IDEALS OF OPERATORS STRICTLY SINGULAR ON SUBSPACES

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Abstract. For each set \( \Omega \) of cardinality at most \( 2^{\omega} \), we give an easy construction of Banach spaces \( X \), such that the algebra \( \mathcal{B}(X) \) of all bounded linear operators on \( X \) contains a lattice of closed ideals, which is order isomorphic with respect to inclusion to the full power set of \( \Omega \).

Introduction and result

Let \( X \) be a Banach space and \( \mathcal{B}(X) \) the Banach algebra of bounded linear operators on \( X \). The ideal structure of \( B(X) \) depends very much on the Banach space \( X \). For example, if \( X = c_0 \) or \( X = l_p \), \( 1 \leq p < \infty \), then \( B(X) \) contains only one nontrivial closed two-sided ideal, the ideal of compact operators.

On the other hand Porta constructed in [1] a Banach space \( X \) and, for each nonempty finite subset \( A \subseteq \mathbb{N} \), a closed two-sided ideal \( H(A) \subseteq \mathcal{B}(X) \) such that the map \( H : A \to H(A) \) is an order isomorphism with respect to set inclusion of the family of finite subsets \( A \subseteq \mathbb{N} \) with the family of closed ideals \( H(A) \subseteq \mathcal{B}(X) \).

In this note we give an extremely easy construction of a lattice of closed operator ideals on a suitable Banach space \( X \), which is order isomorphic to the full power set of certain infinite sets \( \Omega \).

Let us call two Banach spaces \( E, F \) incomparable, if each bounded linear operator \( T : E \to F \) is strictly singular [2, 2.c.1 and 2.c.2]. With this we can show:

Theorem 1. Suppose that the Banach space \( X \) contains a family \( (E_a)_{a \in \Omega} \) of pairwise incomparable complemented infinite-dimensional subspaces \( E_a \subseteq X \). Then for each subset \( A \subseteq \Omega \) a closed two-sided ideal \( I(A) \subseteq \mathcal{B}(X) \) can be defined such that the map \( I : A \to I(A) \) is an order isomorphism with respect to inclusion of the power set of \( \Omega \) with a family of closed ideals \( I(A) \subseteq \mathcal{B}(X) \).

Proof. For each closed subspace \( E \subseteq X \) let \( S(E) \) denote the family of all operators \( t \in B(X) \) such that the restriction \( t|_{E_1} : E_1 \to X \) is strictly singular for each closed subspace \( E_1 \subseteq X \), which is isomorphic to \( E \).

It is easily checked that \( S(E) \) is a closed left ideal in \( B(X) \). We claim that it is in fact a two-sided ideal. Since every element \( a \) in the unital Banach algebra...
$B(X)$ can be written as the sum of two invertibles \( a = \lambda 1 + (a - \lambda 1) \), where \(|\lambda|\) exceeds the spectral radius of \( a \), it will suffice to show that \( S(E)g \subseteq S(E) \) for all invertibles \( g \in \mathcal{B}(X) \).

Let now \( t \in S(E) \) and \( g \in \text{inv}(B(X)) \). If \( E_1 \subseteq X \) is any closed subspace isomorphic to \( E \), then so is the subspace \( g(E_1) \). Consequently the restriction \( t|_{g(E_1)} \) is strictly singular. But then the restriction \( t^g|_{E_1} \) is strictly singular. This shows that \( tg \in S(E) \).

For each index \( \alpha \in \Omega \) choose a projection \( p_\alpha \in \mathcal{B}(X) \) with \( p_\alpha(X) = E_\alpha \) and define

\[
I(A) = \bigcap_{\alpha \in \Omega \setminus A} S(E_\alpha), \quad \text{for each subset } A \subseteq \Omega,
\]

where \( I(A) = \mathcal{B}(X) \), if \( A = \Omega \). \( I(A) \) is an intersection of closed two-sided ideals in \( B(X) \) and hence itself such an ideal. Clearly \( A \subseteq B \subseteq \Omega \) implies \( I(A) \subseteq I(B) \). We claim now that

\[
(1) \quad p_\alpha \in I(A) \iff \alpha \in A, \quad \text{for all subsets } A \subseteq \Omega \text{ and } \alpha \in \Omega.
\]

Note first that \( p_\alpha \notin S(E_\alpha) \), since the restriction of \( p_\alpha \) to the subspace \( E_\alpha \) is not strictly singular. Thus \( p_\alpha \notin I(A) \), if \( \alpha \in \Omega \setminus A \). In other words, \( p_\alpha \in I(A) \Rightarrow \alpha \in A \).

Conversely assume that \( \alpha \in A \). We claim that \( p_\alpha \in I(A) \) and must show that \( p_\alpha \in S(E_\beta) \) for all \( \beta \in \Omega \setminus A \). Indeed, we show that \( p_\alpha \in S(E_\beta) \) for all \( \beta \neq \alpha \).

Suppose that \( \beta \in \Omega, \beta \neq \alpha \), and \( F \) is any closed subspace of \( X \) which is isomorphic to \( E_\beta \). The projection \( p_\alpha \) maps \( F \) into \( E_\beta \). Since the spaces \( E_\beta, E_\alpha \) and hence the spaces \( F, E_\alpha \) are incomparable by assumption, the restriction of \( p_\alpha \) to \( F \) must be strictly singular. This shows that \( p_\alpha \in E_\beta \) and proves (1). According to (1)

\[
A = \{ \alpha \in \Omega : p_\alpha \in I(A) \} \quad \text{for each subset } A \subseteq \Omega.
\]

Thus \( I(A) \subseteq I(B) \) implies \( A \subseteq B \) for all subsets \( A, B \subseteq \Omega \). This concludes the proof. \( \Box \)

**Remarks.** The Banach spaces \( E_p = l_p, \ 1 \leq p < \infty \), are well known to be pairwise incomparable [2, 2.c.1 and 2.c.2]. Suppose now that \( \Omega \) is a set of cardinality at most \( 2^\omega \). We may assume that \( \Omega \) is a subset of the interval [2, 3]. Then the Banach space

\[
X = \left\{ x = (x_p)_{p \in \Omega} : x_p \in l_p, \ \text{for all } p \in \Omega, \ \text{and } \| x \| = \left( \sum_{p \in \Omega} \| x_p \|^2_{l_p} \right)^{1/2} < \infty \right\}
\]

satisfies the assumptions of Theorem 1 for the family of subspaces \( (E_p)_{p \in \Omega} \). Theorem 1 now gives a large family of closed ideals in the Banach algebra \( \mathcal{B}(X) \). It is now interesting to note that

**Proposition 1.** Let \( X \) be a Banach space, and suppose that \( I, J \subseteq \mathcal{B}(X) \) are distinct ideals which are Banach algebras in some norms. Then \( I \) and \( J \) are not isomorphic as complex algebras.
Proof. According to [3, Theorem 2.5.19] any algebra isomorphism $\phi : I \to J$ is the restriction to $I$ of an inner automorphism of $\mathcal{B}(X)$. Consequently $\phi$ leaves $I$ invariant. □

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REFERENCES


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