/2} \varphi(t)\}| dt \\ = \|D_t \{e^{-at} t^m (t^{-1} D)^k t^{-\mu-1/2} \varphi(t)\}\|_{L_1(0, \infty)} \end{aligned}$$

is finite, where $\|\cdot\|_{L_1(0, \infty)}$ denotes the norm on the space $L_1(0, \infty)$. Then we have

$$(10) \quad |\langle f, \varphi \rangle| \leq C \max_{\substack{0 \leq k \leq k_0 \\ 0 \leq m \leq m_0}} \|D_t \{e^{-at} t^m (t^{-1} D)^k t^{-\mu-1/2} \varphi(t)\}\|_{L_1(0, \infty)}.$$

Define an injective map $F_1: M_{a,\mu} \rightarrow F_1 M_{a,\mu}$ by

$$\varphi \rightarrow (D_t \{e^{-at} t^m (t^{-1} D)^k t^{-\mu-1/2} \varphi(t)\}), \quad m = 0, 1, \dots, m_0, \\ k = 0, 1, \dots, k_0.$$

$F_1 M_{a,\mu}$ is endowed with the topology induced on it by the product space $A_{m_0, k_0} = (L_1(0, \infty))^{(m_0+1)(k_0+1)}$.

Define $F_2: F_1 M_{a,\mu} \rightarrow C$ by $F_1 \varphi \rightarrow \langle f, \varphi \rangle$. By virtue of (10), F_2 is a continuous linear mapping.

By applying the Hahn-Banach theorem, F_2 can be extended to A_{m_0, k_0} . Therefore, since A'_{m_0, k_0} is isomorphic to $(L_\infty(0, \infty))^{(k_0+1)(m_0+1)}$ (see Treves [6]), there exist $(k_0+1)(m_0+1)$ bounded measurable functions $g_{m,k}(x)$ ($m = 0, 1, 2, \dots, m_0$, $k = 0, 1, 2, \dots, k_0$), such that

$$\begin{aligned} F_2(F_1 \varphi) = \langle f, \varphi \rangle &= \sum_{m=0, k=0}^{m_0, k_0} \langle g_{m,k}(x), D_x \{e^{-ax} x^m (x^{-1}D)^k x^{-\mu-1/2} \varphi\} \rangle \\ &= \left\langle \sum_{m=0, k=0}^{m_0, k_0} x^{-\mu-1/2} \left(-D \frac{1}{x}\right)^k \{x^m e^{-ax} (-D) g_{m,k}(x)\}, \varphi \right\rangle. \end{aligned}$$

On the other hand, we assume f is defined by (8). Obviously f is linear. Let $\varphi_n \rightarrow 0$ in $M_{a,\mu}$, then $D_x \{e^{-ax} x^m (x^{-1}D)^k x^{-\mu-1/2} \varphi_n\}$ converges to 0 in $L_1(0, \infty)$. This completes the proof.

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