THE EXISTENCE OF BOUNDED INFINITE $DTr$-ORBITS

SHIPING LIU AND RAINER SCHULZ

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Abstract. We construct an indecomposable module over a symmetric algebra whose $DTr$-orbit is infinite and bounded. This yields a counterexample to a conjecture which states that the number of modules in an Auslander-Reiten component having the same length is finite.

Let $\Lambda$ be an Artin algebra, $C$ a connected component of the Auslander-Reiten quiver of $\Lambda$, and $DTr$ the Auslander-Reiten translation [1]. In [8], Ringel asked whether the number of modules having the same length in $C$ is always finite. This is the case when $\Lambda$ is a hereditary algebra [2, 10] or a tame algebra [4]. For an arbitrary algebra $\Lambda$, the question has an affirmative answer if $C$ has at most finitely many nonperiodic $DTr$-orbits [3, 6] or is a regular component of the form $\mathbb{Z}\Delta$ with $\Delta$ one of $A_\infty^\infty$, $B_\infty$, $C_\infty$, or $D_\infty$ [7].

The aim of this paper is to show that the above problem has no affirmative answer in general. We shall construct a local symmetric algebra whose Auslander-Reiten quiver contains a bounded infinite $DTr$-orbit. Our example will be a modification of that given by the second author in a different context [9].

Let $K$ be a field which contains an element $\rho$ of infinite multiplicative order. Let $R$ be the polynomial ring over $K$ in noncommuting variables $X$ and $Y$ modulo the ideal generated by $X^2$, $Y^2$, and $XY - \rho YX$. Then $R$ is a local Frobenius algebra over $K$ with radical $J(R) = xR + yR$, $J(R)^2 = \text{Soc}(R) = xyR$, and $J(R)^3 = 0$, where $x, y$ denote the residue classes of $X$, $Y$, respectively. Let $DR = \text{Hom}_{K}(R, K)$ be the dual of $R$ with the following $R$-$R$-bimodule structure: given $r', r'' \in R$ and $f \in DR$, $(r', f')(r'', f'') = (rr', rf' + fr'')$ for $r, r' \in R$ and $f, f' \in DR$. Then $T$ is the trivial extension algebra of $R$ by $DR$ which is the $K$-vector space $T = R \oplus DR$ with multiplication given by

$$(r, f)(r', f') = (rr', rf' + fr')$$

for $r, r' \in R$ and $f, f' \in DR$. Then $T$ is a local symmetric $K$-algebra with radical

$$J(T) = \{(r, f) | r \in J(R), f \in DR\}$$

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and \( J(T)^4 = 0 \). We choose a \( K \)-basis \( 1, x, y, xy, a, b, c, d \) of \( T \), where \( a, b, c, d \) is the \( K \)-basis of \( DR \) dual to the \( K \)-basis \( 1, x, y, xy \) of \( R \). We find the following multiplication table of \( T \):

\[
\begin{array}{cccccccc}
1 & x & y & xy & a & b & c & d \\
1 & 1 & x & y & xy & a & b & c & d \\
x & x & 0 & xy & 0 & 0 & a & 0 & \rho c \\
y & y & \rho xy & 0 & 0 & 0 & 0 & a & b \\
xy & xy & 0 & 0 & 0 & 0 & 0 & a & 0 \\
a & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & b & a & 0 & 0 & 0 & 0 & 0 & 0 \\
c & c & 0 & a & 0 & 0 & 0 & 0 & 0 \\
d & d & c & \rho b & a & 0 & 0 & 0 & 0 \\
\end{array}
\]

For a right \( T \)-module \( N \), let \( \Omega N \) denote the kernel of a minimal projective cover of \( N \). It is well known that \( \Omega^2 N = DTr N \), since \( T \) is a symmetric algebra. Let now \( M \) be the module \( M = (x + y)T \). We shall compute \( \Omega^i M \) for all \( i \in \mathbb{Z} \). Since \( M \) is the image of the map from \( T \) to \( T \) given by the left multiplication with the element \( x + y \), the module \( \Omega M \) is the right annihilator of \( x + y \) in \( T \). Using the equation

\[
(x + y)(A + Bx + Cy + Dxy + Ea + Fb + Gc + Hd) = 0,
\]

where \( A, B, C, D, E, F, G, H \in K \), one finds \( A = H = 0 \), \( C + B \rho = 0 \), and \( F + G = 0 \). Hence, \( \Omega M = (x + (-\rho) y) T + (-b + c) T \). Note that \( (x + (-\rho) y) d = \rho (-b + c) \). This implies that \( \Omega^i M = (x + (-\rho)^i y) T \). By induction, one can show that \( \Omega^i M = (x + (-\rho)^i y) T \) for all \( i \in \mathbb{Z} \). Using the fact that \( T = K + J(T) \) and \( J(T)^4 = 0 \), one has

\[
M(x + (-\rho)^2 y) J(T) = (x + y) T (x + (-\rho)^2 y) J(T) = (x + y) (x + (-\rho)^2 y) J(T) \neq 0
\]

and

\[
\Omega^i M(x + (-\rho)^2 y) J(T) = (x + (-\rho) y) (x + (-\rho)^2 y) J(T) = 0.
\]

Hence, \( M \) and \( \Omega M \) have different annihilators in \( T \). Therefore, they are not isomorphic. Using the fact that \( \rho \) is not a root of unity, one can similarly show that the module \( \Omega^j M \) is not annihilated by \( (x + (-\rho)^i y) J(T) \) for \( j \neq i \), whereas \( \Omega^i M \) is. Thus the modules \( \Omega^i M \) are pairwise nonisomorphic. Obviously, \( \dim_K \Omega^i M = 4 \) for all \( i \in \mathbb{Z} \). Consequently all modules in the \( DTr \)-orbits of \( M \) have the same dimension over \( K \).

Remarks. (1) We conjecture that a stable component of an Auslander-Reiten quiver is of the form \( \mathbb{Z} A_\infty \) if it contains infinitely many modules of the same length. It has been shown in [6] that this is the case if one of its \( DTr \)-orbits contains infinitely many modules of the same length.

(2) In [9, §4], the \( R \)-module \( M = (x + y)R \) has been used to give an example of an \( \Omega \)-bounded but not \( \Omega \)-periodic module. The following computation shows that \( DTr M \cong M \) in this case. The sequence

\[
R \xrightarrow{(x + (-\rho) y)} R \xrightarrow{(x + y)} M \to 0
\]

induces a sequence

\[
R \xrightarrow{(x + (-\rho)^2 y)} R \xrightarrow{(x + (-\rho)^2 y)} TrM \to 0;
\]
hence, $DTRM = \text{Hom}_K(R(x + (-\rho)^2y), K)$. This right $R$-module is annihilated by $x + (-\rho)y$ and hence, is isomorphic to $(x + y)R$. Note that the last argument does not work if one replaces $R$ with $T$.

(3) Another example of an infinite $DTr$-orbit of bounded modules can be constructed by using Example 3.2 in [5].

**References**