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IDEALS OF OPERATORS STRICTLY SINGULAR ON SUBSPACES

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ABSTRACT. For each set Ω of cardinality at most 2^ω, we give an easy construction of Banach spaces X, such that the algebra $\mathscr{B}(X)$ of all bounded linear operators on X contains a lattice of closed ideals, which is order isomorphic with respect to inclusion to the full power set of Ω.

INTRODUCTION AND RESULT

Let X be a Banach space and $\mathscr{B}(X)$ the Banach algebra of bounded linear operators on X. The ideal structure of $B(X)$ depends very much on the Banach space X. For example, if $X = c_0$ or $X = l_p$, $1 \le p < \infty$, then $B(X)$ contains only one nontrivial closed two-sided ideal, the ideal of compact operators.

On the other hand Porta constructed in [1] a Banach space X and, for each nonempty finite subset $A \subseteq \mathbb{N}$, a closed two-sided ideal $H(A) \subseteq \mathscr{B}(X)$ such that the map $H : A \to H(A)$ is an order isomorphism with respect to set inclusion of the family of finite subsets $A \subseteq \mathbb{N}$ with the family of closed ideals $H(A) \subseteq \mathscr{B}(X)$.

In this note we give an extremely easy construction of a lattice of closed operator ideals on a suitable Banach space X, which is order isomorphic to the full power set of certain infinite sets Ω.

Let us call two Banach spaces E, F *incomparable*, if each bounded linear operator $T : E \to F$ is strictly singular [2, 2.c.1 and 2.c.2]. With this we can show:

Theorem 1. *Suppose that the Banach space X contains a family $(E_\alpha)_{\alpha\in\Omega}$ of pairwise incomparable complemented infinite-dimensional subspaces $E_\alpha \subseteq X$. Then for each subset $A \subseteq \Omega$ a closed two-sided ideal $I(A) \subseteq \mathscr{B}(X)$ can be defined such that the map $I : A \to I(A)$ is an order isomorphism with respect to inclusion of the power set of Ω with a family of closed ideals $I(A) \subseteq \mathscr{B}(X)$.*

Proof. For each closed subspace $E \subseteq X$ let $S(E)$ denote the family of all operators $t \in B(X)$ such that the restriction $t|_{E_1} : E_1 \to X$ is strictly singular for each closed subspace $E_1 \subseteq X$, which is isomorphic to E.

It is easily checked that $S(E)$ is a closed left ideal in $B(X)$. We claim that it is in fact a two-sided ideal. Since every element a in the unital Banach algebra

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$B(X)$ can be written as the sum of two invertibles ($a = \lambda 1 + (a - \lambda 1)$, where $|\lambda|$ exceeds the spectral radius of a), it will suffice to show that $S(E)g \subseteq S(E)$ for all invertibles $g \in \mathscr{B}(X)$.

Let now $t \in S(E)$ and $g \in \mathrm{inv}(B(X))$. If $E_1 \subseteq X$ is any closed subspace isomorphic to E, then so is the subspace $g(E_1)$. Consequently the restriction $t|_{g(E_1)}$ is strictly singular. But then the restriction $tg|_{E_1}$ is strictly singular. This shows that $tg \in S(E)$.

For each index $\alpha \in \Omega$ choose a projection $p_\alpha \in \mathscr{B}(X)$ with $p_\alpha(X) = E_\alpha$ and define

$$I(A) = \bigcap_{\alpha \in \Omega \setminus A} S(E_\alpha), \quad \text{for each subset } A \subseteq \Omega,$$

where $I(A) = \mathscr{B}(X)$, if $A = \Omega$. $I(A)$ is an intersection of closed two-sided ideals in $B(X)$ and hence itself such an ideal. Clearly $A \subseteq B \subseteq \Omega$ implies $I(A) \subseteq I(B)$. We claim now that

$$(1) \qquad p_\alpha \in I(A) \Leftrightarrow \alpha \in A, \quad \text{for all subsets } A \subseteq \Omega \text{ and } \alpha \in \Omega.$$

Note first that $p_\alpha \notin S(E_\alpha)$, since the restriction of p_α to the subspace E_α is not strictly singular. Thus $p_\alpha \notin I(A)$, if $\alpha \in \Omega \setminus A$. In other words, $p_\alpha \in I(A) \Rightarrow \alpha \in A$.

Conversely assume that $\alpha \in A$. We claim that $p_\alpha \in I(A)$ and must show that $p_\alpha \in S(E_\beta)$ for all $\beta \in \Omega \setminus A$. Indeed, we show that $p_\alpha \in S(E_\beta)$ for all $\beta \neq \alpha$.

Suppose that $\beta \in \Omega$, $\beta \neq \alpha$, and F is any closed subspace of X which is isomorphic to E_β. The projection p_α maps F into E_α. Since the spaces E_β, E_α and hence the spaces F, E_α are incomparable by assumption, the restriction of p_α to F must be strictly singular. This shows that $p_\alpha \in E_\beta$ and proves (1). According to (1)

$$A = \{\alpha \in \Omega : p_\alpha \in I(A)\} \quad \text{for each subset } A \subseteq \Omega.$$

Thus $I(A) \subseteq I(B)$ implies $A \subseteq B$ for all subsets A, $B \subseteq \Omega$. This concludes the proof. □

Remarks. The Banach spaces $E_p = l_p$, $1 \leq p < \infty$, are well known to be pairwise incomparable [2, 2.c.1 and 2.c.2]. Suppose now that Ω is a set of cardinality at most 2^ω. We may assume that Ω is a subset of the interval [2, 3]. Then the Banach space

$$X = \left\{ x = (x_p)_{p \in \Omega} : x_p \in l_p, \text{ for all } p \in \Omega, \text{ and } ||x|| = \left(\sum_{p \in \Omega} \|x_p\|_{l_p}^2 \right)^{1/2} < \infty \right\}$$

satisfies the assumptions of Theorem 1 for the family of subspaces $(E_p)_{p \in \Omega}$. Theorem 1 now gives a large family of closed ideals in the Banach algebra $\mathscr{B}(X)$. It is now interesting to note that

Proposition 1. *Let X be a Banach space, and suppose that I, $J \subseteq \mathscr{B}(X)$ are distinct ideals which are Banach algebras in some norms. Then I and J are not isomorphic as complex algebras.*

Proof. According to [3, Theorem 2.5.19] any algebra isomorphism $\phi : I \to J$ is the restriction to I of an inner automorphism of $\mathscr{B}(X)$. Consequently ϕ leaves I invariant. □

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