A CONVERSE OF THE VOLUME-MEAN-VALUE PROPERTY
FOR INVARIANTLY HARMONIC FUNCTIONS

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Abstract. It is shown that the balls are the only domains having the mean value property with respect to the invariantly harmonic functions in the unit ball of $\mathbb{C}^n$.

1. Introduction

Let $B$ be the unit ball in $\mathbb{C}^n$. The Laplacian with respect to the Bergman metric in $B$ is given in coordinates by

$$\Delta = (1 - |z|^2) \sum_{i,j=1}^{n} (\delta_{ij} - z_i \overline{z_j}) D_i D_j.$$

It is also called the invariant Laplacian because for every $f \in C^2(B)$ and every complex automorphism $\Psi$ of $B$ one has $\Delta(f \circ \Psi) = (\Delta f) \circ \Psi$. The group $\text{Aut}(B)$ of such automorphisms is generated by the unitary operators on $\mathbb{C}^n$ and the involutions $\varphi_a$ permuting $a$ and $0$. The functions $f$ such that $\Delta f = 0$ are called $\mathcal{H}$-harmonic (we refer to [7, Chapter 4] for general properties of these functions).

If $f$ is $\mathcal{H}$-harmonic in $B$, the mean-value of $f$ on a sphere centered at $0$ is $f(0)$ [7, p. 51]. Therefore, if $B_r$ denotes the Euclidean ball of radius $r$ centered at $0$ and $dm$ denotes normalized Lebesgue measure in $B$, for a measure $d\mu$ of type $d\mu = \varphi(|\zeta|) dm(\zeta)$ one has

$$f(0) = \frac{1}{\mu(B_r)} \int_{B_r} f d\mu$$

for all $f$ $\mathcal{H}$-harmonic in $B$ (in fact, for all $f$ $\mathcal{H}$-harmonic and $d\mu$-integrable in $B_r$).

This is the case for the measure $d\lambda = (1 - |\zeta|^2)^{-n-1} dm$. The measure $d\lambda$ is invariant under the action of $\text{Aut}(B)$, whence

$$(*) \quad f(a) = \frac{1}{\lambda(B_r)} \int_{\varphi_a(B_r)} f(\zeta) d\lambda(\zeta)$$

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for all \( f \) \( \mathcal{M} \)-harmonic and \( dm \)-integrable in \( \varphi_a(B_r) \); i.e., \( f \) has the mean value property with respect to \( d\lambda \) and the “Bergman balls” \( \varphi_a(B_r) \). In this paper we prove a sort of converse of this fact:

**Theorem 1.** Let \( U \) be open, connected, relatively compact in \( B \), and such that \( \partial U = \partial \bar{U} \). Assume \( a \in U \) is such that

\[
\int_U f(\zeta) d\lambda(\zeta)
\]

for all \( \mathcal{M} \)-harmonic functions \( f \) in a neighbourhood of \( \bar{U} \). Then \( U \) is a Bergman ball centered at \( a \); i.e., \( U = \varphi_a(B_r) \) for some \( r < 1 \).

The analogue of this theorem for harmonic functions in Euclidean space (and Euclidean balls) was proved in [4] (see also [5] and [2]). See [1] for a converse, in another sense, of the mean-value property (*)

2. The proof

We need some known facts about \( \mathcal{M} \)-harmonic functions that we will proceed to recall. There is a Riesz-type decomposition formula, valid at least for \( u \in C^2(B) \),

\[
u(z) = \int_S \frac{1}{S} P(\zeta, z) u(\zeta) d\sigma(\zeta) + \int_B \Delta u(\zeta) G(\zeta, z) d\lambda(\zeta), \quad z \in B.
\]

Here \( S = \partial B \), \( d\sigma \) is the normalized Lebesgue measure on \( S \), \( P(\zeta, z) \) is the invariant (or Poisson-Szegő) kernel, and \( G(\zeta, z) \) is the Green function with pole at \( z \),

\[
G(\zeta, z) = g(|\varphi_z(\zeta)|^2), \quad g(x) = -c_n \int_1^t \frac{(1 - t)^{n-1}}{t^n} dt,
\]

\( c_n \) being a positive constant. Moreover, \( 1 - |\varphi_z(\zeta)|^2 = \frac{(1 - |z|^2)(1 - |\zeta|^2)}{|1 - \overline{z}\zeta|^2} \), so \( G \) is symmetric. From the formula it follows that if \( f \) has compact support and is \( \mathcal{M} \)-harmonic in \( U \), then

\[
f(z) = \int_{B \setminus U} \Delta f(\zeta) G(\zeta, z) d\lambda(\zeta), \quad z \in U.
\]

This means that the functions \( G_z(\zeta) = G(\zeta, z) \), \( z \notin U \), span the space of all \( \mathcal{M} \)-harmonic functions in a neighbourhood of \( \bar{U} \). It also follows from the formula that under suitable conditions on \( \Psi \), the Green potential

\[
G^\Psi(z) = \int_B \Psi(\zeta) G(\zeta, z) d\lambda(\zeta)
\]

satisfies \( \Delta G^\Psi = \Psi \) in the weak sense.

Let us begin the proof of the theorem. Obviously we can assume \( a = 0 \). By what has been said, the hypothesis is equivalent to

\[
cG(0, z) = \int_U G(\zeta, z) d\lambda(\zeta), \quad z \notin U,
\]
with $c = \lambda(U)$. Let us look at the potential

$$h(z) = \int_U G(\zeta, z) \, d\lambda(\zeta), \quad z \in B.$$ 

We know that $h(z) = cG(0, z)$ for $z \notin U$. In $U$, $h$ satisfies $\Delta h = 1$; a computation shows that $\frac{1}{n} \log(1 - |z|^2)$ has invariant Laplacian $1$ in $B$, hence

$$h(z) = \frac{1}{n} \log \frac{1}{1 - |z|^2} + u(z), \quad z \in U,$$ 

with $u$ $\mathcal{M}$-harmonic in $U$.

**Lemma 1.** $h$ is of class $C^1$ in $B$.

**Proof.** One must prove that $\nabla_z G(\zeta, z)$ is locally uniformly integrable. Using the formulas above,

$$|D_i G(\zeta, z)| = c_n \frac{(1 - |\varphi_z(\zeta)|^2)^{n-1}}{|\varphi_z(\zeta)|^{2n}} \frac{1 - |\zeta|^2}{|1 - \zeta z||1 - \zeta|} 
\times |\zeta_i(1 - |z|^2) - z_i(1 - \zeta z)|$$

$$= c_n \frac{(1 - |z|^2)^{n-1}(1 - |\zeta|^2)^{n-1}}{|1 - \zeta z|^2 - (1 - |z|^2)(1 - |\zeta|^2)|^n} \frac{1 - |\zeta|^2}{|1 - \zeta|} 
\times |\zeta_i(1 - |z|^2) - z_i(1 - \zeta z)|.$$ 

Therefore,

$$|\nabla_z G(\zeta, z)| \leq C \frac{|\zeta - z|}{|1 - \zeta z|^2 - (1 - |z|^2)(1 - |\zeta|^2)|^n}, \quad \zeta, z \in B.$$ 

This is a well-known singularity; in fact, it is the same that appears in the Cauchy kernel for the ball. If $e_1, \ldots, e_{n-1}$ is an orthonormal basis of $(\mathbb{C}z)\perp$ and $\zeta - z = \lambda_n z + \sum \lambda_i e_i$, the expression inside the brackets equals

$$|\lambda_n|^2 |z|^2 + (1 - |z|^2) \sum_{i=1}^{n-1} |\lambda_i|^2 \geq (1 - |z|^2)|\zeta - z|^2.$$ 

Hence, for $z$ in a compact set $K \subset B$,

$$|\nabla_z G(\zeta, z)| \leq C_k |\zeta - z|^{1-2n},$$

which proves the lemma. □

By Lemma 1, $u \in C^1(U)$, and since $\partial U = \partial B$, it follows that $u$ coincides on $\partial U$ with

$$cG(0, z) - \frac{1}{n} \log \frac{1}{1 - |z|^2},$$

a radial function, up to order one.

Our objective is to prove that $u$ is constant. At this point, we need to introduce the tangent fields

$$R = \sum_{j=1}^n z_j D_j = N + iT, \quad L_{jk} = z_j D_k - z_k D_j, \quad j \neq k.$$
The fields $T$, $\text{Re} L_{jk}$, and $\text{Im} L_{jk}$ span at each point $z$ the tangent space to the sphere of radius $|z|$. We write

$$\Delta_0 = \sum_{i,j=1}^n (\delta_{ij} - z_i \bar{z}_j)D_i \bar{D}_j.$$ 

By direct computation one proves

**Lemma 2.** (a) $\Delta_0 T - T \Delta_0 = 0$.

(b) If $L$ denotes one of the $L_{jk}$, $\Delta_0 L - L \Delta_0 = 2iLT$.

By Lemma 2(a), $Tu$ is also $\mathcal{M}$-harmonic in $U$. But $u$ is radial on $\partial U$ up to order 1; hence, $Tu = 0$ on $\partial U$. By the maximum principle for $\mathcal{M}$-harmonic functions, $Tu \equiv 0$ in $U$.

Lemma 2(b) implies then that each $L_{jk}u$ is also $\mathcal{M}$-harmonic in $U$. As $u$ is radial on $\partial U$ up to order 1, it follows as before that $L_{jk}u \equiv 0$ in $U$ for all $j, k$.

We have thus proved that

$$Tu = L_{jk}u = 0 \quad \text{in} \quad U.$$ 

Let $B'$ be a ball centered at 0 included in $U$; then $u$ is radial in $B'$. Being $\mathcal{M}$-harmonic in $B'$ and radial, it must be constant. By analytic continuation, $u$ is constant, say $k$, in $U$.

Then

$$cg(|z|^2) - \frac{1}{n} \log \frac{1}{1 - |z|^2} = k, \quad z \in \partial U.$$ 

The left-hand side, as a function of $|z|$, increases from $-\infty$ to a maximum value and then decreases again to $-\infty$ when $|z| \to 1$. Therefore, $|z|$ takes at most two values on $\partial U$, and since $U$ was assumed to contain 0, $U$ must be a ball centered at 0.

### 3. Remarks and Questions

(a) The following approximation theorem can be obtained by combining a general result of [6] on the Runge property for functions annihilated by analytic-hypoelliptic operators and [7, 5.5.4] (see also [3] for a different proof).

**Theorem 2.** Assume $K$ is a compact set in $B$ such that $B \setminus K$ is connected. Then every $\mathcal{M}$-harmonic function $f$ in a neighbourhood of $K$ is the uniform limit on $K$ of a sequence of $\mathcal{M}$-harmonic functions in the whole ball $B$ and continuous on $B$.

Hence if $U$ is as in the theorem and, moreover, $B \setminus U$ is connected, then from the assumption

$$f(a) = \frac{1}{\lambda(U)} \int_U f \, d\lambda$$

for all $f$ $\mathcal{M}$-harmonic in $B$ and continuous on $B$ it follows that $U$ is a Bergman ball centered at $a$. We do not know if this holds true when $B \setminus U$ is not connected.
In the case of harmonic functions on Euclidean space, Shapiro [8] provided us an example of a (doubly connected) open set $A$ of $\mathbb{R}^2$ such that $0 \in A$, $\int_A z^n \, dm(z) = 0 \ \forall n \geq 1$, and $A$ is not a disc centered at 0.

(b) The same method proves the following result: let $U$ be as in the theorem, $0 \in U$, and let $\varphi(t)$, $0 \leq t < 1$, be a continuous strictly positive function. Assume that

$$f(0) = \frac{1}{c} \int_U f(\zeta) \varphi(|\zeta|^2) \, dm(\zeta)$$

with $c = \int_U \varphi(|\zeta|^2) \, dm(\zeta)$ for all $\mathcal{H}$-harmonic functions in $U$. Then $U$ is a ball centered at 0. The only difference is that $\frac{1}{n} \log 1/(1 - |z|^2)$ must be replaced by $\Psi(|z|^2)$, the unique radial function vanishing at zero whose invariant Laplacian is $\varphi(|z|^2)(1 - |z|^2)^{n+1}$. It is easy to check that

$$\Psi'(x) = h(x)g'(x),$$

where $h(x) = \int_0^x \varphi(t)t^{n-1} \, dt$. Then the equation $cg - \Psi = k$ has again at most two solutions, because $(cg - \Psi)' = (c - h)g'$ changes sign only once, $h$ being strictly increasing.

(c) In dimension $n = 1$, the invariant Laplacian is $(1 - |z|^2)DD$; hence, the $\mathcal{H}$-harmonic functions are the usual harmonic functions in the plane. A Euclidean disk therefore has the mean-value property, with respect to the Lebesgue measure. In dimension $n > 1$, by remark (b), the balls are the only domains having the property

$$cf(0) = \int_U f(\zeta) \, dm(\zeta), \quad f \text{ $\mathcal{H}$-harmonic in } U, \ 0 \in U.$$

The question arises whether this holds true for another point: what pairs $(a, U)$ $a \in U$ have the property

$$cf(a) = \int_U f(\zeta) \, dm(\zeta), \quad c = m(U), f \text{ $\mathcal{H}$-harmonic in } U?$$

Does it follow that $a = 0$ (and so $U$ is a ball)?

(d) The theorem in the case of harmonic function in $\mathbb{R}^n$ is still true if $U$ is of finite area but not necessarily bounded. In our case, is our theorem still true if $U$ is no more relatively compact in $B$?

References


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