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LOCALLY UNIFORMLY CONTINUOUS FUNCTIONS

ALEXANDER J. IZZO

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ABSTRACT. It is shown that on every infinite-dimensional separable normed space there exist continuous real-valued functions that are nowhere locally uniformly continuous. An explicit example of such a function on l^p $(1 \leq p < \infty)$ is given. It is also shown that every continuous real-valued function on a metric space can be approximated uniformly by locally uniformly continuous functions.

INTRODUCTION

Although it is trivial that not every continuous function between metric spaces is *uniformly* continuous, every continuous function on Euclidean space (or more generally any locally compact metric space) is *locally* uniformly continuous, i.e., uniformly continuous on a neighborhood of each point of its domain. Thus the question arises as to what can be said about the local uniform continuity of continuous functions on infinite-dimensional normed spaces (or more generally nonlocally compact metric spaces). Using the Baire category theorem, we will show that on every infinite-dimensional separable normed space there exist continuous real-valued functions that are nowhere locally uniformly continuous (i.e., locally uniformly continuous at no point), and that such functions are in fact quite abundant. In addition, an explicit example of such a function on l^p $(1 \leq p < \infty)$ will be given. However, despite the abundance of continuous, nowhere locally uniformly continuous functions, we will also show that every continuous real-valued function on a metric space can be approximated uniformly by locally uniformly continuous functions.

1. EXISTENCE OF CONTINUOUS, NOWHERE LOCALLY UNIFORMLY CONTINUOUS REAL-VALUED FUNCTIONS

We begin by making a few definitions explicit.

Definitions. We say that a function $f : X \to Y$ between metric spaces is **locally uniformly continuous at a point x** if there is a neighborhood U of x on which f is uniformly continuous. If f is locally uniformly continuous at every point of X, we say that f is **locally uniformly continuous**. If f is locally uniformly

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continuous at no point of X, we say that f is **nowhere locally uniformly continuous**. Equivalently, f is nowhere locally uniformly continuous if it is uniformly continuous on no open set of X.

Theorem 1. *Suppose X is an infinite-dimensional separable normed space. Then there is a bounded continuous real-valued function on X that is nowhere locally uniformly continuous. In fact, the collection of such functions is a dense G_δ in the space of bounded continuous real-valued functions under the supremum norm.*

The above theorem will be proven using the Baire category theorem, but first we need a lemma.

Lemma 2. *Suppose X is an infinite-dimensional normed space and U is an open set of X. Then there is a bounded continuous real-valued function on X that is not uniformly continuous on U.*

Proof. Since the closure of U in X is not compact, there is a countable subset $\{x_n\}_{n=1}^{\infty}$ of U with no limit point in X. For each $n = 1, 2, \ldots,$ choose a point y_n in U such that $\|x_n - y_n\| < 1/2^n$. It is clear that the set $S = \{x_n\} \cup \{y_n\}$ is a discrete and closed subset of X. Hence every real-valued function on S is continuous, and by the Tietze extension theorem every bounded continuous real-valued function on S extends to a bounded continuous function on X. In particular, the function that is 1 at each x_n and 0 at each y_n can be extended to a bounded continuous function on X, and it is clear that the resulting function is not uniformly continuous on U. $\square$

Proof of Theorem 1. Let $C_b(X)$ denote the space of all bounded continuous real-valued functions on X equipped with the supremum norm. Note that $C_b(X)$ is a Banach space, i.e., that it is complete. Choose a countable basis $\{U_n\}_{n=1}^{\infty}$ for the topology of X, and for each positive integer n, let $E_n = \{f \in C_b(X) : f$ is uniformly continuous on $U_n\}$. Clearly $\bigcup_{n=1}^{\infty} E_n = \{f \in C_b(X) : f$ is locally uniformly continuous at some point of $X\}$. The uniform limit of a sequence of uniformly continuous functions is uniformly continuous, so each E_n is closed in $C_b(X)$. Moreover, it follows from Lemma 2 that each E_n has empty interior, for if f is a bounded continuous function on X that is uniformly continuous on U_n, and g is a bounded continuous function on X with supremum norm equal to ϵ that is *not* uniformly continuous on U_n, then $f + g$ is an element of $C_b(X)$ that is *not* uniformly continuous on U_n and whose distance from f is ϵ. Now the Baire category theorem shows that the complement of $\bigcup E_n$ in $C_b(X)$ (which consists of the nowhere locally uniformly continuous functions in $C_b(X)$) is a dense G_δ. $\square$

One might be tempted to imagine that if X is an arbitrary metric space that fails to be locally compact at some point x_0, then there must be a continuous real-valued function on X that is not locally uniformly continuous at x_0. However, this is not the case. To see this, suppose in addition that X has the property that whenever (x_n) and (y_n) are sequences in X such that $d(x_n, y_n) \to 0$ as $n \to \infty$, then in fact $x_n \to x_0$ and $y_n \to x_0$. Then every continuous function on X is uniformly continuous. (For a specific example of such a metric space consider the set $\{\frac{1}{2^m} e_n : m, n$ are positive integers$\} \cup \{0\}$ in l^1 where e_n is the sequence all of whose entries are zero except for the n^{th} entry which is one.) However, the reader can easily verify that the arguments given above can be repeated without essential change to show that Theorem 1 remains valid if the hypothesis that X be an infinite-dimensional separable normed space is relaxed to assume merely that X be a separable metric space that is locally compact at no point. A curious feature of the proof of Theorem 1 is that we proved the existence of continuous, nowhere locally uniformly continuous functions without "directly" proving the existence of a function that fails to be locally uniformly continuous at even one point. Actually it is not difficult to modify the proof of Lemma 2 to give a "direct" proof of the existence of a continuous function that fails to be locally uniformly continuous. In fact we have the following:

Theorem 3. *Suppose X is a metric space that fails to be locally compact at some point x_0 and has no isolated point. Then there is a bounded continuous real-valued function on X that is not locally uniformly continuous at x_0.*

Proof. Since X is not locally compact at x_0, we can choose for each positive integer k a countable subset $\{x_{k,n}\}_{n=1}^{\infty}$ of $B(x_0, 1/k)$ (the open ball of radius $1/k$ centered at x_0) such that $\{x_{k,n}\}_{n=1}^{\infty}$ has no limit point in X. Since X has no isolated point, we can choose for each ordered pair of positive integers k and n, a point $y_{k,n}$ in $B(x_0, 1/k)$ such that $d(x_{k,n}, y_{k,n}) < 1/2^n$. It is easy to see that the only limit point of the set S that consists of all the $x_{k,n}$'s and $y_{k,n}$'s is the point x_0. Thus the set $S \cup \{x_0\}$ is a closed set whose only limit point is x_0. Hence the Tietze extension theorem shows that the function that is $1/2^k$ at $x_{k,n}$ $(n = 1, 2, \ldots)$, 0 at each $y_{k,n}$, and 0 at x_0 can be extended to a bounded continuous function on X, and it is clear that the resulting function is not locally uniformly continuous at x_0. □

We next give an explicit example of a continuous, nowhere locally uniformly continuous function on l^p $(1 \le p < \infty)$, the space of p-summable sequences of *real* numbers with the usual l^p norm. For x in l^p, we will denote the nth term of x by x_n.

Example. The function $f : l^p \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=1}^{\infty} |x_n|^p \cos(nx_n)$$

is continuous but nowhere locally uniformly continuous.

Proof. First note that the series used to define f converges absolutely for every x in l^p, so f is well defined. To show that f is continuous, fix an arbitrary point a in l^p and an $\epsilon > 0$. Choose a positive integer N such that $\sum_{n=N}^{\infty} |a_n|^p < \epsilon$, and let $g : l^p \to \mathbb{R}$ be defined by

$$g(x) = \sum_{n=1}^{N-1} |x_n|^p \cos(nx_n).$$

The function g is obviously continuous, so we can choose $\delta > 0$ such that $\delta < \epsilon^{1/p}$ and $|g(x) - g(a)| < \epsilon$ whenever $\|x - a\|_p < \delta$. Obviously $(\sum_{n=N}^{\infty} |x_n - a_n|^p)^{1/p} < \delta$ whenever $\|x - a\|_p < \delta$. Thus whenever $\|x - a\|_p < \delta$ we have (by the triangle inequality in l^p)

$$\Big(\sum_{n=N}^{\infty} |x_n|^p\Big)^{1/p} \le \Big(\sum_{n=N}^{\infty} |x_n - a_n|^p\Big)^{1/p} + \Big(\sum_{n=N}^{\infty} |a_n|^p\Big)^{1/p} < \delta + \epsilon^{1/p} < 2\epsilon^{1/p}.$$

Thus whenever $\|x-a\|_p<\delta$ we have

$$\begin{aligned}
|f(x)-f(a)| &= \left|\left(g(x)+\sum_{n=N}^{\infty}|x_n|^p\cos(nx_n)\right)-\left(g(a)+\sum_{n=N}^{\infty}|a_n|^p\cos(na_n)\right)\right| \\
&\le |g(x)-g(a)|+\left|\sum_{n=N}^{\infty}|x_n|^p\cos(nx_n)-\sum_{n=N}^{\infty}|a_n|^p\cos(na_n)\right| \\
&< \epsilon+\sum_{n=N}^{\infty}\Big||x_n|^p\cos(nx_n)-|a_n|^p\cos(na_n)\Big| \\
&\le \epsilon+\sum_{n=N}^{\infty}\left(|x_n|^p+|a_n|^p\right) \\
&< \epsilon+2^p\epsilon+\epsilon.
\end{aligned}$$

Therefore, f is continuous at a, and hence is continuous.

To show that f is not locally uniformly continuous, fix an arbitrary point a in l^p and an $r>0$. Let $\epsilon=(r/2)^p$. We will show that f is not uniformly continuous on $B(a,r)$ by showing that for every $\delta>0$ there exist points y and z in $B(a,r)$ with $\|y-z\|_p<\delta$ and $|f(y)-f(z)|>\epsilon$. So fix $\delta>0$, and without loss of generality assume that $\delta<r/4$. Choose a positive integer N such that $N\delta>2\pi$. Then also $Nr/4>2\pi$, so it is easy to see that we can choose a real number c_1 with $|c_1|>r/2$ such that $|a_N-c_1|<3r/4$ and $\cos(Nc_1)=1$. We can also choose a real number c_0 such that $|c_1-c_0|<\delta$ and $\cos(Nc_0)=0$. Now let y be the point in l^p given by

$$y_n=\begin{cases} a_n & \text{if } n\neq N, \\ c_1 & \text{if } n=N, \end{cases}$$

and z be the point in l^p given by

$$z_n=\begin{cases} a_n & \text{if } n\neq N, \\ c_0 & \text{if } n=N. \end{cases}$$

Then $\|a-y\|_p=|a_N-c_1|<3r/4$, and

$$\|a-z\|_p=|a_N-c_0|\le|a_N-c_1|+|c_1-c_0|<\frac{3r}{4}+\delta<r,$$

so y and z are both in $B(a,r)$, and moreover $\|y-z\|_p=|c_1-c_0|<\delta$. However,

$$\begin{aligned}
|f(y)-f(z)| &= \left|\sum_{n=1}^{\infty}|y_n|^p\cos(ny_n)-\sum_{n=1}^{\infty}|z_n|^p\cos(nz_n)\right| \\
&= \Big||c_1|^p\cos(Nc_1)-|c_0|^p\cos(Nc_0)\Big| \\
&= |c_1|^p>\left(\frac{r}{2}\right)^p=\epsilon.
\end{aligned}$$

Thus f is not uniformly continuous on $B(a,r)$, and hence is nowhere locally uniformly continuous. □

When p is an integer we can omit the absolute values in the definition of f and consider the real-valued function on l^p given by $h(x)=\sum_{n=1}^{\infty}x_n^p\cos(nx_n)$.

The above proof can be repeated with the obvious alterations to show that h is a continuous, nowhere locally uniformly continuous function.

2. Density of locally uniformly continuous real-valued functions

As mentioned in the introduction, despite the abundance of continuous, nowhere locally uniformly continuous functions, every continuous real-valued function on a metric space can be approximated uniformly by locally uniformly continuous functions. Explicitly, the following result holds.

Theorem 4. *Suppose X is a metric space, f is a continuous real-valued function on X, and $\epsilon > 0$ is given. Then there is a locally uniformly continuous real-valued function g on X such that*

$$|f(x) - g(x)| < \epsilon$$

for every x in X.

Before proving this theorem, we develop some preliminaries. If X is a metric space and A is a subset of X, define the distance $d_A(x)$ from a point x in X to A by $d_A(x) = \inf\{d(x, a) : a \in A\}$.

Lemma 5. *If A and B are disjoint closed sets in a metric space X and $\rho : X \to \mathbb{R}$ is defined by*

$$\rho(x) = \frac{d_A(x)}{d_A(x) + d_B(x)}$$

then ρ is a locally uniformly continuous function.

Proof. It is straightforward to verify that the numerator and denominator in the definition of ρ are uniformly continuous, and that the denominator never vanishes. Moreover, the function $F : \mathbb{R} \times (\mathbb{R}\backslash\{0\}) \to \mathbb{R}$ given by

$$F(s, t) = s/t$$

is obviously locally uniformly continuous since every continuous function on a locally compact metric space is locally uniformly continuous. Thus ρ is a composition of locally uniformly continuous functions and hence is locally uniformly continuous. □

Lemma 6. *If $\{U_\alpha\}_{\alpha\in J}$ is an indexed family of open sets covering a metric space X, then there is a partition of unity subordinate to $\{U_\alpha\}$ consisting of locally uniformly continuous functions.*

Proof. First consider the special case when the covering $\{U_\alpha\}$ is locally finite. By the "shrinking lemma" (see [M, p. 224]) we can choose an open covering $\{V_\alpha\}$ of X such that $\overline{V}_\alpha \subset U_\alpha$ for each α. Applying the "shrinking lemma" again, choose an open covering $\{W_\alpha\}$ of X such that $\overline{W} \subset V_\alpha$ for each α. For each α, define $\psi_\alpha : X \to [0, 1]$ by

$$\psi_\alpha(x) = \frac{d_{X\backslash V_\alpha}(x)}{d_{X\backslash V_\alpha}(x) + d_{\overline{W}_\alpha}(x)}.$$

Then $\psi_\alpha(\overline{W}_\alpha) = \{1\}$ and $\psi_\alpha(X\backslash V_\alpha) = \{0\}$, and by the preceding lemma, ψ_α is locally uniformly continuous. Note also that

$$(\operatorname{supp} \psi_\alpha) \subset \overline{V}_\alpha \subset U_\alpha.$$

In particular, the sum $\sum_{\alpha\in J}\psi_\alpha(x)$ is locally finite, and hence defines a locally uniformly continuous function Ψ on X. Because the collection $\{W_\alpha\}$ covers X, we have that $\Psi(x)$ is strictly positive for each x. As in the proof of the preceding lemma, it follows that for each α, the function given by

$$\phi_\alpha(x)=\frac{\psi_\alpha(x)}{\Psi(x)}$$

is locally uniformly continuous. It is easy to check that the collection $\{\phi_\alpha\}_{\alpha\in J}$ is the desired partition of unity.

Now consider the general case. Since X is a metric space, it is paracompact [M, Chapter 6, Theorem 4.3], so there is a locally finite open refinement $\{N_\beta\}_{\beta\in K}$ of $\{U_\alpha\}_{\alpha\in J}$. Choose a mapping $s:K\to J$ so that $N_\beta\subset U_{s(\beta)}$ for each β in K. Let U'_α be the union of those N_β for which $s(\beta)=\alpha$. Then it is easy to check that $\{U'_\alpha\}$ is locally finite and clearly $U'_\alpha\subset U_\alpha$. By the preceding paragraph, there is a partition of unity subordinate to $\{U'_\alpha\}$ consisting of locally uniformly continuous functions and obviously such a partition of unity is also subordinate to $\{U_\alpha\}$. □

Proof of Theorem 4. For each point y in X choose an open neighborhood U_y such that $f(U_y)\subset\big(f(y)-\epsilon,\,f(y)+\epsilon\big)$. Then $\{U_y\}_{y\in X}$ is an indexed open covering of X. By the preceding lemma, there is a partition of unity $\{\phi_y\}$ subordinate to $\{U_y\}$ consisting of locally uniformly continuous functions. Now define $g:X\to\mathbb{R}$ by

$$g(x)=\sum_{y\in X}f(y)\phi_y(x).$$

Since the supports of the ϕ_y form a locally finite collection, the sum makes sense, and each point of X has a neighborhood on which g is a finite sum of locally uniformly continuous functions. Hence g is a well-defined locally uniformly continuous function. Moreover, for each point x in X we have

$$\begin{aligned}|f(x)-g(x)|&=\left|\sum_{y\in X}f(x)\phi_y(x)-\sum_{y\in X}f(y)\phi_y(x)\right|\\&=\left|\sum_{y\in X}(f(x)-f(y))\phi_y(x)\right|\\&\le\sum_{y\in X}|f(x)-f(y)|\phi_y(x)\\&<\sum_{y\in X}\epsilon\phi_y(x)\qquad(\text{as } |f(x)-f(y)|<\epsilon \text{ whenever } \phi_y(x)\neq 0)\\&=\epsilon.\quad\square\end{aligned}$$

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Reference

[M] J. R. Munkres, *Topology: a first course*, Prentice-Hall, Englewood Cliffs, NJ, 1975.

Department of Mathematics, Brown University, Providence, Rhode Island 02912