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PATHWISE CONNECTIVITY OF THE SPATIAL NUMERICAL RANGE

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ABSTRACT. In this paper we prove that the spatial numerical range of a given operator on a *separable* Banach space is pathwise connected.

In 1977 Luna [1] gave a partial answer to a question of Bonsall and Duncan [2] by proving that the spatial numerical range of a bounded linear operator is pathwise connected in the case of a *reflexive* Banach space. Later in 1985 Weigel [3] simplified considerably Luna's proof. In the present paper we give a nearly elementary proof of the same fact in the case of a *separable* Banach space.

Let E be a complex Banach space, and let E^* be the conjugate space of E. We set, for $x \in E$,

$$D(x) = \{y \in E^*;\ \|y\| = \|x\| \text{ and } y(x) = \|x\|^2\}.$$

The set-valued mapping $x \mapsto D(x)$ is usually called the *duality mapping* of E. The set $D(x)$ is a nonempty convex and w^*-compact subset of E^*. The *graph* of D is defined to be the set

$$\Gamma = \Gamma_E = \{(x, y);\ y \in D(x)\}.$$

Theorem 1. *Let E be a separable Banach space. Then the graph Γ_E of the duality mapping of E is a pathwise connected set.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in \Gamma$. Without real loss of generality we may assume that $\|x_1\| = \|x_2\| = 1$. Let $V = \bigvee\{x_1, x_2\}$ be the subspace of E generated by x_1, x_2, and let $s(X)$ be the unit sphere of a Banach space X. It is sufficient to prove that the set $G = \{(x, y);\ x \in s(V),\ y \in D(x)\}$ is pathwise connected.

Claim 1. *G is compact.* Let $((x_i, y_i))_{i\in I}$ be a net in G. There is a subnet J of I such that $(x_j)_{j\in J}$ converges in norm to some element x. We can choose a subnet K of J for which $(y_k)_{k\in K}$ converges in w^*-topology to some y. We must show that $y \in D(x)$. But we have

$$0 \leq \|y(x) - 1\| \leq \|y(x) - y_k(x)\| + \|y_k(x) - y_k(x_k)\| \to 0.$$

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Claim 2. *G is connected.* This can be proved as in [2].

Claim 3. *G is a matrizable continuum.* Since E is separable, the w^*-topology on $s(E^*)$ is metrizable; hence, our statement follows from the Claims 1 and 2.

Claim 4. *G is locally connected.* Note that $s(V)$ is locally connected. Indeed, since $\dim V < \infty$, $s(V)$ is homeomorphic to the unit sphere of a finite-dimensional Hilbert space, which is, obviously, locally connected. Now let $(x, y) \in G$ and U be a neighbourhood of (x, y). Then $U \supset (U_1 \times U_2) \cap G$, where U_1 is a neighbourhood of x and

$$U_2 = \{y' \in E^* ;\ |(y' - y)(x_n)| < \varepsilon\},$$

$x_n \in E$, $n = 1, 2, \ldots, m$, is a convex neighbourhood of y. Since $s(V)$ is locally connected and connected, the set of closed connected neigbourhoods of x forms a base at the point x, so we may assume that U_1 is a closed connected set. But then $(U_1 \times U_2) \cap G$ is connected, which can be proved in the same way as in [2].

Claim 5. *G is pathwise connected.* This follows from Claims 3 and 4 and [4, 6.3.11].

The proof is completed. □

We do not know whether it is possible to omit the separability condition.

As a corollary we get the following:

Theorem 2. *Let E be a separable Banach space, and let T be a bounded linear operator on E. Then the spatial numerical range of T, i.e., the set*

$$W(T) = \{y(Tx);\ \|x\| = \|y\| = y(x) = 1\},$$

is pathwise connected.

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