

## HECKE ALGEBRA ON HOMOGENEOUS TREES AND RELATIONS WITH TOEPLITZ AND HANKEL OPERATORS

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**ABSTRACT.** We consider the Hecke algebra on homogeneous trees. We prove that it is a maximal abelian subalgebra of some operator algebras if the degree of the tree is greater than 2. There we show the influence of geometry of the tree on that fact. If the degree is 2 (for example, in the case of integers) then we show that operators which commute with the Hecke algebra can be uniquely represented as a sum of Hankel and Toeplitz matrices.

### INTRODUCTION

The aim of this paper is to study some operator algebras which appear naturally on homogeneous trees. One of them, called the *Hecke algebra*, is abelian. We investigate when it is a maximal abelian subalgebra of bigger operator algebras.

In the case of the tree of integers the operators which commute with all Hecke operators are essentially *Toeplitz* and *Hankel* operators. Other homogeneous trees differ from that case in their geometrical structure, and therefore we get the maximality property.

### 1. HOMOGENEOUS TREES

A nice introduction to trees is the book of Serre [S]. We mention here only some basic properties.

A tree  $\Gamma$  consists of the set  $X$  of vertices and the set  $E$  of edges. It does not include any loop; thus, for every two vertices  $x, y \in X$  there exists the only path of minimal *length* (= number of edges included in it) connecting them. By *path* we mean a sequence  $x_1, x_2, \dots, x_m$  of  $m$  different vertices such that  $(x_i, x_{i+1})$  is an edge for  $1 \leq i \leq m-1$ ,  $1 \leq m$ -arbitrary. We say that the tree  $\Gamma$  is *homogeneous* if the number of edges is constant at any vertex. Equivalently, we say that each vertex has the same number of nearest neighbours (= at distance 1). This fixed number one calls the *degree* of the tree—we will denote it by  $\deg(\Gamma) = q$ ,  $q \geq 2$ .

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The homogeneous tree of degree 2 can be obtained by considering either the integer group  $\mathbb{Z}$  or the free product of two copies of  $\mathbb{Z}_2$ . In both cases the set  $X$  of vertices consists of elements of the group and an edge is: in the first case a pair  $(i, j)$  if and only if  $|i - j| = 1$ , while in the second one a pair  $(x, y)$  if and only if  $y^{-1}x$  is one of the generators or their inverses.

The most interesting example is that the free group  $\mathbb{F}_r$  on  $r$  free generators may be thought of as being a homogeneous tree of degree  $2r$ . Therefore, our results apply, in particular, to that group.

An important difference between homogeneous and other trees is that only in the first case do Hecke operators form an algebra, even an abelian one. This algebra was noticed by Cartier [C] and also in the book of Serre—they gave a simple formula for the generating function. This work was motivated by the paper of Pytlik [P] in which he proved that radial functions form a maximal abelian subalgebra in the convolution algebras  $C^p(\mathbb{F}_r)$  on  $l^p(\mathbb{F}_r)$ .

### 2. HECKE ALGEBRA

Let  $d$  denote the combinatorial distance function on the tree  $\Gamma$ : the distance between two vertices is equal to the length of the unique shortest path connecting them. Define

$$\chi_n(x, y) = \begin{cases} 1 & \text{if } d(x, y) = n, \\ 0 & \text{otherwise,} \end{cases}$$

where  $x, y \in X, n \in \mathbb{N} \cup \{0\}$ . We may think of those  $\chi_n(x, y)$  as kernels of operators  $\chi_n$  acting on the space  $K(X)$  of finitely supported functions on  $X$  by the formula:

$$(*) \quad \chi_n(f)(x) = \sum_{y \in X} \chi_n(x, y)f(y) = \sum_{y \in X, d(x, y)=n} f(y).$$

Assume that our tree  $\Gamma$  is homogeneous of degree  $q, q \geq 2$ ; then we have the following well-known multiplication formulas (see [S]):

$$\begin{aligned} \chi_n \circ \chi_0 &= \chi_0 \circ \chi_n = \chi_n \quad \text{for every } n \in \mathbb{N}, \\ \chi_1 \circ \chi_1 &= q\chi_0 + \chi_2, \\ \chi_1 \circ \chi_n &= (q - 1)\chi_{n-1} + \chi_{n+1} \quad \text{for every } n \geq 2. \end{aligned}$$

We will denote by  $H(X)$  the linear span of  $\chi_n, n \in \mathbb{N}$ , on  $X$ . It follows that the algebra of finitely supported radial functions on the free group  $\mathbb{F}_k$  is isomorphic to  $H(X)$  where  $\text{deg}(X) = 2k$  (see [P]). From the above formulas it follows that this algebra, called the *Hecke algebra*, is abelian and generated by  $\chi_1$ . Thus the question of whether some operator with kernel  $\varphi$  commutes with all  $H(X)$  is equivalent to its commuting with  $\chi_1$ . Let us observe that this is further equivalent to the following equation:

$$(**) \quad \forall x, y \in X \quad \sum_{z \in X, d(z, y)=1} \varphi(x, z) = \sum_{t \in X, d(x, t)=1} \varphi(t, y).$$

### 3. GEOMETRIC APPROACH

Let us denote by  $\mathcal{F}(X)$  the algebra of those operators  $\varphi$  on  $K(X)$  whose kernels satisfy

$$\exists m \in \mathbb{N} \quad \varphi(x, y) = 0 \quad \text{if } d(x, y) > m.$$

One sees that  $\mathbf{H}(X) \subseteq \mathcal{F}(X)$  is an abelian subalgebra.

**Theorem 3.1.** *If  $X$  is a homogeneous tree of degree  $q \geq 3$ , then  $\mathbf{H}(X)$  is a maximal abelian subalgebra in  $\mathcal{F}(X)$ ; in other words, if  $\varphi \in \mathcal{F}(X)$  commutes with  $\chi_1$ , then  $\varphi \in \mathbf{H}(X)$ .*

*Proof.* By definition there exists  $m \in \mathbb{N}$  such that  $\varphi(x, y) = 0$  if  $\mathbf{d}(x, y) > m$ . Take arbitrary  $x_0, y_0 \in X$  such that  $\mathbf{d}(x_0, y_0) = m + 1$ , and consider the path  $x_0, x_1, \dots, x_m, y_0$  of length  $m + 1$  connecting  $x_0$  and  $y_0$  (it is unique!). Then the formula (\*\*) written for  $x = x_0, y = y_0$  simplifies to the equality

$$(1) \quad \varphi(x_0, x_m) = \varphi(x_1, y_0)$$

because other summands vanish. Now take two different vertices  $y_1$  and  $z_1$  in the neighbourhood of  $y_0$ , that is,  $\mathbf{d}(y_0, y_1) = \mathbf{d}(y_0, z_1) = 1$ , other than  $x_m$ . This is possible because, by the assumption on  $q$ ,  $y_0$  has at least three neighbour vertices. Let us consider any two paths  $y_0, y_1, \dots, y_m$  and  $y_0, z_1, \dots, z_m$  each of length  $m$ . Then we have  $\mathbf{d}(x_i, y_i) = \mathbf{d}(x_i, z_i) = \mathbf{d}(z_{m+1-i}, y_i) = m + 1$  for  $0 \leq i \leq m$ . Therefore, if we write (\*\*) first with  $x = x_i, y = y_i$ , successively for  $i = 1, 2, \dots, m$ , then with  $x = y_i, y = z_{m+1-i}$ , successively for  $i = m, m - 1, \dots, 1$  and then with  $x = z_i, y = x_i$ , successively for  $i = m, m - 1, \dots, 1$  then we get simple equalities which may be written in the following sequence:

$$(2) \quad \begin{aligned} \varphi(x_0, x_m) &= \varphi(x_1, y_0) = \varphi(x_2, y_1) = \dots = \varphi(x_m, y_{m-1}) \\ &= \varphi(y_0, y_m) = \varphi(z_1, y_{m-1}) = \varphi(z_2, y_{m-2}) \\ &= \dots = \varphi(z_m, y_0) = \varphi(z_{m-1}, x_m) = \varphi(z_{m-2}, x_{m-1}) \\ &= \dots = \varphi(z_1, x_2) = \varphi(y_0, x_1) = \varphi(x_m, x_0). \end{aligned}$$

From these equalities we deduce first that  $\varphi$  is symmetric on the pairs  $(x, y)$  at distance  $m$ . Then we explain how the above procedure implies that  $\varphi$  is constant on the set  $E_m$  of such pairs. To do that let us introduce some geometric interpretation. For a while we will look at the value  $\varphi(x_0, x_m)$  as an arrow laying on the tree and having the beginning point at  $x_0$  and the terminal point at  $x_m$ . Moreover, let each equality of the form (1) say that we can move the arrow from the place determined by the left-hand side to the place determined by the right-hand side. Then (2) says that our arrow can be moved first from  $(x_0, x_m)$  to  $(y_0, y_m)$ , then backward to  $(z_m, y_0)$ , and then forward again to  $(x_m, x_0)$ . If we now take an arbitrary pair  $(x, y)$  such that  $\mathbf{d}(x, y) = m$  then we can move our arrow to that pair as well as to the pair  $(y, x)$ . What it means is that the value of  $\varphi$  is the same at all these pairs, so  $\varphi$  is constant on the set  $E_m$ .

**Remark 3.2.** The above procedure is only possible for  $q \geq 3$  (actually, we need at least one vertex of degree greater than 2), thus the tree of integers is excluded (one cannot turn the arrow back on such a tree!).

It remains to show how induction on  $m$  works. To do that we take an arbitrary pair  $(x, y)$  such that  $\mathbf{d}(x, y) = m$  and write the commutation equality  $(\varphi \circ \chi_1)(x, y) = (\chi_1 \circ \varphi)(x, y)$  like in (\*\*):

$$\sum_{z \in X, \mathbf{d}(z, y)=1} \varphi(x, z) = \sum_{t \in X, \mathbf{d}(x, t)=1} \varphi(t, y).$$

On both sides,  $q$  values of  $\varphi$  are computed on pairs with distance  $m + 1$  between its initial and terminal points, which is constant. Therefore, we can cancel them to get only one term on each side—the values of  $\varphi$  at pairs that have initial and terminal vertices at distance  $m - 1$  similarly like in (1). This allows us to proceed inductively.

Let us mention that this theorem is a special case of what we will prove further by using different methods. Above we have presented which geometric properties are hidden under the assumption that something commutes with the Hecke algebra.

#### 4. INTEGERS

As we have noticed before, in the case of  $q = 2$  the vertices of the 2-homogeneous tree can be identified with the group of integers, so we now assume that  $X = \mathbb{Z}$ . Moreover, we will write  $a_{j,k}$  instead of  $\varphi(j, k)$ . Observe that

$$\chi_1(j, k) = \begin{cases} 1 & \text{if } |j - k| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In this notation “commuting of a matrix  $a$  with  $\chi_1$ ” can be written as:

$$(3) \quad \forall j, k \in \mathbb{Z} \quad a_{j,k-1} + a_{j,k+1} = a_{j-1,k} + a_{j+1,k}.$$

There are two important examples of matrices satisfying (3):

- (a) *Hankel* matrices  $h = (h_{j,k})$ —constant on the diagonals from upper right to lower left; in other words,  $h_{j,k} = u(j + k)$  for some doubly infinite sequence  $u$  of complex numbers;
- (b) *Toeplitz* matrices  $t = (t_{j,k})$ —constant on the diagonals from upper left to lower right; in other words,  $t_{j,k} = v(k - j)$  for some doubly infinite sequence  $v$  of complex numbers.

Let us introduce some Banach space structure on  $\mathcal{F}(\mathbb{Z})$ . Observe that its elements act as bounded operators on each  $l^p(\mathbb{Z})$  for  $p \geq 1$ . In particular, we can consider their simultaneous action on  $l^1(\mathbb{Z})$  and  $l^\infty(\mathbb{Z})$ , which gives the following norm:  $\|a\|$  is the least constant  $C \geq 0$  such that:

- (i)  $\sup_j \sum_k |a_{j,k}| \leq C$ ,
- (ii)  $\sup_k \sum_j |a_{j,k}| \leq C$ .

Let  $\mathcal{Z}_1(\mathbb{Z})$  denote the set of all matrices  $a$  such that  $\|a\| < +\infty$ . By the interpolation argument, it is an algebra acting on  $l^2(\mathbb{Z})$ . Let  $\mathcal{H}_1(\mathbb{Z})$  and  $\mathcal{T}_1(\mathbb{Z})$  denote the subsets in  $\mathcal{Z}_1(\mathbb{Z})$  of Hankel and Toeplitz matrices respectively, and let  $\mathcal{M}_1(\mathbb{Z})$  stand for those elements in  $\mathcal{Z}_1(\mathbb{Z})$  which are constant on the sets  $E_m$  (Hecke subalgebra). Then one can see that  $\mathcal{M}_1(\mathbb{Z})$  is a proper subset of  $\mathcal{T}_1(\mathbb{Z})$  and that  $\mathcal{T}_1(\mathbb{Z}) \cap \mathcal{H}_1(\mathbb{Z}) = \{0\}$ .

If we denote by  $(\mathcal{M}_1)'(\mathbb{Z})$  the commutant of  $\mathcal{M}_1(\mathbb{Z})$  in  $\mathcal{Z}_1(\mathbb{Z})$  then we have the following:

**Theorem 4.1.**  $(\mathcal{M}_1)'(\mathbb{Z}) = \mathcal{H}_1(\mathbb{Z}) \oplus \mathcal{T}_1(\mathbb{Z})$ .

*Proof.* We have to prove that if a matrix  $a$  satisfies (3) then it can be uniquely represented as a sum of a Hankel matrix from  $\mathcal{H}_1(\mathbb{Z})$  and a Toeplitz matrix from  $\mathcal{T}_1(\mathbb{Z})$ . For this purpose we show first that a matrix  $a$  satisfying (3) has limits along its diagonals, equal at plus and minus infinity on the same diagonal; the limits along diagonals from upper right to lower left will form the Hankel

part of  $a$  and the limits along diagonals from upper left to lower right will form its Toeplitz part. Now let us fix a matrix  $a$  in  $\mathcal{X}_1(\mathbb{Z})$  and assume that it satisfies (3). We start with

**Lemma 4.2.** *For every  $k \in \mathbb{Z}$  there exist limits*

$$\begin{aligned} \lim_{n \rightarrow +\infty} a_{-n,n+k} &= \lim_{n \rightarrow -\infty} a_{-n,n+k} = v(k), \\ \lim_{n \rightarrow +\infty} a_{n,n+k} &= \lim_{n \rightarrow -\infty} a_{n,n+k} = u(k). \end{aligned}$$

*Proof of Lemma 4.2.* As a first consequence of (3) we get that the matrix  $\tilde{t}$  defined by  $\tilde{t}_{j,k} = a_{j,k-1} - a_{j+1,k}$  is in  $\mathcal{S}_1(\mathbb{Z})$ . This implies that  $a_{-n+1,n+1+k} - a_{-n,n+k} = a_{-2n+1,k+1} - a_{-2n,k}$ . Moreover,  $\sup_k \sum_j |a_{j,k}| < +\infty$ ; thus, in particular,  $\lim_n a_{-n+1,n+1+k} - a_{-n,n+k} = 0$ . The limit exists when  $n \rightarrow +\infty$  as well as when  $n \rightarrow -\infty$ . Now boundedness of the sequence  $(a_{j,k})$  and convergence of the series  $\sum_{n=-\infty}^{+\infty} (a_{-n+1,n+1+k} - a_{-n,n+k})$  imply that there exists the limit  $\lim_{n \rightarrow +\infty} a_{-n,n+k} = \lim_{n \rightarrow -\infty} a_{-n,n+k} = v(k)$ .

Similarly if one observes that the matrix  $\tilde{h}$  defined by  $\tilde{h}_{j,k} = a_{j,k-1} - a_{j-1,k}$  is in  $\mathcal{H}_1(\mathbb{Z})$  then almost the same argument as above shows that the second type of limits exist. This proves the lemma.

Now we are going to show that the matrices  $t_{j,k} = u(k-j)$  and  $h_{j,k} = v(k+j)$  give the desired decomposition of  $a$ . To prove that  $t, h \in \mathcal{X}_1(\mathbb{Z})$  it suffices to prove that  $u, v \in l^1(\mathbb{Z})$ . But this is true because for fixed  $N > 0$  we have

$$\sum_{k=1}^N |u(k)| = \sum_{k=1}^N \lim_{n \rightarrow +\infty} |a_{-n,n+k}| \leq \limsup_{n \rightarrow +\infty} \sum_{k=1}^{\infty} |a_{-n,n+k}| \leq \|a\|.$$

It is left to check that  $\forall j, k \in \mathbb{Z}, a_{j,k} = u(k-j) + v(k+j)$ . By induction (3) implies that

$$\begin{aligned} a_{0,0} - a_{n,n} &= a_{0,0} - a_{1,1} + a_{1,1} - a_{2,2} + \dots + a_{n-1,n-1} - a_{n,n} \\ &= a_{1,-1} - a_{2,0} + a_{3,-1} - a_{4,0} + \dots + a_{2n-1,-1} - a_{2n,0}, \\ a_{0,0} - a_{n,-n} &= a_{0,0} - a_{1,-1} + a_{2,0} - a_{3,-1} + \dots + a_{2(n-1),0} - a_{2n-1,-1}. \end{aligned}$$

So we get  $a_{0,0} - a_{n,n} = a_{n,-n} - a_{2n,0}$ , and by taking  $\lim_{n \rightarrow +\infty}$  of both sides we obtain  $a_{0,0} - u(0) = v(0)$ . Thus  $a_{0,0} = t_{0,0} + h_{0,0}$ . The same method applies for other  $j, k \in \mathbb{Z}$ . This finishes the proof of the theorem.

*Remark 3.* One can easily prove that  $\mathcal{S}_1(\mathbb{Z})$  is a maximal abelian subalgebra in  $\mathcal{X}_1(\mathbb{Z})$ . Moreover, as we noticed in the beginning of this section, every Toeplitz matrix, in particular from  $\mathcal{F}(\mathbb{Z})$ , commutes with all Hecke matrices. Thus  $H(\mathbb{Z})$  cannot be equal to its commutant in  $\mathcal{F}(\mathbb{Z})$ . From this it follows that Theorem 3.1 is true only if  $q \geq 3$ .

### 5. GENERAL CASE

From now on we will assume that  $\Gamma$  is the homogeneous tree of degree  $q$  with  $q \geq 2$ . Let  $\mathcal{S}_1(X)$  denote the completion of  $\mathcal{F}(X)$  with respect to the following norm:  $\|\varphi\|$  is the least constant  $C \geq 0$  such that:

$$\sup_x \sum_y |\varphi(x, y)| \leq C, \quad \sup_y \sum_x |\varphi(x, y)| \leq C.$$

Observe that  $\varphi \in \mathcal{F}_1(X)$  if and only if

$$\sup_x \sum_{y, \mathbf{d}(x,y) \geq n} |\varphi(x, y)| \rightarrow_n 0, \quad \sup_y \sum_{x, \mathbf{d}(x,y) \geq n} |\varphi(x, y)| \rightarrow_n 0.$$

As an easy consequence of the above observation we get that  $\mathcal{F}_1(X)$  is an algebra. We should stress the difference between  $\mathcal{F}_1(\mathbb{Z})$  and  $\mathcal{Z}_1(\mathbb{Z})$  in the case  $X = \mathbb{Z}$ ; it follows that the inclusion  $\mathcal{F}_1(\mathbb{Z}) \subset \mathcal{Z}_1(\mathbb{Z})$  is proper because, for example, the matrix  $a = (a_{j,k})$  defined by

$$a_{j,k} = \begin{cases} 1 & \text{if } k = 2^j, \\ 0 & \text{otherwise} \end{cases}$$

is in the difference.

*Remark.* We would like to point out that in the book [R] of Renault it is proved that (see Proposition 4.7)  $C^*(G^o)$  is a maximal abelian subalgebra of  $C_{red}^*(G)$  for a grupoid  $G$ . One can recognize that in our case this means that the algebra of diagonal operators  $C^*(G^o)$  is maximal abelian. Therefore, one sees that our results concern different subalgebras.

We need additional notation: if  $x, y \in X$  are two distinct fixed vertices at the distance  $r \geq 1$  then by  $\bar{x}$  and  $\bar{y}$  we will denote the vertices uniquely determined by the conditions  $\mathbf{d}(x, \bar{x}) = \mathbf{d}(y, \bar{y}) = 1$  and  $\mathbf{d}(x, \bar{y}) = \mathbf{d}(y, \bar{x}) = \mathbf{d}(x, y) - 1 = r - 1$ . In other words, if we go along the (unique!) shortest path from  $x$  to  $y$  then  $\bar{x}$  is the first vertex we pass and  $\bar{y}$  is the last one we pass before reaching  $y$ .

One can observe that  $\mathcal{M}_1(X)$  is the completion of the Hecke algebra  $\mathbf{H}(X)$  in  $\mathcal{F}_1(X)$ . We are going to prove that  $\mathcal{M}_1(X)$  is a maximal abelian subalgebra of  $\mathcal{F}_1(X)$ . Our main tool is the following:

**Lemma 5.1.** *Let  $\varphi \in \mathcal{F}_1(X)$  commute with  $\chi_1$ , and let  $x_0, y_0 \in X$  satisfy  $\mathbf{d}(x_0, y_0) = r, r \geq 1$ . Then for every  $n \in \mathbb{N}$ ,*

$$\varphi(x_0, \bar{y}_0) - \varphi(\bar{x}_0, y_0) = \frac{1}{q^n} \cdot \sum_{x,y} (\varphi(x, \bar{y}) - \varphi(\bar{x}, y))$$

where the summation is over all  $x, y \in X$  such that

$$(4) \quad \mathbf{d}(x_0, x) = \mathbf{d}(y, y_0) = n, \quad \mathbf{d}(x, y) = 2n + r,$$

and  $\bar{x}_0, \bar{y}_0$  are defined by the pair  $(x_0, y_0)$ .

*Proof.* Induction on  $n$ . For  $n = 1$  we have

$$\begin{aligned} \varphi \circ \chi_1(x_0, y_0) &= \varphi(x_0, \bar{y}_0) + \sum_y \varphi(x_0, y) = \varphi(x_0, \bar{y}) + \frac{1}{q} \cdot \sum_{x,y} \varphi(\bar{x}, y), \\ \chi_1 \circ \varphi(x_0, y_0) &= \varphi(\bar{x}_0, y_0) + \sum_x \varphi(x, y_0) = \varphi(\bar{x}_0, y_0) + \frac{1}{q} \cdot \sum_{x,y} \varphi(x, \bar{y}) \end{aligned}$$

where  $x, y$  in the summations satisfy  $\mathbf{d}(x_0, x) = \mathbf{d}(y, y_0) = 1, \mathbf{d}(x, y_0) = \mathbf{d}(x_0, y) = r + 1$ . For all such  $x, y$  we have  $\bar{x} = x_0, \bar{y} = y_0$ . Thus by subtraction side-by-side of the above formulas we get the required equality.

Now assume that the lemma holds for some  $n \in \mathbb{N}$ ; then, using the case  $n = 1$ , for each pair  $x, y \in X$  satisfying (4) we may write

$$\varphi(x, \bar{y}) - \varphi(\bar{x}, y) = \frac{1}{q} \cdot \sum_{x', y'} (\varphi(x', \bar{y}') - \varphi(\bar{x}', y'))$$

with  $x', y'$  defined by  $\mathbf{d}(x, x') = \mathbf{d}(y, y') = 1, \mathbf{d}(x', y') = \mathbf{d}(x, y) = 2$ . Therefore,

$$\begin{aligned} \varphi(x_0, \bar{y}_0) - \varphi(\bar{x}_0, y_0) &= \frac{1}{q^n} \cdot \sum_{x, y} \frac{1}{q} \cdot \sum_{x', y'} (\varphi(x', \bar{y}') - \varphi(\bar{x}', y')) \\ &= \frac{1}{q^{n+1}} \cdot \sum_{z, w} (\varphi(z, \bar{w}) - \varphi(\bar{z}, w)) \end{aligned}$$

where  $\mathbf{d}(z, x_0) = \mathbf{d}(w, y_0) = n + 1, \mathbf{d}(z, w) = r + 2(n + 1)$ . This proves the lemma.

**Theorem 5.2.**  $\mathcal{M}_1(X)$  is a maximal abelian subalgebra in  $\mathcal{F}_1(X)$ .

*Proof.* Let us assume that  $\varphi \in \mathcal{F}_1(X)$  commutes with  $\chi_1$ ; then by Lemma 5.1 for arbitrary  $x_0, y_0 \in X$  such that  $\mathbf{d}(x_0, y_0) = r \geq 1$  we may write

$$\varphi(x_0, \bar{y}_0) - \varphi(\bar{x}_0, y_0) = \frac{1}{q^n} \cdot \sum_{x, y} (\varphi(x, \bar{y}) - \varphi(\bar{x}, y))$$

where  $x, y$  are defined by (4). Therefore, we can estimate

$$\begin{aligned} |\varphi(x_0, \bar{y}_0) - \varphi(\bar{x}_0, y_0)| &\leq \frac{1}{q^n} \cdot q^n \cdot \sup_x \sum_y |(\varphi(x, \bar{y}) - \varphi(\bar{x}, y))| \\ &\quad + \frac{1}{q^n} \cdot q^n \cdot \sup_y \sum_x |(\varphi(x, \bar{y}) - \varphi(\bar{x}, y))| \end{aligned}$$

where in the summations  $\mathbf{d}(x, y) = 2n + r - 1$ .

The right-hand side tends to zero when  $n \rightarrow +\infty$ , so we conclude that the left-hand side must be zero and thus  $\varphi(x_0, \bar{y}_0) = \varphi(\bar{x}_0, y_0)$ .

Now if  $q = 2$  then we conclude that  $\varphi$  is a Toeplitz matrix which may not be symmetric, hence also not Hecke. However, if  $q \geq 3$  then by repeating the geometric trick of changing the direction of an arrow of fixed length, used in the proof of Theorem 3.1, one can prove the symmetry of  $\varphi$ . This implies also that  $\varphi$  is constant on each  $E_r, r = 1, 2, \dots$ , and therefore  $\varphi \in \mathcal{M}_1(X)$ .

*Remark 5.3.* In the above proof it suffices to assume that  $\varphi$  satisfies only

$$\sup_x \sum_{y, \mathbf{d}(x, y)=n} |\varphi(x, y)| \rightarrow_n 0, \quad \sup_y \sum_{x, \mathbf{d}(x, y)=n} |\varphi(x, y)| \rightarrow_n 0.$$

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