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ON THE K-GROUPS OF CERTAIN C*-ALGEBRAS USED IN E-THEORY

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ABSTRACT. Let A be a C*-algebra. We denote by A_∞ the quotient of the C*-algebra of bounded continuous functions $[1, \infty) \to A$ by the ideal of the functions which vanish at ∞. We show that the the canonical map $A \to A_\infty$ gives an isomorphism between K-groups, provided that A is stable.

In [CH] the notion of *asymptotic homomorphism* was introduced, one of the many remarkable achievements being a simpler description for the operations in K-theory. Roughly speaking, modulo suspensions and stabilizations of the E-category is obtained from the homotopy category of separable C*-algebras by "inverting certain arrows".

In this note we shall examine one type of homomorphisms which should be inverted in E-theory. By computing the K-groups that are involved we hope to bring some evidence that one can expect those homomorphisms to be E-equivalences.

To be more specific, let A be a C*-algebra . We denote by $C^b([1, \infty), A)$, or for short C^bA, the C*-algebra of continuous and bounded functions $f : [1, \infty) \longrightarrow A$. By A_∞ one denotes the quotient C^bA/CA, where CA (sometimes denoted $C_0([1, \infty), A)$) is the ideal of all functions in C^bA which vanish at infinity. Let $\imath : A \longrightarrow A_\infty$ be the "standard embedding" given by $\imath(a) = [\operatorname{const} a] \pmod{CA}$. Let us take, using the Bartle-Graves Theorem, a section $p : A_\infty \longrightarrow C^bA$ for the map $\pi_A : C^bA \longrightarrow A_\infty$ with the following properties:

(i) $p(\lambda x) = \lambda p(x)$ for all $x \in A$, $\lambda \in \mathbb{C}$;

(ii) p is continuous.

If we take $\phi_t^p : A_\infty \longrightarrow A$ to be the map defined, for each $t \in [1, \infty)$, by $\phi_t^p(x) = p(x)(t)$, then using the terminology of [CH] we get an asymptotic homomorphism from A_∞ to A.

It is easy to show that $\phi_t^p \circ \imath : A \longrightarrow A$ gives an asymptotic homomorphism which is homotopic to Id_A. So $\imath : A \longrightarrow A_\infty$ has a left inverse in $\mathrm{E}(A_\infty, A)$. As noted in [CH], every asymptotic homomorphism $\psi_t : B \longrightarrow A$ is equivalent to one of the form $\phi_t^p \circ \psi$ with $\psi : B \longrightarrow A_\infty$ a $*$-homomorphism.

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The natural question that arises is then: *For a stable C*-algebra A, is $\iota : A \longrightarrow A_\infty$ an equivalence, i.e., is the asymptotic homomorphism $\iota \circ \phi_t^p$ homotopic (possibly after further stabilizations and suspensions) to* Id_A ?

Note that A_∞ is not separable, so this problem is somehow "out of E-theory".

In this note we shall prove the following.

Theorem. *For stable C*-algebras A the embeddings $\iota : A \longrightarrow A_\infty$ induce isomorphisms at the level of K-groups.*

Let us see the analogy with a more familiar situation. We know that to any extension

$$0 \to A \otimes \mathscr{K} \to B \to C \to 0$$

one can associate its *Busby invariant* which is a $*$-homomorphism $C \longrightarrow Q(A \otimes \mathscr{K})$ (Here for any C*-algebra B we denote by $Q(B)$ the quotient $M(B)/B$, where $M(B)$ is the multiplier algebra of B.) Conversely, to any $*$-homomorphism $C \longrightarrow Q(A \otimes \mathscr{K})$ one can associate an extension by taking the pull-back of the fundamental extension

$$\begin{tikzcd} A \otimes \mathscr{K} \arrow[r] & M(A \otimes \mathscr{K}) \arrow[r] & Q(A \otimes \mathscr{K}) \\ & & C \arrow[u] \end{tikzcd}$$

In E-theory to any extension $A \longrightarrow B \longrightarrow C$ one associates an asymptotic homomorphism $SC \longrightarrow A$, and conversely, to any asymptotic homomorphism one can associate an extension by taking the pull-back

$$\begin{tikzcd} SA \arrow[r] & C_\circ^b A \arrow[r] & A_\infty \\ & & SC \arrow[u] \end{tikzcd}$$

Here $C_\circ^b A = \{f \in C^b A : f(1) = 0\}$. Note that here we get actually a suspension.

So it appears that what should be relevant for E-theory are (one usually works with stabilizations) extensions like

$$SA \otimes \mathscr{K} \longrightarrow C_\circ^b(A \otimes \mathscr{K}) \longrightarrow (A \otimes \mathscr{K})_\infty .$$

We shall think of this extension as an analog of the extension

$$SA \otimes \mathscr{K} \longrightarrow M(SA \otimes \mathscr{K}) \longrightarrow Q(SA \otimes \mathscr{K}) .$$

Our result says that, exactly as for the multiplier algebras like $M(B \otimes \mathscr{K})$, the K-groups of algebras like $C_\circ^b(B \otimes \mathscr{K})$ are trivial. It can be easily shown that this statement is equivalent to the Theorem. Moreover, for K_0-groups the result holds even for nonstable algebras, that is, $K_0(C_\circ^b B) = 0$ for any C*-algebra B. Instead, for nonstable C*-algebras B the group $K_1(C_\circ^b B)$ may not be trivial.

Finally we shall see that $\iota : A \longrightarrow A_\infty$ induces, after suspension, an inverse for the connecting homomorphism $\partial : K_*(A_\infty) \to K_{*+1}(SA) = K_*(A)$.

1

We begin by examining the exactness properties for the functors $C^b : A \longmapsto C^b A$ and $C_\circ^b : A \longmapsto C_\circ^b A$.

Proposition. *The correspondences C^b and $C^b_\circ$ are exact functors.*

Proof. Let $0 \to J \to A \xrightarrow{\pi} B \to 0$ be an exact sequence of C*-algebras. The sequences

$$0 \longrightarrow C^b J \longrightarrow C^b A \xrightarrow{C^b\pi} C^b B \longrightarrow 0$$

$$0 \longrightarrow C^b_\circ J \longrightarrow C^b_\circ A \xrightarrow{C^b_\circ\pi} C^b_\circ B \longrightarrow 0$$

are obviously exact at A and J. The only problem can occur at B. Let $p : B \to A$ be a continuous map with $p(\lambda b) = \lambda p(b)$ for all $\lambda \in \mathbb{C}$, $b \in B$, such that $\pi \circ p = \mathrm{Id}_B$. (Here we used, of course, the Bartle-Graves Theorem.) The homogeneity and continuity of p give the existence of a constant $M > 0$ such that $\|p(b)\| \le M\|b\|$ for all $b \in B$. So we can define the map $\tilde{p} : C^b B \to C^b A$ by $\tilde{p}(f) = p \circ f$, which gives a section for $C^b\pi$ and for $C^b_\circ\pi$ as well. This proves the surjectivity of the maps $C^b\pi$ and $C^b_\circ\pi$. □

2

The functor C^b allows one to define a new functor $A \longmapsto A_\infty$ in the following way. For $\psi : A \to B$ a $*$-homomorphism one defines $\psi_\infty : A_\infty \to B_\infty$ as the unique $*$-homomorphism that makes the diagram

$$\begin{array}{ccc} C^b A & \xrightarrow{\pi_A} & A_\infty \\ {\scriptstyle C^b\psi}\big\downarrow & & \big\downarrow{\scriptstyle \psi_\infty} \\ C^b B & \xrightarrow{\pi_B} & B_\infty \end{array}$$

commutative. Since $(C^b\psi)(CA) \subset CB$, everything is correctly defined.

Lemma. *If $\psi : A \to B$ is injective (resp. surjective) then so is $\psi_\infty : A_\infty \to B_\infty$.*

Proof. Suppose ψ is injective. Take $x \in A_\infty$ such that $\psi_\infty(x) = 0$. If we take $f \in C^b A$ with $x = \pi_A(f)$, we get $\pi_B(\psi \circ f) = 0$, that is, $\psi \circ f \in CB$. This reads $\lim_{t\to\infty}(\psi \circ f)(t) = 0$. But the injectivity of ψ yields $\|(\psi \circ f)(t)\| = \|f(t)\|$, which gives $\lim_{t\to\infty} f(t) = 0$. That is, $f \in CA$, so $x = 0$. This shows the injectivity of ψ_∞.

If ψ is surjective, the surjectivity of ψ_∞ follows from Proposition 1 since $C^b\psi$ is surjective. □

3

The above property is used to prove the following.

Proposition. *The functor $A \longmapsto A_\infty$ is exact.*

Proof. This is a standard fact which follows from the exactness of the functors $A \mapsto C^b A$, $A \mapsto C^b_\circ A$. □

4

Next we will rephrase the statement of the Theorem using the canonical embedding $A \to C^b A$. To be more precise, let us define $\gamma_A : A \to C^b A$ by

$\gamma_A(a)(t) = a$ for all $a \in A$, $t \in [1, \infty)$. Define also $\sigma_A : C^b A \to A$ by $\sigma_A(f) = f(1)$. The statement we are going to prove is:

Theorem. (i) *For any C^*-algebra A the map $(\gamma_A)_* : K_0(A) \to K_0(C^b A)$ is an isomorphism.*

(ii) *If A is stable, the the map $(\gamma_A)_* : K_1(A) \to K_1(C^b A)$ is also an isomorphism.*

The first step in proving this result is the following fact. Note that $\sigma \circ \gamma = \mathrm{Id}_A$. This proves the injectivity part of the Theorem, that is: *For any C^*-algebra A the map $(\gamma_A)_* : K_*(A) \to K_*(C^b A)$ is injective.*

So, what remanins to be shown is the surjectivity.

5

Before we go on with the proof, let us remark first that the stability assumption made in part (ii) of Theorem 4 is essential. To see this we examine the following (trivial) example. Take $A = \mathbb{C}$, and let $u \in C^b\mathbb{C}$ be the unitary given as $u(t) = e^{it}$, $t \in [1, \infty)$. The first thing one observes is that u cannot be connected to 1 by a path of invertible elements in $C^b\mathbb{C}$. Indeed, if this were the case it would follow that u can be written as a product of exponentials. Since we work with commutative algebras, this would give an element $f \in C^b\mathbb{C}$ with $u = e^f$. But this is clearly impossible. Let us turn our attention now to the class $[u]$ of u in $K_1(C^b\mathbb{C})$. If $[u] = 0$, this means that there exists a path $(U_s)_{s\in[0,1]}$ in some $\mathrm{GL}_n(C^b\mathbb{C})$ with

$$U_0 = \begin{pmatrix} u & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

But if we take $v_s = \det(U_s) \in C^b\mathbb{C}$, we again get a path of invertibles in $C^b\mathbb{C}$ that connects u to 1, contradicting the previous remark. This proves that $[u] \neq 0$, so $K_1(C^b\mathbb{C}) \neq \{0\}$.

6. Proof of Theorem 4, part (i)

Suppose first that A is unital. Let $\alpha \in K_0(C^b A)$. Since $\mathrm{Mat}_n(C^b A) \simeq C^b(\mathrm{Mat}_n(A))$, we can suppose $\alpha = [P]$ with P a projection in $C^b A$. This means that P is a continuous function $P : [1, \infty) \to A$ such that $P(t)$ is a projection for all $t \in [1, \infty)$.

Using Lemma 3.8. from [EK], there is a continuous function $[1, \infty) \ni t \mapsto V(t) \in A$ such that $V(t)$ is a partial isometry and $P(t) = V(t)^* V(t)$ for all t. So if we take $p = P(1) \in A$, this shows that P is Murray-von Neumann equivalent to $\gamma(p)$, so $\alpha \in \mathrm{Ran}\, \gamma_*$.

In the nonunital case take $\tilde{A}$ the algebra obtained from A by adjoining the unit. Take $\epsilon : \tilde{A} \to \mathbb{C}$ as the canonical map and $\nu : \mathbb{C} \to \tilde{A}$ as the obvious section for ϵ. Applying the C^b functor to the split exact sequence $A \to \tilde{A} \underset{\nu}{\overset{\epsilon}{\rightleftarrows}} \mathbb{C}$

we get a split exact sequence $C^b A \to C^b(\tilde{A}) \underset{C^b\nu}{\overset{C^b\epsilon}{\rightleftarrows}} C^b\mathbb{C}$.

But then if we connect the corresponding sequences for K_0-groups we get a commutative diagram like

$$\begin{array}{ccccc} K_0(C^bA) & \to & K_0(C^b(\tilde{A})) & \rightleftarrows & K_0(C^b\mathbb{C}) \\ \uparrow (\gamma_A)_* & & \uparrow (\gamma_{\tilde{A}})_* & & \uparrow (\gamma_{\mathbb{C}})_* \\ K_0(A) & \to & K_0(\tilde{A}) & \rightleftarrows & K_0(\mathbb{C}) \end{array}$$

which gives the desired result. □

7

To prove part (ii) of Theorem 4, one cannot expect to use a similar argument for invertibles. The simple case $A = \mathbb{C}$ shows that that stability condition is necessary. The result which we shall use is the following.

Lemma. *If A is stable, then $K_1(C^b(M(A))) = \{0\}$.*

Proof. Let $\alpha \in K_1(C^b(M(A)))$. Since $\mathrm{Mat}_n(M(A)) \simeq M(\mathrm{Mat}_n(A)) \simeq M(A)$, we can assume $\alpha = [u]$ for some unitary $u \in C^b(M(A))$. That is, u is a unitary map $u : [1, \infty) \to M(A)$.

Take then, for all $t \in [1, \infty)$, the unitary

$$V(t) = \begin{pmatrix} u(t) & & & \\ & u(t)^* & & \\ & & u(t) & \\ & & & \ddots \end{pmatrix} \in M(A).$$

Clearly $V \in C^b(M(A))$. With the usual "rotation trick" V can be connected to both

$$\begin{pmatrix} u & & \\ & 1 & \\ & & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}.$$

That is $[u] = [V] = 0$. □

8. Proof of Theorem 4, part (ii)

We shall reduce the proof to a "K_0-situation" for which we can apply the first part of the Theorem.

Suppose A is stable. Take the exact sequences $C^bA \to C^b(M(A)) \to C^b(Q(A))$ and $A \to M(A) \to Q(A)$, and connect them through the corresponding γ-maps.

By the above Lemma the homomorphism $\partial : K_0(C^b(Q(A)) \to K_1(C^bA)$ is surjective. The same is true for the connecting homomorphism $\partial : K_0(Q(A)) \to K_1(A)$ (here we use the fact that $K_*(M(A)) = \{0\}$). But by naturality we have a commutative diagram

$$\begin{array}{ccc} K_0(C^b(Q(A)) & \xrightarrow{\partial} & K_1(C^bA) \\ {\scriptstyle (\gamma_{Q(A)})_*}\uparrow & & {\scriptstyle (\gamma_A)_*}\uparrow \\ K_0(Q(A)) & \xrightarrow{\partial} & K_1(A) \end{array}$$

in which three arrows are surjective. Clearly this enforces the surjectivity of $(\gamma_A)_*$. □

9

We turn our attention now to the original statement of the Theorem. In fact, everything follows from:

Corollary. (i) *For any C^*-algebra A we have* $K_0(C^b_\circ A) = \{0\}$.
(ii) *For any stable C^*-algebra A we have* $K_1(C^b_\circ A) = \{0\}$.

Indeed, from the Theorem it follows that $\sigma_A : C^b A \to A$ induces an isomorphism between the K-groups. But $C^b_\circ A = \operatorname{Ker} \sigma_A$, which according to the exact sequence of K-theory, gives the desired result.

Let us regard CA as the algebra of functions $f : [1, \infty) \to A$ with $f(1) = 0$ and such that $\lim_{t \to \infty} f(t)$ exists. If we consider the obvious embedding $CA \to C^b_\circ A$ given by this description, then we have two exact sequences connected as follows:

$$\begin{array}{ccccc} SA & \to & C^b_\circ A & \to & A_\infty \\ \| & & \uparrow & & \uparrow \iota \\ SA & \to & CA & \to & A \end{array}$$

By naturality, using the above Corollary we get that $\iota_* : K_*(A) \to K_*(A_\infty)$ is an isomorphism.

Finally, if we take a "piece" of the exact sequence of K-groups we see that we have a commutative diagram like

$$\begin{array}{ccc} K_*(A_\infty) & \xrightarrow{\partial} & K_{*+1}(SA) \\ \iota_* \uparrow & & \| \\ K_*(A) & \underset{\partial}{\to} & K_{*+1}(SA) \end{array}$$

This shows precisely, that modulo suspension, ι gives an inverse for $\partial : K_*(A_\infty) \to K_{*+1}(SA)$.

Remark. Lemma 7 has another alternative proof, which gives a stronger result. This is:

Lemma. *Let A be a stable C^*-algebra and let $B = S(C^b(M(A)))$. Then the embedding $B \to \mathrm{Mat}_2(B)$ given by $x \longmapsto x \otimes e_{11}$ is homotopic to the null map.*

Proof. Because of the isomorphism $M(A) \simeq \mathrm{Mat}_2(M(A))$, we can view the above embedding as the map described in the following way. If we consider an element $x \in B$ as a function $f : (0, 1) \times [1, \infty) \to M(A)$, then our embedding sends x to the function $g : (0, 1) \times [1, \infty) \to M(A)$ given by $g(s, t) = \begin{pmatrix} f(s,t) & 0 \\ 0 & 0 \end{pmatrix}$. Take $\eta : B \to B$ to be the map given by $\eta(f)(s, t) = f(1 - s, t)$. Then our embedding is homotopic to

$$f \longmapsto \begin{pmatrix} f & & & \\ & \eta(f) & & \\ & & f & \\ & & & \ddots \end{pmatrix},$$

which is clearly homotopic to the null map. (What we use here is that $\mathrm{Id}_B \oplus \eta$ and $\eta \oplus \mathrm{Id}_B$ are null homotopic.) □

Comments. In the above proof, as well as in the proof of Lemma 7, we have used the following fact: *If A is stable, then one has a* (*canonical*) $*$*-homomorphism*

$\kappa : M(M(A) \otimes \mathscr{K}) \to M(A)$, *and all the computations we have made took place in* Ran κ.

Another way is to use (see, for example, [Mi], Proposition 2.1) a sequence of orthogonal projections inside $M(A)$ each of them equivalent to 1 and summing up to 1 (in the strict topology).

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