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AN OPERATOR BOUND RELATED TO FEYNMAN-KAC FORMULAE

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ABSTRACT. Those Fourier matrix multiplier operators which are convolutions with respect to a matrix valued measure are characterised in terms of an operator bound. As an application, the finite-dimensional distributions of the process associated with Dirac equation are shown to be unbounded on the algebra of cylinder sets.

In its traditional incarnation, the Feynman-Kac formula is a means of expressing a perturbation to the heat semigroup in terms of an integral with respect to Wiener measure. It has been useful in proving estimates in quantum physics [Si], and an analogue of the formula is an important tool of quantum field theory [G-J]. Perturbations of the groups of operators associated with certain classes of hyperbolic differential equations can also be represented in terms of integrals with respect to σ-additive operator valued measures along the lines of the Feynman-Kac formula [I2], [J1]. To establish the existence of σ-additive operator valued measures associated with particular evolution equations, the following question arises. Suppose that Σ is a locally compact abelian group with a given Haar measure λ. Let $n = 1, 2, \dots$ and let $T : L^2(\Sigma, \mathbb{C}^n) \to L^2(\Sigma, \mathbb{C}^n)$ be a Fourier matrix multiplier operator acting on the space $L^2(\Sigma, \mathbb{C}^n)$ of all (λ-equivalence classes of) functions square integrable with respect to λ and with values in $\mathbb{C}^n$. This means that if the Fourier transform of a function $g \in L^2(\Sigma, \mathbb{C}^n)$ is denoted by $\hat{g}$, then there exists a bounded Borel measurable function $\Phi_T : \Gamma \to \mathscr{L}(\mathbb{C}^n)$ from the group Γ dual to Σ into the space of linear maps $\mathscr{L}(\mathbb{C}^n)$ on $\mathbb{C}^n$, such that for every $f \in L^2(\Sigma, \mathbb{C}^n)$, the equality $(Tf)\hat{}(\gamma) = \Phi_T(\gamma)\hat{f}(\gamma)$ holds for almost all $\gamma \in \Gamma$. Let Q be the spectral measure acting on $L^2(\Sigma, \mathbb{C}^n)$ of multiplication by the characteristic functions of Borel subsets of Σ.

When is it true that for some $C > 0$, the inequality

$$(1)\qquad \left\| \sum_{j=1}^{k} Q(g_j)TQ(f_j) \right\| \le C \left\| \sum_{j=1}^{k} f_j \otimes g_j \right\|_\infty$$

holds for all bounded scalar valued Borel measurable functions f_j, g_j, $j =$

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$1, \ldots, k$, defined on Σ and all $k = 1, 2, \ldots$? Here $u \otimes v$ denotes the function $(u \otimes v)(x, y) = u(x)v(y)$ and the norm $\|\cdot\|_\infty$ denotes the L^∞-norm with respect to $\lambda \otimes \lambda$ on $\Sigma \times \Sigma$. Another way of stating inequality (1) is that the bilinear map $(f, g) \mapsto Q(g)TQ(f)$, $f, g \in C_0(\Sigma)$, is continuous for the topology of bi-equicontinuous convergence. It is not surprising that (1) holds if and only if the operator T is convolution with respect to a matrix-valued measure; see Theorem 1 below, where the result is formulated for an arbitrary separable Hilbert space H in place of $\mathbb{C}^n$.

To see what the operator bound (1) has to do with the Feynman-Kac formula, suppose that $\Sigma = \mathbb{R}^d$ and let S be a C_0-semigroup of continuous linear operators acting on $L^2(\mathbb{R}^d, \mathbb{C}^n)$. If H is the infinitesimal generator of S, then $S(t)u_0$, $t \geq 0$, is the solution of the initial-value problem $u'(t) = Hu(t)$, $u(0) = u_0$ in the case that u_0 belongs to the domain of H. For example, with $H = 1/2\Delta$ for the Laplacian Δ acting in $L^2(\mathbb{R}^d, \mathbb{C})$, S is the heat semigroup and the corresponding the initial value problem is the heat equation $\partial_t u(x, t) = 1/2\Delta u(x, t)$, $u(\cdot, 0) = u_0$.

Denote the collection of *all* functions $\omega : [0, \infty) \to \mathbb{R}^d$ by Ω. Let $t > 0$, $m = 1, 2, \ldots,$ $0 < t_1 < \cdots < t_m < t$, and suppose that $B_1, \ldots, B_m$ are Borel subsets of $\mathbb{R}^d$. For each subset E of Ω of the form $E = \{\omega \in \Omega : \omega(t_1) \in B_1, \ldots, \omega(t_m) \in B_m\}$, the operator $M_t(E) \in \mathscr{L}\left(L^2(\mathbb{R}^d, \mathbb{C}^n)\right)$ is defined by the formula

$$(2) \qquad M_t(E) = S(t - t_m)Q(B_m)S(t_m - t_{m-1}) \cdots Q(B_2)S(t_2 - t_1)Q(B_1)S(t_1).$$

Then as the times $t_1, \ldots, t_m$, the Borel sets $B_1, \ldots, B_m$ and $m = 1, 2, \ldots$ vary, but t is fixed, the sets E form a semialgebra $\mathscr{S}_t$ of subsets of Ω and the expression (2) defines an additive operator valued set function M_t, defined on $\mathscr{S}_t$ and acting on $L^2(\mathbb{R}^d, \mathbb{C}^n)$. Furthermore, the additivity of the set function M_t ensures that it has a unique extension, also denoted by M_t, to the algebra $\alpha(\mathscr{S}_t)$ of subsets of Ω generated by $\mathscr{S}_t$. The idea of associating an operator valued set function with an arbitray semigroup and a spectral measure is due to I. Kluvánek [K].

Set $X_s(\omega) = \omega(s)$ for all $s \geq 0$ and $\omega \in \Omega$, and for a finite ordered subset $J = \{t_1, \ldots, t_m\}$ of $(0, t)$, put $X_J(\omega) = (X_{t_1}(\omega), \ldots, X_{t_m}(\omega))$. Then $M_t \circ X_J^{-1}(C)$ is defined to be equal to the operator $M_t(X_J^{-1}(C))$ for any set C which is the finite union of product sets.

The following phenomona may occur.

(i) For some $0 < t_1 < t_2 < t$, the set function $M_t \circ X_{\{t_1, t_2\}}{}^{-1}$ is unbounded on the algebra generated by product sets in $\mathbb{R}^d$;

(ii) for all *finite* $J \subseteq [0, t]$, the set function $M_t \circ X_J^{-1}$ is bounded on the algebra generated by product sets, but M_t is unbounded on the algebra $\alpha(\mathscr{S}_t)$;

(iii) the additive set function M_t is bounded on the whole algebra $\alpha(\mathscr{S}_t)$.

Briefly, we can say in the language of random processes that the finite-dimensional distributions of X_s, $s \geq 0$, with respect to M_t are unbounded in case (i), the finite-dimensional distributions of X_s, $s \geq 0$, are bounded in case (ii), but M_t is unbounded (on $\alpha(\mathscr{S}_t)$) and M_t is *bounded* in case (iii). The group $S(t) = e^{i\Delta t/2}$, $t \in \mathbb{R}$, of operators is an example of case (i), $S(t) = e^{z\Delta t/2}$, $t \geq 0$, is an example of case (ii) whenever $z \in \mathbb{C}$, $\Im(z) \neq 0$, and $\Re(z) > 0$, and (iii) occurs if $\Re(z) > 0$ and $\Im(z) = 0$; see [J2] and the references therein. Only in case (iii) is M_t the restriction of a σ-additive operator valued measure defined on the σ-algebra $\sigma(\mathscr{S}_t)$ of cylinder sets generated by $\mathscr{S}_t$.

In the case that the semigroup S commutes with translations, it follows from the characterisation of operator norm inequality (1) that case (ii) obtains if and only if for each $0 \leq s < t$, the operator $S(s)$ is convolution with a matrix valued measure; see Corollary 1.

Section 1 is devoted to the just mentioned characterisation of operators satisfying the inequality (1). In section 2, it is noted that the finite-dimensional distributions associated with the Dirac equation in four space-time dimensions are unbounded, although it is known that case (iii) applies to the Dirac equation in two space-time dimensions [I1],[J1]. More generally, a result from the theory of matrix multipliers [B] enables a characterisation of those hyperbolic systems of partial differential equations with constant matrix coefficients where case (ii) applies; these are the hyperbolic systems treated by T. Ichinose [I2], for which the operator valued set functions M_t are automatically bounded and define σ-additive operator valued measures.

1. An operator norm inequality

Throughout this section, Σ is a locally compact abelian group with a given Haar measure λ. The group dual to Σ is denoted by Γ, and its Haar measure is denoted by λ'. The value $\gamma(\sigma)$ of a character $\gamma \in \Gamma$ at $\sigma \in \Sigma$ is written as $\langle \sigma, \gamma \rangle$. The measure λ' is so normalised that the Fourier-Plancherel formula is valid, that is, for all $f \in L^1(\Sigma)$, set $\hat{f}(\gamma) = \int_\Sigma f(\sigma)\overline{\langle \sigma, \gamma \rangle}\, d\lambda(\sigma)$ for all $\gamma \in \Gamma$; then $f(\sigma) = \int_\Sigma \hat{f}(\gamma)\langle \sigma, \gamma \rangle\, d\lambda'(\gamma)$ for almost all $\sigma \in \Sigma$ if $f \in L^1(\Sigma)$ and $\hat{f} \in L^1(\Gamma)$, and $\|f\|_2 = \|\hat{f}\|_2$ if $f \in L^1(\Sigma) \cap L^2(\Sigma)$.

The Borel σ-algebra of a locally compact Hausdorff space X is denoted by $\mathscr{B}(X)$. The Banach space of all continuous functions on X vanishing at infinity, with the uniform norm, is denoted by $C_0(X)$. The variation of a Borel measure $\mu : \mathscr{B}(X) \to \mathbb{C}$ is denoted by $|\mu|$.

Let $(\Lambda, \mathscr{T}, \mu)$ be a measure space. Suppose that H is a separable Hilbert space with inner product $(\cdot, \cdot)$, antilinear in the second variable, and norm $\|\cdot\|_H$. Let $L^2(\mu, H)$ be the Hilbert space of (μ-equivalence classes of) strongly μ-measurable functions $f : \Lambda \to H$ such that $\|f\|_2 \equiv (\int_\Lambda \|f(\xi)\|_H^2)\, d\mu(\xi))^{1/2}$ is finite. Here "strongly μ-measurable" means the limit μ-a.e. of countably valued H-valued $\mathscr{T}$-measurable functions. Because H is assumed to be separable, it is enough to assume that the function $\xi \mapsto (f(\xi), h)$, $\xi \in \Lambda$, is μ-measurable for each $h \in H$ [D-U, II.1.2], that is, f is *scalarly measurable*. The inner product of $L^2(\mu, H)$ is defined by $(f, g) = \int_\Lambda (f(\sigma), g(\sigma))\, d\mu(\sigma)$, $f, g \in L^2(\mu, H)$. For the applications of section two, H is $\mathbb{C}^n$. The space of continuous linear operators on H is denoted by $\mathscr{L}(H)$; it is equipped with the topology of strong operator convergence.

We now extend some notions from commutative harmonic analysis to the vector valued setting. The Fourier-Plancherel formula also applies to $L^2(\Sigma, H)$, because $L^2(\Sigma, H)$ is isomorphic to the Hilbert space tensor product of $L^2(\Sigma)$ and H so that the Fourier transform $\hat{\ }: L^2(\Sigma, H) \to L^2(\Gamma, H)$ is an isometry.

A continuous linear operator $T : L^2(\Sigma, H) \to L^2(\Sigma, H)$ is called a *Fourier* *multiplier operator* if there exists a function $\Phi : \Gamma \to \mathscr{L}(H)$, such that for each $h \in H$, the function $\Phi h : \gamma \mapsto \Phi(\gamma)h$, $\gamma \in \Gamma$, is strongly λ'-measurable, Φ is λ'-essentially bounded in the operator norm of $\mathscr{L}(H)$ and for each $f \in L^2(\Sigma, H)$, the equality $(Tf)\hat{}(\gamma) = \Phi(\gamma)\hat{f}(\gamma)$ holds for λ'-almost all $\gamma \in \Gamma$. The separability of H ensures that the function $\gamma \mapsto \|\Phi(\gamma)\|$, $\gamma \in \Gamma$, is measurable and that $T : L^2(\Sigma, H) \to L^2(\Sigma, H)$ is a Fourier multiplier operator if and only if it commutes with each unitary operator U_τ, $\tau \in \Sigma$, sending $f \in L^2(\Sigma, H)$ to the function $\sigma \mapsto f(\sigma\tau)$, $\sigma \in \Sigma$. In the case that $H = \mathbb{C}^n$, the values of Φ may be viewed as $(n \times n)$-matrices relative to the standard basis of $\mathbb{R}^n$ and for $f \in L^2(\Sigma, \mathbb{C}^n)$, $(\Phi(\gamma)\hat{f}(\gamma))_j = \sum_{k=1}^n \Phi(\gamma)_{jk}\hat{f}(\gamma)_k$, for each $j = 1, \dots, n$.

Lemma 1. *For any function $\Phi : \Gamma \to \mathscr{L}(H)$, such that for each $h \in H$, the function $\Phi h : \gamma \mapsto \Phi(\gamma)h$, $\gamma \in \Gamma$, is strongly λ'-measurable and Φ is λ'-essentially bounded in the operator norm of $\mathscr{L}(H)$, there exists a unique bounded linear operator $T_\Phi : L^2(\Sigma, H) \to L^2(\Sigma, H)$ such that for each $f \in L^2(\Sigma, H)$, the equality $(T_\Phi f)\hat{}(\gamma) = \Phi(\gamma)\hat{f}(\gamma)$ holds for almost all $\gamma \in \Gamma$. The operator norm of T_Φ is equal to* $\operatorname{ess.sup}_{\gamma\in\Gamma}\|\Phi(\gamma)\|$.

Proof. The function $\gamma \mapsto \Phi(\gamma)g(\gamma)$, $\gamma \in \Gamma$, is strongly measurable for each $g \in L^2(\Gamma, H)$, because if s_n, $n = 1, 2, \dots$, are countably H-valued measurable functions converging off a λ'-null set N to g, then for all $\gamma \in \Sigma \setminus N$, $\lim_{n\to\infty}\Phi(\gamma)s_n(\gamma) = \Phi(\gamma)g(\gamma)$, because $\Phi(\gamma)$ is a continuous linear operator on H; moreover, each of the functions $\gamma \mapsto \Phi(\gamma)s_n(\gamma)$, $\gamma \in \Gamma$, $n = 1, 2, \dots$, is strongly λ'-measurable and so is their pointwise limit.

Let $\|\Phi\|_\infty$ denote the essential supremum $\operatorname{ess.sup}_{\gamma\in\Gamma}\|\Phi(\gamma)\|$. If $f \in L^2(\Sigma, H)$, then $\|\Phi\hat{f}\|_2 \le \|\Phi\|_\infty\|\hat{f}\|_2$. By the Plancherel theorem, the equality $(T_\Phi f)\hat{}(\gamma) = \Phi(\gamma)\hat{f}(\gamma)$ for almost all $\gamma \in \Gamma$ defines an element $T_\Phi f$ of $L^2(\Sigma, H)$ such that $\|T_\Phi f\|_2 = \|\Phi\hat{f}\|_2 \le \|\Phi\|_\infty\|\hat{f}\|_2 = \|\Phi\|_\infty\|f\|_2$, so the mapping $f \mapsto T_\Phi f$, $f \in L^2(\Sigma, H)$, is a bounded linear operator T_Φ on $L^2(\Sigma, H)$ with norm at most $\|\Phi\|_\infty$.

To see that the norm $\|T_\Phi\|$ of T_Φ is actually equal to the essential supremum $\|\Phi\|_\infty$, let u_j, $j = 1, 2, \dots$, be a countable dense subset of the closed unit ball of H. For all $u, v \in H$, set $\Phi_{u,v}(\gamma) = (\Phi(\gamma)u, v)$ for every $\gamma \in \Gamma$. Let $\epsilon > 0$, and for each $j, k = 1, 2, \dots$, set $A_{j,k} = \{\gamma \in \Gamma : |\Phi_{u_j,v_k}(\gamma)| > \|\Phi\|_\infty - \epsilon\}$. If $\xi \in \cap_{j,k}A^c_{j,k}$, then $\|\Phi(\xi)\| \le \|\Phi\|_\infty - \epsilon$, because $\Phi(\xi)$ is a continuous linear operator on H. By the definition of $\|\Phi\|_\infty$, the set $\cap_{j,k}A^c_{j,k}$ cannot be a set of full λ'-measure. Consequently, there exists $j, k = 1, 2, \dots$ such that the set $A_{j,k}$ is not λ'-null, so $\|\Phi_{u_j,v_k}\|_\infty > \|\Phi\|_\infty - \epsilon$. The equality $\sup_{\|u\|,\|v\|\le 1}\|\Phi_{u,v}\|_\infty = \|\Phi\|_\infty$ now follows, because for all $u, v \in H$ with $\|u\|, \|v\| \le 1$, $\|\Phi_{u,v}\|_\infty \le \|\Phi\|_\infty$. For each $u, v \in H$, define $(T_\Phi u, v) : L^2(\Sigma) \to L^2(\Sigma)$ by

$$\big((T_\Phi u, v)f\big)(\sigma) = (T_\Phi(uf)(\sigma), v), \qquad f \in L^2(\Sigma).$$

From the scalar case, $\|T_\Phi\| = \sup_{\|u\|,\|v\|\le 1}\|(T_\Phi u, v)\| = \sup_{\|u\|,\|v\|\le 1}\|\Phi_{u,v}\|_\infty = \|\Phi\|_\infty$, so the equality $\|T_\Phi\| = \|\Phi\|_\infty$ is valid. □

Of particular interest is the following special class of Fourier multiplier operators on $L^2(\Sigma, H)$. Suppose that $T : L^2(\Sigma, H) \to L^2(\Sigma, H)$ is a bounded linear operator for which there exists a regular operator valued measure μ :

$\mathscr{B}(\Sigma) \to \mathscr{L}(H)$, such that for every $u, v \in H$ and $f \in L^2(\Sigma)$, the equality $(T(uf)(\sigma), v) = \int_\Sigma f(\sigma\tau^{-1})\, d(\mu u, v)(\tau)$ holds for λ-almost all $\sigma \in \Sigma$. Here $(\mu u, v)$ is the scalar measure $(\mu u, v) : A \mapsto (\mu(A)u, v), A \in \mathscr{B}(\Sigma)$, and the regularity of μ means that for every $u \in H, A \in \mathscr{B}(\Sigma)$, and $\epsilon > 0$, there exists a compact subset K of A such that $\|\mu(B)u\| < \epsilon$ for all $B \subseteq A \setminus K$. A reference for integration with respect to vector measures is [D-U]. The set of all finite linear combinations of H-valued functions uf is dense in $L^2(\Sigma, H)$, so μ uniquely determines T. The operator T is said to be the operator of *convolution with respect to* μ.

For any regular operator valued measure $\mu : \mathscr{B}(\Sigma) \to \mathscr{L}(H)$, the continuous operator valued function $\Phi : \Gamma \to \mathscr{L}(H)$ defined by $\Phi(\gamma) = \int_\Sigma \overline{\langle \sigma, \gamma \rangle}\, d\mu(\sigma)$ is called the *Fourier-Stieltjes transform* of μ. The *total variation* $V(\mu)$ of a measure $\mu : \mathscr{B}(\Sigma) \to \mathscr{L}(H)$ is the (possibly infinite) supremum of the numbers $\sum_{j=1}^n \|\mu(A_j)\|$ over all partitions $A_j, j = 1, \dots, n, n = 1, 2, \dots$, of Σ by Borel sets. If $V(\mu) < \infty$, then μ is said to have *finite variation.* For a finite-dimensional Hilbert space H, all $\mathscr{L}(H)$-valued measures have finite variation, but if H is, say, $L^2([0, 1])$, then the spectral measure of multiplication by characteristic functions has infinite variation.

The next proposition ensures that any regular $\mathscr{L}(H)$-valued measure μ uniquely defines a bounded linear operator on $L^2(\Sigma, H)$ of convolution with respect to μ.

Proposition 1. *Suppose that $T : L^2(\Sigma, H) \to L^2(\Sigma, H)$ is the operator of convolution with respect to a regular operator valued measure $\mu : \mathscr{B}(\Sigma) \to \mathscr{L}(H)$. Then T is the Fourier multiplier operator T_Φ for the Fourier-Stieltjes transform Φ of μ. Conversely, if Φ is the Fourier-Stieltjes transform of a regular operator valued measure $\mu : \mathscr{B}(\Sigma) \to \mathscr{L}(H)$, then T_Φ is the operator of convolution with respect to μ.*

Proof. Let $u, v \in H$ and $f \in L^2(\Sigma)$. Suppose first that T is the operator convolution with μ. The scalar measure $B \mapsto (\mu(B)u, v), B \in \mathscr{B}(\Sigma)$, is denoted by $(\mu u, v)$. Then $(T(uf), v) = f * (\mu u, v)$, so for almost all $\gamma \in \Gamma, (T(uf), v)\hat{}(\gamma) = ([T(uf)]\hat{}(\gamma), v) = \Phi_{u,v}(\gamma)\hat{f}(\gamma)$. Here $\Phi_{u,v}$ is the Fourier-Stieltjes transform of the scalar measure $(\mu u, v)$, so for every $\gamma \in \Gamma, \Phi_{u,v}(\gamma) = (\Phi(\gamma)u, v)$ for the Fourier-Stieltjes transform Φ of μ. The equality $T(uf) = T_\Phi(uf)$ therefore holds for all $u \in H$ and $f \in L^2(\Sigma)$, so equality holds for the closure in $L^2(\Sigma, H)$ of linear combinations of all such functions uf, namely, on the whole of $L^2(\Sigma, H)$.

Conversely, suppose that Φ is the Fourier-Stieltjes transform of μ and let $u, v \in H$ and $f \in L^2(\Sigma)$. According to Lemma 1, the $\mathscr{L}(H)$-valued function Φ defines the bounded linear operator $T_\Phi : L^2(\Sigma, H) \to L^2(\Sigma, H)$. Then for almost all $\gamma \in \Gamma$,

$$([T_\Phi(uf)]\hat{}(\gamma), v) = (T_\Phi(uf), v)\hat{}(\gamma) = \Phi_{u,v}(\gamma)\hat{f}(\gamma) = (f * (\mu u, v))\hat{}.$$

It follows that T_Φ is convolution with μ. $\square$

The following are examples of continuous linear maps $T : L^2(\Sigma, H) \to L^2(\Sigma, H)$, each of which is the operator of convolution with respect to an operator valued measure with infinite variation.

Example 1. (i) Set $H = L^2(\mathbb{R})$. Then $L^2(\mathbb{R}, H)$ can be identified with $L^2(\mathbb{R}^2)$ by identifying the H-valued function $x \mapsto g(x)h$, $x \in \mathbb{R}$ with $(x, y) \mapsto g(x)h(y)$, $x, y \in \mathbb{R}^2$. Let $Q : \mathscr{B}(\mathbb{R}) \to \mathscr{L}(H)$ be defined by $Q(B)f = \chi_B f$, for every $B \in \mathscr{B}(\mathbb{R})$ and every $f \in L^2(\mathbb{R}, H)$. Then the operator $T : L^2(\mathbb{R}, H) \to L^2(\mathbb{R}, H)$ of convolution with Q corresponds to the isometry $f \mapsto f \circ u$, $f \in L^2(\mathbb{R}^2)$ with

$$u(x, y) = (x - y, y), \qquad x, y \in \mathbb{R}.$$

The measure Q has infinite variation.

(ii) The following example is more interesting. Let $F_t(x) = \frac{1}{i\pi}\operatorname{sgn} x . e^{-|x|t}$ for all $x \in \mathbb{R}$ and $t \geq 0$. Denote by $\widehat{F}_t(D)$ the operator of convolution with F_t on $L^2(\mathbb{R})$ for each $t > 0$. The notation is motivated by considering the Fourier transform

$$(\widehat{F}_t(D)f)\widehat{\ }(\xi) = \widehat{F}_t(\xi)\hat{f}(\xi)$$

for $f \in L^2(\mathbb{R})$ and almost all $\xi \in \mathbb{R}$. The operator of multiplication of the Fourier transform by ξ corresponds to $D = \frac{1}{i}\frac{d}{dx}$, so $\widehat{F}_t(D)$ is actually the operator obtained from the bounded function $\widehat{F}_t$ by the functional calculus for the selfadjoint operator D. The function $t \mapsto \widehat{F}_t(D)$, $t \geq 0$, is integrable in $\mathscr{L}(H)$, so let

$$\mu(A) = \int_{A\cap[0,\infty)} \widehat{F}_t(D)\,dt \quad \text{for all } A \in \mathscr{B}(\mathbb{R}).$$

The Fourier-Stieltjes transform Φ of μ is the bounded operator valued function $\xi \mapsto \int_0^\infty e^{-i\xi t}\widehat{F}_t(D)\,dt$, $\xi \in \mathbb{R}$, and according to Proposition 1, the Fourier multiplier operator T_Φ is the operator of convolution with respect to μ. The measure μ has infinite variation.

Note that $\mu([0, \infty))$ is the Hilbert transform on $L^2(\mathbb{R})$. The details concerning this example, and others constructed from singular integral operators on $L^2(\mathbb{R})$, may be gleaned from [J-O].

The next result gives a characterisation of Fourier multiplier operators which are convolutions with respect to an $\mathscr{L}(H)$-valued measure.

Theorem 1. *Let Σ be a locally compact abelian group, H a separable Hilbert space, and suppose that $T : L^2(\Sigma, H) \to L^2(\Sigma, H)$ is a Fourier multiplier operator. Then T is the operator of convolution with respect to a regular operator-valued measure $\mu : \mathscr{B}(\Sigma) \to \mathscr{L}(H)$ if and only if there exists a constant $C > 0$ such that for every $\phi, \psi \in L^2(\Sigma)$ and $u, v \in H$,*

$$\left|\sum_{j=1}^{k}(T(f_j\phi u), \overline{g_j}\psi v)\right| \leq C\|u\|_H\|v\|_H\|\phi\|_2\|\psi\|_2\left\|\sum_{j=1}^{k} f_j \otimes g_j\right\|_\infty \tag{3}$$

for all $f_j, g_j \in L^\infty(\Sigma)$, $j = 1, \ldots, k$, and $k = 1, 2, \ldots$.

Proof. Suppose that T is convolution with respect to an $\mathscr{L}(H)$-valued measure μ. Let $\phi, \psi \in L^2(\Sigma)$, $u, v \in H$, and let $\alpha = \phi u$, $\beta = \psi v$ be elements of

$L^2(\Sigma, H)$. Then

$$\begin{aligned}
&\left|\sum_{j=1}^{k}(T(f_j\alpha), \overline{g_j}\beta)\right| \\
&\quad = \left|\sum_{j=1}^{k}\int_\Sigma\left[\int_\Sigma (f_j\phi)(\sigma\tau^{-1})\, d(\mu u, v)(\tau)\right](g_j\overline{\psi})(\sigma)\, d\lambda(\sigma)\right| \\
&\quad = \left|\sum_{j=1}^{k}\int_\Sigma\left[\int_\Sigma (f_j\phi)(\sigma\tau^{-1})(g_j\overline{\psi})(\sigma)\, d\lambda(\sigma)\right]\, d(\mu u, v)(\tau)\right| \\
&\quad \le \left\|\sum_{j=1}^{k} f_j\otimes g_j\right\|_\infty \int_\Sigma\left[\int_\Sigma |\phi(\sigma\tau^{-1})||\psi(\sigma)|\, d\lambda(\sigma)\right]\, d|(\mu u, v)|(\tau) \\
&\quad \le \|\mu\|.\|u\|_H\|v\|_H\|\,|\phi| * |\psi|\,\|_\infty\left\|\sum_{j=1}^{k} f_j\otimes g_j\right\|_\infty \\
&\quad \le \|\mu\|.\|u\|_H\|v\|_H\|\phi\|_2\|\psi\|_2\left\|\sum_{j=1}^{k} f_j\otimes g_j\right\|_\infty
\end{aligned}$$

by the Fubini-Tonelli theorem. Here the semivariation [D-U, I.1.11] $\|\mu\|$ is defined by the formula

$$\|\mu\| = \sup\{|(\mu p, q)|(\Sigma) : p, q \in H, \|p\|_H \le 1, \|q\|_H \le 1\}.$$

Now suppose that (3) holds and that T is a Fourier multiplier operator. Let $\phi, \psi \in L^2(\Sigma), u, v \in H$, and set $\alpha = \phi u$, $\beta = \psi v$. Then there exists an essentially bounded operator valued function $\Phi : \Sigma \to \mathscr{L}(H)$ such that $T = T_\Phi$.

The collection of all complex functions $\sum_{j=1}^k f_j \otimes g_j$, $f_j, g_j \in C_0(\Sigma)$, $j = 1, \dots, k$, $k = 1, 2, \dots$, is dense in $C_0(\Sigma\times\Sigma)$ by the Stone-Weierstrass theorem. The inequality (3) therefore shows that there exists a unique continuous linear map $\Psi_{\alpha,\beta} : C_0(\Sigma\times\Sigma) \to \mathbb{C}$ such that $\Psi_{\alpha,\beta}(f\otimes g) = (T(f\alpha), \overline{g}\beta)$ for all $f, g \in C_0(\Sigma)$ and $\|\Psi_{\alpha,\beta}\| \le C\|\alpha\|_2\|\beta\|_2$. The Riesz representation theorem ensures that there exists a unique regular Borel measure $\nu_{\alpha,\beta} : \mathscr{B}(\Sigma\times\Sigma) \to \mathbb{C}$ such that $\Psi_{\alpha,\beta}(\Lambda) = \int_{\Sigma\times\Sigma}\Lambda(\sigma, \tau)\, d\nu_{\alpha,\beta}(\sigma, \tau)$ for all $\Lambda \in C_0(\Sigma\times\Sigma)$. Another appeal to the Riesz representation theorem shows that there exists a unique regular Borel measure $\mu_{\alpha,\beta} : \mathscr{B}(\Sigma) \to \mathbb{C}$ such that

$$\int_\Sigma \Lambda(\sigma)\, d\mu_{\alpha,\beta}(\sigma) = \int_{\Sigma\times\Sigma}\Lambda(\tau\sigma^{-1})\, d\nu_{\alpha,\beta}(\sigma, \tau)$$

for all $\Lambda \in C_0(\Sigma)$ and whose total variation $|\mu_{\alpha,\beta}|(\Sigma)$ is bounded by $C\|\alpha\|_2\|\beta\|_2$. Let $w : \Sigma \to \mathbb{C}$ be a function such that $\hat{w} \in L^1(\Gamma)$. Then $w \in C_0(\Sigma)$ and by

Fubini's theorem and the Plancherel formula,

$$\begin{aligned}
\int_\Sigma w(\sigma)\,d\mu_{\alpha,\beta}(\sigma) &= \int_{\Sigma\times\Sigma} w(\tau\sigma^{-1})\,d\nu_{\alpha,\beta}(\sigma,\tau)\\
&= \int_{\Sigma\times\Sigma}\left[\int_\Gamma \langle\tau\sigma^{-1},\gamma\rangle\hat{w}(\gamma)\,d\lambda'(\gamma)\right]d\nu_{\alpha,\beta}(\sigma,\tau)\\
&= \int_\Gamma\left[\int_{\Sigma\times\Sigma}\langle\tau\sigma^{-1},\gamma\rangle\,d\nu_{\alpha,\beta}(\sigma,\tau)\right]\hat{w}(\gamma)\,d\lambda'(\gamma)\\
&= \int_\Gamma\left[\int_{\Sigma\times\Sigma}\langle\tau,\gamma\rangle\overline{\langle\sigma,\gamma\rangle}\,d\nu_{\alpha,\beta}(\sigma,\tau)\right]\hat{w}(\gamma)\,d\lambda'(\gamma)\\
&= \int_\Gamma (T(\overline{\gamma}\alpha),\overline{\gamma}\beta)\hat{w}(\gamma)\,d\lambda'(\gamma)\\
&= \int_\Gamma\left[\int_\Gamma(\Phi(\xi)(\overline{\gamma}\alpha)\hat{}(\xi),(\overline{\gamma}\beta)\hat{}(\xi))\,d\lambda'(\xi)\right]\hat{w}(\gamma)\,d\lambda'(\gamma)\\
&= \int_\Gamma\left[\int_\Gamma(\Phi(\xi)\hat{\alpha}(\xi\gamma),\hat{\beta}(\xi\gamma))\hat{w}(\gamma)\,d\lambda'(\gamma)\right]d\lambda'(\xi)\\
&= \int_\Gamma\left[\int_\Gamma(\Phi(\xi)\hat{\alpha}(\xi\gamma^{-1}),\hat{\beta}(\xi\gamma^{-1}))\hat{w}(\gamma^{-1})\,d\lambda'(\gamma)\right]d\lambda'(\xi).
\end{aligned}$$

Hence, the absolute value of the last integral is bounded by $C\|w\|_\infty\|\alpha\|_2\|\beta\|_2$.

Now let $\mathscr{U}$ denote the family of neighbourhoods of the identity in Γ directed by inclusion. If $f_U \geq 0$ vanishes off $U \in \mathscr{U}$ and satisfies $\int_\Sigma f_U\,d\lambda = 1$, then $\{f_U\}_{U\in\mathscr{U}}$ is an approximate unit for $L^1(\Gamma)$. For each $U \in \mathscr{U}$, let $g_U \in L^2(\Sigma)$ be the essentially unique function satisfying $\hat{g}_U = f_U^{1/2}$. If we set $\phi = g_U$ and $\psi = R\overline{g_U}$, then $\|\alpha\|_2 = \|u\|_H$ and $\|\beta\|_2 = \|v\|_H$ and the integral above becomes $\int_\Gamma(\Phi(\xi)u, v)f_U*(R\hat{w})(\xi)\,d\lambda'(\xi)$, where R is the inversion operator given by $Rp(\gamma) = p(\gamma^{-1})$, for all measurable functions p and $\gamma \in \Gamma$.

Because $\lim_{U\in\mathscr{U}} f_U*\hat{w} = \hat{w}$ in $L^1(\Gamma)$, it follows that

$$\left|\int_\Gamma(\Phi(\xi)u, v)\hat{w}(\xi^{-1})\,d\lambda'(\xi)\right| \leq C\|w\|_\infty\|u\|_2\|v\|_2.$$

Moreover, the measures $\mu_{\alpha,\beta}$ converge in the total variation norm to a measure $\mu_{u,v}$ satisfying $|\mu_{u,v}|(\Sigma) \leq C\|u\|_H\|v\|_H$. Consequently, there exists an operator valued measure $\mu:\mathscr{B}(\Sigma)\to\mathscr{L}(H)$ such that $(\mu u, v) = \mu_{u,v}$ for all $u, v \in H$. The σ-additivity of μ in the strong operator topology of $\mathscr{L}(H)$ follows from the Orlicz-Pettis theorem [D-U, I.4.4]. The regularity of μ follows from the regularity of $\mu_{u,v}$ for all $u, v \in H$ and [D-U, I.2.4].

The identity $\int_\Gamma \Phi(\xi^{-1})\hat{w}(\xi)\,d\lambda'(\xi) = \int_\Sigma w(\sigma)\,d\mu(\sigma)$ is valid for all $w:\Sigma\to\mathbb{C}$ such that $\hat{w} \in L^1(\Gamma)$, so by the Plancherel formula, $\int_\Gamma\Phi(\xi^{-1})u(\xi)\,d\lambda'(\xi) = \int_\Sigma \check{u}(\sigma)\,d\mu(\sigma)$ for the inverse Fourier transform $\check{u}$ of a function $u \in L^1(\Gamma)$. By Fubini's theorem, it follows that Φ is the Fourier-Stieltjes transform of the regular operator valued measure $\mu:\mathscr{B}(\Sigma)\to\mathscr{L}(H)$. An appeal to Proposition 1 establishes that T is convolution with μ. □

Corollary 1. *Let Σ be a locally compact abelian group, H a separable Hilbert space, and $\mu:\mathscr{B}(\Sigma)\to\mathscr{L}(H)$ a regular $\mathscr{L}(H)$-valued measure with finite variation. Let Q be the spectral measure of multiplication by Borel subsets of*

Σ, acting on $L^2(\Sigma, H)$. Suppose that $T : L^2(\Sigma, H) \to L^2(\Sigma, H)$ is the operator of convolution with respect to μ. Then there exists a constant $C > 0$ such that

$$\left\| \sum_{j=1}^{k} Q(g_j)TQ(f_j) \right\| \leq C \left\| \sum_{j=1}^{k} f_j \otimes g_j \right\|_\infty \tag{4}$$

for all $f_j, g_j \in L^\infty(\Sigma)$, $j = 1, \ldots, k$, and $k = 1, 2, \ldots$.

Proof. The proof is along the lines of the first part of the proof of Theorem 1. Suppose that ξ_j, $j = 1 \ldots, k$, are the characteristic functions of pairwise disjoint Borel subsets of Σ; similarly for ζ_j, $j = 1, \ldots, k$. Then for $u_j, v_j \in H$, $j = 1, \ldots, k$, $\sum_{j=1}^k u_j\xi_j$ and $\sum_{j=1}^k v_j\zeta_j$ are H-valued simple functions. Some of these vectors may be zero. Then

$$\begin{aligned}
&\left| \sum_{l=1}^{m} \sum_{i,j=1}^{k} (T(u_i f_l \xi_i), v_j \overline{g_l} \zeta_j) \right| \\
&\quad = \left| \sum_{l=1}^{m} \sum_{i,j=1}^{k} \int_\Sigma \left[\int_\Sigma (f_l\xi_i)(\sigma\tau^{-1})(g_l\zeta_j)(\sigma)\, d\lambda(\sigma) \right] d(\mu u_i, v_j)(\tau) \right| \\
&\quad \leq \left\| \sum_{l=1}^{m} f_l \otimes g_l \right\|_\infty \times \sum_{i,j=1}^{k} \int_\Sigma \left[\int_\Sigma |\xi_i(\sigma\tau^{-1})||\zeta_j(\sigma)|\, d\lambda(\sigma) \right] d|(\mu u_i, v_j)|(\tau) \\
&\quad \leq V(\mu) \left\| \sum_{l=1}^{m} f_l \otimes g_l \right\|_\infty \sup_\tau \sum_{i,j=1}^{k} \int_\Sigma |\xi_i(\sigma\tau^{-1})|.\|u_i\|_H |\zeta_j(\sigma)|.\|v_j\|_H\, d\lambda(\sigma) \\
&\quad \leq V(\mu) \left\| \sum_{i=1}^{k} u_i\xi_i \right\|_2 \left\| \sum_{j=1}^{k} v_j\zeta_j \right\|_2 \left\| \sum_{l=1}^{m} f_l \otimes g_l \right\|_\infty .
\end{aligned}$$

Simple functions are dense in $L^2(\Sigma, H)$, so the required estimate follows by continuity. □

Remarks. (i) Clearly, inequality (4) implies (3). Furthermore, if H is finite dimensional, then (4) holds for any $\mathscr{L}(H)$-valued measure, because all such measures automatically have finite variation.

(ii) Inequality (4) does not even imply that for each $h \in H$, the H-valued measure μh has finite variation. For example, if Q is the spectral measure and T is the operator of Example 1 (i), then $\sum_{j=1}^k (Q(g_j)TQ(f_j)\phi)(x, y) = \sum_{j=1}^k f_j(x-y)g_j(x)\phi(x-y, y)$, for all $\phi \in L^2(\mathbb{R}^2)$, so (4) is clearly satisfied. But, as mentioned earlier, Qh has finite variation in H only if h is the zero vector.

2. Evolution equations

The *Dirac operator* is defined in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ by means of the differential expression

$$D = c\sum_{j=1}^{3} \alpha_j p_j + \alpha_4 mc^2,$$

where $c > 0$ is the velocity of light, $m > 0$ is the mass of the particle, $p_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j}$, and

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3, \qquad \alpha_4 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}.$$

Here $\sigma_1, \sigma_2, \sigma_2$ are the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\sigma_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is the 2×2 identity matrix.

Then D defines a selfadjoint operator, and so, a unitary group $S(t) = e^{iDt}$, $t \in \mathbb{R}$, of operators acting on $L^2(\mathbb{R}^3, \mathbb{C}^4)$. The question arises as to whether or not the finite-dimensional distributions of the process defined by formula (2) are bounded.

Proposition 2. *Let $t > 0$ and let M_t be the set function defined by formula (2) for the operators $S(t) = e^{iDt}$, $t \in \mathbb{R}$. Let X_s, $s \geq 0$, be the evaluation maps defined on the space Ω of all paths $\omega : [0, \infty) \to \mathbb{R}^3$. Then for any $0 < t_1 < t_2 < t$, the set function $M_t \circ X^{-1}_{\{t_1, t_2\}}$ is unbounded on the algebra generated by all product sets $A \times B$, with $A, B \in \mathscr{B}(\mathbb{R}^3)$.*

Proof. According to (2), $M_t \circ X^{-1}_{\{t_1, t_2\}}(A \times B) = S(t-t_2)Q(B)S(t_2-t_1)Q(A)S(t_1)$ for all $A, B \in \mathscr{B}(\mathbb{R}^3)$. It is enough to show the unboundedness of the collection of all operators $\sum_{j=1}^k Q(B_j)S(t_2-t_1)Q(A_j)$ with $A_j \times B_j$, $A_j, B_j \in \mathscr{B}(\mathbb{R}^3)$, $j = 1, \dots, k$, $k = 1, 2, \dots$, a partition of $\mathbb{R}^3 \times \mathbb{R}^3$. Suppose otherwise, that for each $u, v \in \mathbb{C}^n$ the set function $m : A \times B \mapsto (Q(B)S(t_2 - t_1)Q(A)u, v)$ is actually the restriction to product sets of a bounded additive set function on the algebra generated by product sets. Then [J1, Proposition 2] shows that m is the restriction of a Borel measure on $\mathbb{R}^3 \times \mathbb{R}^3$, so for $T = S(t_2 - t_1)$, the inequality (3) is certainly satisfied. By Theorem 1, it follows that $S(t_2 - t_1)$ is convolution with respect to a matrix valued measure.

However, if $S(t_2-t_1)$ were convolution with respect to a matrix valued measure μ, then $S(t_2-t_1)$ would map the subspace $L^2(\mathbb{R}^3, \mathbb{C}^4) \cap L^\infty(\mathbb{R}^3, \mathbb{C}^4)$ of $L^2(\mathbb{R}^3, \mathbb{C}^4)$ into itself, continuously for the uniform norm. An elementary proof showing that this is not the case is given in [Z, II.1]. The operator $S(t_2-t_1)$ is actually convolution with respect a matrix valued distribution of order one, explicitly calculated in [R]. This contradiction shows that the original assumption that $M_t \circ X^{-1}_{\{t_1, t_2\}}$ is bounded must be false. The result is also a consequence of Theorem 2 below. □

The matrices $\alpha_1, \alpha_2, \alpha_3$ in the Dirac operator do not commute. It turns out that this is the source of the unboundedness of the finite-dimensional distributions of the associated process.

Suppose that $A_1, \dots, A_d$ are hermitian $n \times n$ matrices and B is any $n \times n$ matrix. Then $H = \sum_{j=1}^d A_j \partial_{x_j} + B$ is the infinitesimal generator of a group S of operators on $L^2(\mathbb{R}^d, \mathbb{C}^n)$. The operator ∂_{x_j} is viewed as the generator of the group of translation operators in the j-th direction, acting on $L^2(\mathbb{R}^d, \mathbb{C}^n)$.

Theorem 2. *Let* $t > 0$ *and let* M_t *be the set function defined by formula* (2) *for the operators* $S(t) = e^{iHt}$, $t \in \mathbb{R}$. *Let* X_s, $s \geq 0$, *be the evaluation maps defined on the space* Ω *of all paths* $\omega : [0, \infty) \to \mathbb{R}^3$. *Then for some* $0 < t_1 < t_2 < t$, *the set function* $M_t \circ X^{-1}_{\{t_1, t_2\}}$ *is bounded on the algebra generated by all product sets* $A \times B$, $A, B \in \mathscr{B}(\mathbb{R}^3)$, *if and only if the matrices* $A_1, \ldots, A_d$ *commute. If the matrices* $A_1, \ldots, A_d$ *do commute, then* M_t *is actually bounded.*

Proof. First suppose that $M_t \circ X^{-1}_{\{t_1, t_2\}}$ is bounded. Then as in the proof of Proposition 2, it follows from Theorem 1 that $S(t_2 - t_1)$ is convolution with respect to a matrix valued measure μ. Moreover, the spectral calculus for self-adjoint operators ensures that $S(t_2 - t_1)$ is the Fourier multiplier operator T_Φ for the matrix valued function

$$\Phi(\xi) = \exp\left(\left(i\sum_{j=1}^{d} A_j\xi_j + B\right)(t_2 - t_1)\right), \qquad \xi \in \mathbb{R}^d.$$

By Proposition 1, Φ is the Fourier-Stieltjes transform of μ.

According to [B, Lemma 1 (iii)], Φ belongs to the space of matrix multipliers on $L^1(\mathbb{R}^d, \mathbb{C}^n)$, and hence, by [B, Lemma 7], so does the function

$$\xi \mapsto \exp\,(i\sum_{j=1}^{d} A_j\xi_j(t_2 - t_1)), \qquad \xi \in \mathbb{R}^d.$$

An appeal to [B, Theorem 1] ensures that $A_1, \ldots, A_d$ commute.

If the matrices $A_1, \ldots, A_d$ commute, then we are in the situation considered in [I2] and [J1, Section 2], and the conclusion follows from the results there. □

The point of Theorem 2 is that the boundedness of the finite-dimensional distributions of the process associated with a symmetric hyperbolic system already forces the system to be of a very restricted type.

References

[B] P. Brenner, *The Cauchy problem for symmetric hyperbolic systems in* L_p, Math. Scand. **19** (1966), 27–37.

[D-U] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Surveys Monographs, vol. 15, Amer. Math. Soc., Providence, RI, 1977.

[G-J] J. Glimm and A. Jaffe, *Quantum physics: A functional integral point of view*, Springer-Verlag, New York, 1981.

[I1] T. Ichinose, *Path integral for the Dirac equation in two space-time dimensions*, Proc. Japan Acad. Ser. A Math. Sci. **58** (1982), 290–293.

[I2] ———, *Path integral for a hyperbolic system of first order*, Duke Math. J. **51** (1984), 1–36.

[J1] B. Jefferies, *Processes associated with evolution equations*, J. Funct. Anal. **91** (1990), 259–277.

[J2] ———, *On the additivity of unbounded set functions*, Bull. Austral. Math. Soc. **45** (1992), 223-236.

[J-O] B. Jefferies and S. Okada, *Pettis integrals and singular integral operators*, Illinois J. Math. **38** (1994).

[K] I. Kluvánek, *Operator valued measures and perturbations of semi-groups*, Arch. Rational Mech. Anal. **81** (1983), 161–180.

[R] G. Rosen, *Feynman path summation for the Dirac equation: An underlying one-dimensional aspect of relativistic particle motion*, Phys. Rev. A (3) **28** (1983), 1139–1140.

[S] L. Schwartz, *Radon measures in arbitrary topological spaces*, Oxford Univ. Press., Tata Institute of Fundamental Research, Bombay, 1973.

[Si] B. Simon, *Functional integration and quantum physics*, Academic Press, New York, San Francisco, and London, 1979.

[Z] T. Zastawniak, *Path integrals for the Dirac equation—some recent developments in the mathematical theory*, Stochastic Analysis, Path Integration and Dynamics (K. D. Elworthy and J-C Zambrini, eds.), Pitman Res. Notes in Math., vol. 200, Longman Sci. Tech., Harlow, 1989, pp. 243–263.

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