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L^p-BOUNDEDNESS OF THE HILBERT TRANSFORM AND MAXIMAL FUNCTION ASSOCIATED TO FLAT PLANE CURVES

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ABSTRACT. We give a sufficient condition for the Hilbert transform and maximal function associated to a flat plane convex curve $\Gamma(t) = (t, \gamma(t))$ to be bounded on L^p, $1 < p < \infty$. Our result includes the previously known sufficient conditions, i.e., γ' doubling or h, defined by $h(t) = t\gamma'(t) - \gamma(t)$, $t > 0$, infinitesimally doubling, as special cases.

1. INTRODUCTION

Let $\Gamma\colon \mathbb{R} \to \mathbb{R}^2$ be a curve in $\mathbb{R}^2$ with $\Gamma(0) = 0$. Associated to Γ are the Hilbert transform, $\mathscr{H}_\Gamma$, and maximal function, $\mathscr{M}_\Gamma$, defined by

$$\mathscr{H}_\Gamma f(x) = \text{p.v.} \int_{-\infty}^{\infty} f(x - \Gamma(t))\,\frac{dt}{t}$$

and

$$\mathscr{M}_\Gamma f(x) = \sup_{r>0} \frac{1}{r} \int_0^r |f(x - \Gamma(t))|\,dt,$$

respectively. Here p.v. is used to indicate a principal-value integral.

For odd convex curves $\Gamma(t) = (t, \gamma(t))$, which may be flat (i.e., vanish to infinite order) at the origin, there are two well-known theorems, giving conditions on Γ under which $\mathscr{H}_\Gamma$ and $\mathscr{M}_\Gamma$ are bounded on $L^p(\mathbb{R}^2)$, $1 < p < \infty$.

Theorem 1.1 [1]. *Let $\Gamma\colon \mathbb{R} \to \mathbb{R}^2$ and $\Gamma(t) = (t, \gamma(t))$ be an odd convex curve, of class $C^2(0, \infty)$, such that $\gamma(0) = \gamma'(0) = 0$. Suppose $\exists\lambda$, $1 < \lambda < \infty$, such that for $t \in (0, \infty)$,*

$$\gamma'(\lambda t) \geq 2\gamma'(t). \tag{1}$$

Then

$$\|\mathscr{H}_\Gamma f\|_p \leq C\|f\|_p \quad \text{and} \quad \|\mathscr{M}_\Gamma f\|_p \leq C\|f\|_p, \qquad 1 < p < \infty.$$

Theorem 1.2 [2]. *Let $\Gamma\colon \mathbb{R} \to \mathbb{R}^2$ and $\Gamma(t) = (t, \gamma(t))$ be an odd convex curve, of class $C^2(0, \infty)$, such that $\gamma(0) = 0$. Let the function h be given by, for*

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$t \in (0, \infty)$, $h(t) = t\gamma'(t) - \gamma(t)$ and suppose $\exists \varepsilon_0 > 0$ such that for $t \in (0, \infty)$,

$$h'(t) \geq \varepsilon_0 \frac{h(t)}{t}. \tag{2}$$

Then

$$\|\mathscr{H}_\Gamma f\|_p \leq C\|f\|_p \quad \text{and} \quad \|\mathscr{M}_\Gamma f\|_p \leq C\|f\|_p, \qquad 1 < p < \infty.$$

Remarks. The theorem of [1] also gives the result that for even curves $\mathscr{H}_\Gamma$ is bounded on $L^p(\mathbb{R}^2)$ if and only if (1) holds.

If (1) holds we say that γ' doubles, whilst (2) is known as the h infinitesimally doubling condition. The condition in Theorem 1.1 that $\gamma \in C^2(0, \infty)$ may be relaxed to allow piecewise-linear curves, after also a technical adjustment to (1). For example the polygonalized version of (t, t^2) obtained by joining points of the form $t = 2^j$, $j \in \mathbb{Z}$, by straight-line segments is covered by Theorem 1.1; for such a curve (2) clearly fails. On the other hand the curve given by $\gamma(t) = t \ln t$, for $t > 1$, satisfies (2) but not (1). Our result covers a strictly larger class of curves than either of the above theorems.

Theorem 1.3. *Let $\Gamma\colon \mathbb{R} \to \mathbb{R}^2$ and $\Gamma(t) = (t, \gamma(t))$ be an odd convex curve, of class $C^2(0, \infty)$ such that $\gamma(0) = \gamma'(0) = 0$. Suppose $\exists \varepsilon_0 > 0$ and λ, $1 < \lambda < \infty$, such that for $t \in (0, \infty)$*

$$\max\left\{\frac{1}{\varepsilon_0} h'(t), \frac{1}{2}\left(\gamma'(\lambda t) - 2\frac{\gamma(t)}{t}\right)\right\} \geq \frac{h(t)}{t}. \tag{3}$$

Then $\|\mathscr{H}_\Gamma f\|_p \leq C\|f\|_p$ and $\|\mathscr{M}_\Gamma f\|_p \leq C\|f\|_p$, $1 < p < \infty$.

Condition (3) says precisely that at each $t \in (0, \infty)$ at least one of $\gamma'(\lambda t) \geq 2\gamma'(t)$ and $h'(t) \geq \varepsilon_0 \frac{h(t)}{t}$ must hold. To illustrate that our theorem is indeed stronger than either Theorem 1.1 or Theorem 1.2 we give the following example, motivated by the previous observation that each of the curves $(t, t \ln t)$ and the polygonalized version of (t, t^2) satisfies exactly one of (1) and (2). So we define $\gamma(t)$ for $t \in [4, \infty)$ to be such that, for $j = 1, 2, \dots$,

$$\gamma'(t) = \begin{cases} 2^{2^j}, & t \in [2^{2^j}, 2^{2^j+1}], \\ \dfrac{1}{\ln 2^{2^j-1}}\left(2^{2^{j+1}} \ln \dfrac{t}{2^{2^j+1}} + 2^{2^j} \ln \dfrac{2^{2^{j+1}}}{t}\right), & t \in (2^{2^j+1}, 2^{2^{j+1}}). \end{cases}$$

It is easily checked that $(t, \gamma(t))$ is convex, $\gamma'(e^3 t) \geq 2\gamma'(t)$ for $t \in [2^{2^j}, 2^{2^j+1}]$, $h'(t) \geq \frac{h(t)}{t}$ for $t \in (2^{2^j+1}, 2^{2^{j+1}})$ and moreover that neither of (1) and (2) holds.

The proof of Theorem 1.3 is basically a combination of the ideas used in proving Theorems 1.1 and 1.2. In [2] an "annular" Littlewood-Paley decomposition is developed via the family of dilation matrices $\{A_k\}$ given by $A_k = A(2^k)$ where

$$A(t) = \begin{pmatrix} t & 0 \\ \gamma(t) & h(t) \end{pmatrix}.$$

The hypothesis used to do this is that the Rivière condition is satisfied, i.e., $\exists \alpha$ such that $\|A_{k+1}^{-1} A_k\| \leq \alpha < 1$, or equivalently that h doubles, a weaker condition than both (1) and (2). Here $\|\cdot\|$ is the operator (matrix) norm.

In §2 we shall show that (3) is sufficient to give such a Littlewood-Paley decomposition. The proof of Theorem 1.2 also depends on suitable decay estimates for the Fourier transform of certain measures associated to Γ. These estimates are a consequence of (2) so we can no longer expect them to be satisfied. We consider those points $\zeta \in \mathbb{R}^2$ where the decay estimates may fail as being 'bad' and in §3 we develop a conical Littlewood-Paley theory to deal with these 'bad' ζ, in the spirit of [1]. Finally, in §4, we combine these ideas to complete the proof.

Remark. The method of proof of Theorem 1.3 may easily be adapted to give, for odd curves, a new proof of Theorem 1.1 and also of the result of [4] and [5] that h doubling is sufficient to give L^2-boundedness of $\mathscr{H}_\Gamma$ and $\mathscr{M}_\Gamma$.

First we give two preliminary lemmas. Observe that since Γ is convex and $\gamma(0) = \gamma'(0) = 0$ we have that h and γ' are increasing and nonnegative on $(0, \infty)$. Assuming (3) we put

$$E := \{t \in (0, \infty) \colon \gamma'(\lambda t) \geq 2\gamma'(t)\} \quad \text{and} \quad F := \left\{t \in (0, \infty) \colon h'(t) \geq \varepsilon_0 \frac{h(t)}{t}\right\};$$

then $E \cup F = (0, \infty)$.

Lemma 1.4. *Let γ be convex on $(0, \infty)$. Then $t \in E \Rightarrow \gamma'(\lambda^2 s) \geq 2\gamma'(s)$, $\forall s \in [\frac{t}{\lambda}, t]$.*

Proof. $s \in [\frac{t}{\lambda}, t] \Rightarrow \gamma'(\lambda^2 s) \geq \gamma'(\lambda^2 \frac{t}{\lambda}) = \gamma'(\lambda t) \geq 2\gamma'(t) \geq 2\gamma'(s)$.

Lemma 1.5. *Let γ be convex on $(0, \infty)$ and $E \cup F = (0, \infty)$. Defining*

$$X := \{k \in \mathbb{Z} \colon \gamma'(\lambda^{k+2}) \geq 2\gamma'(\lambda^k)\},$$
$$Y := \left\{k \in \mathbb{Z} \colon h'(t) \geq \varepsilon_0 \frac{h(t)}{t}, \ \forall t \in [\lambda^k, \lambda^{k+1}]\right\},$$

we have $X \cup Y = \mathbb{Z}$.

Proof. If $E \cap [\lambda^k, \lambda^{k+1}] = \varnothing$ then $[\lambda^k, \lambda^{k+1}] \subseteq F$ and so $k \in Y$. If $E \cap [\lambda^k, \lambda^{k+1}] \neq \varnothing$ then $\exists t_0 \in [\lambda^k, \lambda^{k+1}]$ such that $\gamma'(\lambda t_0) \geq 2\gamma'(t_0)$. Then, by Lemma 1.4, $\gamma'(\lambda^2 s) \geq 2\gamma'(s)$, $\forall s \in [\frac{t_0}{\lambda}, t_0]$. In particular, $\lambda^k \in [\frac{t_0}{\lambda}, t_0]$ so $\gamma'(\lambda^{k+2}) \geq 2\gamma'(\lambda^k)$ and hence $k \in X$.

2. An "annular" Littlewood-Paley theory

Let us define dilation matrices $\{A_k\}$ as in [2]. So we let

$$A(t) = \begin{pmatrix} t & 0 \\ \gamma(t) & h(t) \end{pmatrix}$$

and put $A_k = A(\lambda^k)$.

Lemma 2.1. *Suppose γ is convex and $E \cup F = (0, \infty)$. Then $\exists \alpha$ such that*

$$\|A_{k+2}^{-1} A_k\| \leq \alpha < 1.$$

Proof.

$$A_{k+2}^{-1} A_k = \begin{pmatrix} \dfrac{1}{\lambda^2} & 0 \\ \dfrac{-\gamma(\lambda^{k+2}) + \lambda^2 \gamma(\lambda^k)}{\lambda^2 h(\lambda^{k+2})} & \dfrac{h(\lambda^k)}{h(\lambda^{k+2})} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}.$$

By Lemma 1.5 we have $\mathbb{Z} = X \cup Y$. If $k \in X$ then it is easily shown that $a_{22} = \frac{h(\lambda^k)}{h(\lambda^{k+2})} \leq \frac{1}{2}$, whilst if $k \in Y$ we obtain $a_{22} \leq \frac{1}{1+\varepsilon_0 \ln \lambda} < 1$.

In the proof of the analogous result in [2] (Proposition 3.1), the off-diagonal term, a_{21}, is shown to be controlled by a_{22}. In fact, convexity is sufficient to bound a_{21}; a simple computation shows that

$$\gamma'(\lambda^k) \le \frac{\gamma(\lambda^{k+2}) - \gamma(\lambda^k)}{\lambda^k(\lambda^2 - 1)} \le \gamma'(\lambda^{k+2}) \Rightarrow 0 \le -a_{21} \le \frac{\lambda^2 - 1}{\lambda^2},$$

which completes the proof.

Lemma 2.1 allows us to develop an "annular" Littlewood-Paley theory as in [2]. We define $\phi \in C_0^\infty(\mathbb{R}^2)$ such that $0 \le \hat{\phi} \le 1$ and

$$\hat{\phi}(\zeta) = \begin{cases} 1, & |\zeta| \le 1, \\ 0, & |\zeta| \ge 2. \end{cases}$$

Then the Littlewood-Paley operators ψ_k are given by

$$\hat{\psi}_k(\zeta) = \hat{\phi}(A_k^*\zeta) - \hat{\phi}(A_{k+2}^*\zeta),$$

i.e.,

$$\psi_k(x) = \det A_k^{-1}\phi(A_k^{-1}x) - \det A_{k+2}^{-1}\phi(A_{k+2}^{-1}x),$$

and standard convergence arguments, together with Lemma 2.1, show that for $f \in L^p$, $\sum_k \psi_k * f$ converges to f in L^p-norm, $1 \le p < \infty$. Thus we may write

$$\sum_k \psi_k * f = f. \tag{4}$$

We also have the following corollary to Lemma 2.1; for a proof see [2, Propositions 3.2 and 3.3].

Corollary 2.2. *Suppose γ is convex and $E \cup F = (0, \infty)$. Then*

(a) $|A_{k+2}^*\zeta| \ge \frac{1}{\alpha}|A_k^*\zeta| > |A_k^*\zeta|$, $k \in \mathbb{Z}$, $\zeta \in \mathbb{R}^2\backslash\{0\}$.

(b) $\hat{\psi}_k(\zeta) \ne 0 \Rightarrow |A_{k+2}^*\zeta| > 1$ *and* $|A_k^*\zeta| < 2$.

(c) $\exists M \in \mathbb{N}$ *such that, for any* $\zeta \in \mathbb{R}^2\backslash\{0\}$, $\hat{\psi}_k(\zeta) \ne 0$ *for at most* M *indices* k.

We may now follow the arguments of [2] to see that the operator with convolution kernel $\sum_k \pm\psi_k$ is bounded on L^2 and thus that the Calderón-Zygmund theory [2, Theorem 2.3] may be applied to give

$$\left\| \left(\sum_k |\psi_k * f|^2 \right)^{1/2} \right\|_p \le C\|f\|_p, \qquad 1 < p < \infty, \tag{5}$$

and hence also

$$\left\| \sum_k \psi_k * f_k \right\|_p \le C \left\| \left(\sum_k |\psi_k * f_k|^2 \right)^{1/2} \right\|_p, \qquad 1 < p < \infty. \tag{6}$$

Further, following a similar argument to that used to obtain (6) from (5) (see the proof of Theorem 3.5 in [2]) we may also obtain

$$\left\| \left(\sum_k |\psi_k * f_k|^2 \right)^{1/2} \right\|_p \le C \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_p, \qquad 1 < p < \infty. \tag{7}$$

3. Decay estimates and a conical Littlewood-Paley theory

We adopt the standard technique of defining measures σ_k, μ_k associated to Γ by

$$\int f\,d\sigma_k = \int_{\lambda^k \le |t| \le \lambda^{k+1}} f(\Gamma(t))\,\frac{dt}{t}$$

and

$$\int f\,d\mu_k = \frac{1}{\lambda^k(\lambda-1)}\int_{\lambda^k}^{\lambda^{k+1}} f(\Gamma(t))\,dt,$$

with associated Fourier transforms

$$\hat{\sigma}_k(\zeta) = \int_{\lambda^k \le |t| \le \lambda^{k+1}} e^{i\zeta\cdot\Gamma(t)}\frac{dt}{t}$$

and

$$\hat{\mu}_k(\zeta) = \frac{1}{\lambda^k(\lambda-1)}\int_{\lambda^k}^{\lambda^{k+1}} e^{i\zeta\cdot\Gamma(t)}\,dt.$$

Then $\mathscr{H}_\Gamma f = \sum_k \sigma_k * f$ and $\mathscr{M}_\Gamma f \le C\sup_k|\mu_k * f|$. Thus L^p-boundedness of $\mathscr{M}_\Gamma$ follows from that of $\sup_k|\mu_k * f|$. The proof of Theorem 1.2 depended on the following decay estimates:

$$|\hat{\sigma}_k(\zeta)| \le C|A_k^*\zeta|^{-\varepsilon}, \qquad |\hat{\mu}_k(\zeta)| \le C|A_k^*\zeta|^{-\varepsilon}, \tag{8}$$

$$|\hat{\sigma}_k(\zeta)| \le C|A_{k+1}^*\zeta|, \qquad |\hat{\mu}_k(\zeta)-1| \le C|A_{k+1}^*\zeta|. \tag{9}$$

The estimates (9) are easily verified but (8) follows from the h infinitesimally doubling condition and therefore we may no longer expect these estimates to hold. We do, however, have the following.

Lemma 3.1. *Let $\Gamma(t) = (t, \gamma(t))$ be a curve in $\mathbb{R}^2$ with $\gamma \in C^2(0,\infty)$ convex. Then*

(a) $|\zeta\cdot\Gamma'(t)| \ge \frac{C}{\lambda^k}|A_k^*\zeta|$, $\forall t \in [\lambda^k, \lambda^{k+1}) \Rightarrow |\hat{\mu}_k(\zeta)|$, $|\hat{\sigma}_k(\zeta)| \le C|A_k^*\zeta|^{-1}$.

(b) $h'(t) \ge \varepsilon_0\frac{h(t)}{t}$, $\forall t \in [\lambda^k, \lambda^{k+1}] \Rightarrow |\hat{\mu}_k(\zeta)|$, $|\hat{\sigma}_k(\zeta)| \le C|A_k^*\zeta|^{-1/2}$.

(c) $|\hat{\mu}_k(\zeta)-1| \le C|A_{k+1}^*\zeta|$, $|\hat{\sigma}_k(\zeta)| \le C|A_{k+1}^*\zeta|$.

Proof. (a) This follows by a straightforward application of Van der Corput's lemma. See [7, p. 197].

(b) and (c) See the proofs of Propositions 4.2 and 4.4 in [2]. Although for (b) we have only the assumption that $h'(t) \ge \varepsilon_0\frac{h(t)}{t}$, $\forall t \in [\lambda^k, \lambda^{k+1}]$, rather than $\forall t \in (0,\infty)$, it is easily checked that this is sufficient.

From the above lemma we see that the desired decay estimates are satisfied except when $k \in X$ and $\exists t_0 \in [\lambda^k, \lambda^{k+1})$ such that $|\zeta\cdot\Gamma'(t_0)| \le \frac{\varepsilon}{\lambda^k}|A_k^*\zeta|$, where here $\varepsilon > 0$ may be as small as we like. So we need to consider, for $k \in X$,

$$\bigcup_{t\in[\lambda^k,\lambda^{k+1})}\left\{\zeta : |\zeta\cdot\Gamma'(t)| < \frac{\varepsilon}{\lambda^k}|A_k^*\zeta|\right\} := \bigcup_{t\in[\lambda^k,\lambda^{k+1})} C_k^t;$$

we shall assume henceforth that $\varepsilon < \frac{1}{3}$.

Proposition 3.2. *Let $\Gamma(t) = (t, \gamma(t))$ be a curve in $\mathbb{R}^2$ with γ convex. If*

$$C_k := \left\{\zeta = (\xi, \eta) : \frac{1}{2}\gamma'(\lambda^k) \le \left|\frac{\xi}{\eta}\right| = -\frac{\xi}{\eta} \le 2\gamma'(\lambda^{k+2})\right\}$$

then

$$\bigcup_{t\in[\lambda^k,\lambda^{k+1})} C_k^t \subseteq C_k, \qquad k\in\mathbb{Z}.$$

Proof. Since γ is convex, $\zeta\cdot\Gamma'$ must be monotone. We suppose that $\zeta\cdot\Gamma'$ is monotone-increasing on $[\lambda^k, \lambda^{k+1})$. Then

$$\zeta\cdot\Gamma'(\lambda^k)\le\zeta\cdot\Gamma'(t)\le\Gamma'(\lambda^{k+1}), \quad \forall t\in[\lambda^k,\lambda^{k+1}).$$

So if $\zeta\in C_k^t$, some $t\in[\lambda^k,\lambda^{k+1})$, it follows that

$$\zeta\cdot\Gamma'(\lambda^k)<\frac{\varepsilon}{\lambda^k}|A_k^*\zeta| \quad\text{and}\quad \zeta\cdot\Gamma'(\lambda^{k+1})>-\frac{\varepsilon}{\lambda^k}|A_k^*\zeta|. \tag{10}$$

Now $\frac{1}{\lambda^k}|A_k^*\zeta|\le|\xi|+\gamma'(\lambda^k)|\eta|\le|\xi|+\gamma'(\lambda^{k+1})|\eta|$; using this in (10) we arrive at $\frac{1-\varepsilon}{1+\varepsilon}\gamma'(\lambda^k)<|\frac{\xi}{\eta}|<\frac{1+\varepsilon}{1-\varepsilon}\gamma'(\lambda^{k+1})$. The result follows for $\varepsilon<\frac{1}{3}$. We may argue similarly if $\zeta\cdot\Gamma'$ is monotone-decreasing on $[\lambda^k,\lambda^{k+1})$.

We now wish to develop a Littlewood-Paley theory for the family of cones $\{C_k\}_{k\in X}$. We define a function $\Phi\in C_0^\infty(\mathbb{R})$ such that

$$\Phi(t)=\begin{cases}1, & t<\varepsilon,\\ 0, & t>2\varepsilon,\end{cases}$$

and put

$$\Phi_k(\zeta)=\Phi\left(\frac{\xi+\gamma'(\lambda^k)\eta}{|\xi|+\gamma'(\lambda^k)|\eta|}\right)\Phi\left(\frac{-\xi-\gamma'(\lambda^{k+1})\eta}{|\xi|+\gamma'(\lambda^{k+1})|\eta|}\right).$$

Then we certainly have

$$\Phi_k(\zeta)=\begin{cases}1, & \zeta\in C_k,\\ 0, & \text{off } \frac{1}{3}\gamma'(\lambda^k)<|\frac{\xi}{\eta}|<3\gamma'(\lambda^{k+2}).\end{cases}$$

Recalling that if $k\in X$ then $\gamma'(\lambda^{k+2})\ge2\gamma'(\lambda^k)$ it is now easily checked that $\sum_{k\in X}\pm\Phi_k$ satisfies the hypotheses of the Marcinkiewicz multiplier theorem (see [6, p. 108]). If we now associate operators T_k to these Φ_k, i.e., $(\widehat{T_kf})(\zeta)=\Phi_k(\zeta)\hat f(\zeta)$, then a standard Rademacher function argument (see [6, p. 104]) leads us to

$$\left\|\left(\sum_{k\in X}|T_kf|^2\right)^{1/2}\right\|_p\le C\|f\|_p, \qquad 1<p<\infty. \tag{11}$$

4. Proof of Theorem 1.3

We consider first the maximal function and recall that it is enough to show that $\|\sup_k|\mu_k*f|\,\|_p\le C\|f\|_p$, $1<p<\infty$. We begin by noting that

$$\left\|\sup_k|\mu_k*f|\right\|_p\le\left\|\sup_{k\in X}|\mu_k*f|\right\|_p+\left\|\sup_{k\in Y}|\mu_k*f|\right\|_p=A+B.$$

Following the approach in [2] or [1] we prove first the L^2 result, then, using the Littlewood-Paley decompositions, we give a bootstrapping step and hence the L^p-result. The estimate for B follows as in [2], using (5), Corollary 2.2, and Lemma 3.1(b), (c), so we omit it. The argument for A is similar but also uses the conical Littlewood-Paley inequality (16). First we take the case $p = 2$:

$$A \leq \left\| \sup_{k\in X} |\mu_k * T_k f| \right\|_2 + \left\| \sup_{k\in X} |\mu_k * (I - T_k) f| \right\|_2 = D + E$$

and

$$D \leq \left\| \left(\sum_{k\in X} |\mu_k * T_k f|^2 \right)^{1/2} \right\|_2 \leq \left\| \left(\sum_{k\in X} |T_k f|^2 \right)^{1/2} \right\|_2 \leq C\|f\|_2,$$

using Plancherel's theorem, (11) for $p = 2$, and that the μ_k have unit mass.

For E we introduce a measure ν_k with which to compare μ_k. So we let $\rho \in C_0^\infty$ be such that $0 \leq \rho \leq 1$, $\int \rho = 1$, and define $\nu_k(x) = \rho(A_{k+1}^{-1}x) \det A_{k+1}^{-1}$. Then it is easily shown that

$$(12) \qquad |\hat{\nu}_k(\zeta) - 1| \leq C|A_{k+1}^* \zeta|, \qquad |\hat{\nu}_k(\zeta)| \leq C|A_k^* \zeta|^{-1},$$

and

$$(13) \qquad \sup_k |\nu_k * f| \leq C\mathscr{M} f,$$

where $\mathscr{M}$ is the Hardy-Littlewood maximal function associated to the balls $A_k B$, B the unit ball in $\mathbb{R}^2$, and hence by [2, Proposition 2.2] satisfies

$$(14) \qquad \|\mathscr{M} f\|_p \leq C\|f\|_p, \qquad 1 < p < \infty.$$

Then

$$\begin{aligned} E &\leq \left\| \sup_{k\in X} |(\mu_k - \nu_k) * (I - T_k) f| \right\|_2 + \left\| \sup_{k\in X} |\nu_k * f| \right\|_2 + \left\| \sup_{k\in X} |\nu_k * T_k f| \right\|_2 \\ &\leq C\|f\|_2 + \left\| \sup_{k\in X} |(\mu_k - \nu_k) * (I - T_k) f| \right\|_2, \end{aligned}$$

using (13), (14) for the second term and the same argument as for D for the third term on the right-hand side. We recall that by (4) we may write

$$(\mu_k - \nu_k) * (I - T_k) f = \sum_l (\mu_k - \nu_k) * (I - T_k) f * \psi_{k+l}$$

and so

$$\sup_{k\in X} |(\mu_k - \nu_k) * (I - T_k) f| \leq \sum_l G_l f$$

where

$$G_l f = \left\{ \sum_{k\in X} |(\mu_k - \nu_k) * \psi_{k+l} * (I - T_k) f|^2 \right\}^{1/2}.$$

Then, as in the proof of [2, Proposition 5.1] we may combine Lemma 3.1(a) and (c), Corollary 2.2, and the estimates (12) for ν_k to obtain

$$\hat{\psi}_{k+l}(\zeta) \neq 0, \ (1 - \phi_k(\zeta)) \neq 0 \Rightarrow |\hat{\mu}(\zeta)| \leq C\alpha^{|l|}, \qquad k \in X,$$

and hence, by Plancherel's theorem,

$$(15) \qquad \|G_l f\|_2 \leq C\alpha^{|l|}\|f\|_2, \quad \text{for each } l,$$

and the estimate for E follows on summing over l.

To pass to the L^p-result we have the following lemma.

Lemma 4.1. *If*

$$\left\|\sup_k |\mu_k * f|\right\|_{\tilde{p}} \le C\|f\|_{\tilde{p}}, \quad \textit{some } 1 < \tilde{p} \le 2, \tag{16}$$

then

$$\left\|\left(\sum_k |\mu_k * f_k|^2\right)^{1/2}\right\|_p \le C\left\|\left(\sum_k |f_k|^2\right)^{1/2}\right\|_p, \quad \forall p, \frac{2\tilde{p}}{\tilde{p}+1} < p < \frac{2\tilde{p}}{\tilde{p}-1}.$$

For a proof see the proof of Lemma 3 in [3]; the proof uses only the positivity of the μ_k and thus the lemma is also valid for the ν_k. Moreover, the lemma will still hold if only those $k \in X$ are considered. So, assuming (16), we have

$$\begin{aligned}\left\|\sup_{k\in X} |\mu_k * f|\right\|_p \le{}& \left\|\left(\sum_{k\in X} |\mu_k * T_k f|^2\right)^{1/2}\right\|_p + \left\|\sup_{k\in X} |(\mu_k - \nu_k) * (I - T_k) f|\right\|_p \\ &+ \left\|\sup_{k\in X} |\nu_k * f|\right\|_p + \left\|\left(\sum_{k\in X} |\nu_k * T_k f|^2\right)^{1/2}\right\|_p .\end{aligned}$$

Then applying Lemma 4.1 and (16) to the first and last terms and (13) and (14) to the third term it remains to show that

$$\left\|\sup_{k\in X} |(\mu_k - \nu_k) * (I - T_k) f|\right\|_p \le C\|f\|_p, \quad \forall p, \frac{2\tilde{p}}{\tilde{p}+1} < p < \frac{2\tilde{p}}{\tilde{p}-1}. \tag{17}$$

The result then follows by a bootstrapping argument starting with the result for L^2. To obtain (17) we note that it is enough to obtain

$$\|G_l f\|_p \le C\|f\|_p, \quad \forall\, p, \frac{2\tilde{p}}{\tilde{p}+1} < p < \frac{2\tilde{p}}{\tilde{p}-1},$$

interpolation with (15) then completing the proof. Now

$$\begin{aligned}\|G_l f\|_p &\le \left\|\left(\sum_{k\in X} |\mu_k * (I - T_k) f * \psi_{k+l}|^2\right)^{1/2}\right\|_p \\ &\quad + \left\|\left(\sum_{k\in X} |\nu_k * (I - T_k) f * \psi_{k+l}|^2\right)^{1/2}\right\|_p \\ &\le C\left\|\left(\sum_{k\in X} |(I - T_k) f * \psi_{k+l}|^2\right)^{1/2}\right\|_p \\ &\le C\left\|\left(\sum_{k\in X} |f * \psi_{k+l}|^2\right)^{1/2}\right\|_p + \left\|\left(\sum_{k\in X} |T_k f * \psi_{k+l}|^2\right)^{1/2}\right\|_p \\ &\le C\|f\|_p,\end{aligned}$$

using Lemma 4.1, (5), (7), and (16). This completes the proof for $\mathscr{M}_\Gamma$. The proof for $\mathscr{H}_\Gamma$ now follows easily, using the oddness of Γ, exactly as in [2].

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