SOME ESTIMATES OF THE KOBAYASHI METRIC
IN THE NORMAL DIRECTION

SIQI FU

(Communicated by Eric Bedford)

ABSTRACT. In this paper, we study the behavior of the Kobayashi metric in the normal direction near a Levi-pseudoconvex boundary point of a smoothly bounded domain without assuming global pseudoconvexity. As a corollary, we obtain a characterization of pseudoconvexity by the rate of the growth of the Kobayashi metric in the normal direction.

I. Introduction and theorems

Let \( \Omega \subset \subset \mathbb{C}^n \) be a bounded domain with smooth boundary near \( z_0 \in \partial \Omega \). Let \( U \) be a neighborhood of \( z_0 \). Let \( r(z) \) be a local defining function of \( \Omega \) on \( U \), i.e.,

\[
\Omega \cap U = \{ z \in U \mid r(z) < 0 \}
\]

and \( r(z) \in C^\infty (U) \), \( \nabla r(z)|_{\partial \Omega \cap U} \equiv (\partial r(z)/\partial z_1, \partial r(z)/\partial z_2, \ldots, \partial r(z)/\partial z_n) \neq 0 \).

Let \( \Delta \) be the unit disc and let \( \Delta_\gamma \equiv \{ \gamma \zeta ; \zeta \in \Delta \} \). The Kobayashi metric of \( \Omega \) is defined by

\[
F_\Omega (z, X) = \inf \{ \frac{1}{\lambda} \mid \exists f: \Delta \rightarrow \Omega \text{ is holomorphic and } f(0) = z, f'(0) = \lambda X, \lambda > 0 \}.
\]

For \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \), set \( z' = (z_1, z_2, \ldots, z_{n-1}) \). Let \( d(z) = \text{dist}(z, \partial \Omega) \) and let \( \pi(z) \) be the projection to the boundary for \( z \) near \( z_0 \) such that \( d(z) = |z - \pi(z)| \). Let \( N_{\pi(z)} \) be the inward normal direction at \( \pi(z) \). Denote

\[
H_{z_0} = \left\{ X \in \mathbb{C}^n \mid \langle \partial r(z_0), X \rangle \equiv \sum_{i=1}^{n} \frac{\partial r(z_0)}{\partial z_i} (z_0) X_i = 0 \right\}.
\]

We call \( z_0 \in \partial \Omega \) a Levi-pseudoconvex point if

\[
\sum_{i, j=1}^{n} \frac{\partial^2 r(z_0)}{\partial z_i \partial z_j} X_i \overline{X_j} \geq 0 \quad \text{for all} \quad X \in H_{z_0}.
\]

Received by the editors March 24, 1993.

1991 Mathematics Subject Classification. Primary 32H15.

Key words and phrases. Kobayashi metric, pseudoconvex domain.
In [K], Krantz has proven that if \( \Omega \) is a bounded domain with smooth boundary near \( z_0 \), then there exists constant \( C > 0 \) such that

\[
F_\Omega(z, \nabla r(\pi(z))) \geq C \left| \frac{\nabla r(\pi(z))}{d^{3/4}(z)} \right| \quad \text{for } z \in \Omega \text{ near } z_0.
\]

In this note, we shall prove the following theorems.

**Theorem A.** Let \( \Omega \subset \subset \mathbb{C}^n \) be a bounded domain with smooth boundary near \( z_0 \in \partial \Omega \). Suppose that there exist \( \alpha > 3/4 \), \( C > 0 \), \( X \in \mathbb{C}^n \setminus H_{z_0} \), and \( \{z_k\}_{k=1}^\infty \) with \( z_k \to z_0 \) nontangentially (i.e., \( z_k \) stays in some cone \( \Lambda \) with vertex at \( z_0 \) and axis \( N_{z_0} \)) such that

\[
(1.1) \quad F_\Omega(z_k, X) \geq C \frac{|(\partial r(z_0), X)|}{d^{\alpha}(z_k)} \quad \text{for all } k \in \mathbb{N}.
\]

Then \( z_0 \) is a Levi-pseudoconvex point.

**Theorem B.** Let \( \Omega \subset \subset \mathbb{C}^n \) be a bounded domain with smooth boundary near \( z_0 \in \partial \Omega \), and let \( \Lambda \) be a cone with vertex at \( z_0 \) and axis \( N_{z_0} \). Assume that \( z_0 \) is the origin. Suppose that the local defining function of \( \Omega \) has the following form near \( z_0 \):

\[
r(z) = \text{Re } z_n + \theta(|z'|^m + |z_n| |z|).
\]

Then there exist a neighborhood \( V \) of \( z_0 \) and a constant \( C > 0 \) such that

\[
(1.2) \quad F_\Omega(z, X) \geq C \frac{|X_n|}{d^{1-\frac{1}{2m}}(z)}
\]

for \( z \in \Lambda \cap \Omega \cap V \) and all \( X \in \mathbb{C}^n \). Furthermore, there exists \( C_1 > 0 \) such that

\[
(1.3) \quad F_\Omega(z, X) \geq C_1 \frac{|X_n|}{d^{1-\frac{1}{2m}}(z)}
\]

for \( z \in \Lambda \cap \Omega \cap V \) and \( X \in \mathbb{C}^n \) with \( |X| \leq K|X_n| \) (\( C_1 \) may depend on the constant \( K \)).

As corollaries of Theorem A and Theorem B, we obtain

**Corollary 1.** Let \( \Omega \subset \subset \mathbb{C}^n \) be a bounded domain with smooth boundary near \( z_0 \in \partial \Omega \), and let \( \Lambda \) be a cone with vertex at \( z_0 \) and axis \( N_{z_0} \). Then \( z_0 \) is a Levi-pseudoconvex point if and only if there exist a neighborhood \( V \) of \( z_0 \), \( \alpha > 3/4 \), and \( C > 0 \) such that

\[
(1.4) \quad F_\Omega(z, \nabla r(z_0)) \geq C \frac{|\nabla r(z_0)|}{d^{\alpha}(z)}
\]

for all \( z \in \Lambda \cap \Omega \cap V \).

**Corollary 2.** Let \( \Omega \subset \subset \mathbb{C}^2 \) be a bounded domain with smooth pseudoconvex boundary near \( z_0 \in \partial \Omega \) and let \( \Lambda \) be a cone with vertex at \( z_0 \) and axis \( N_{z_0} \). Then for each \( \alpha \in (0, 1) \), there exist a neighborhood \( V \) of \( z_0 \) and a constant \( C > 0 \) such that

\[
(1.5) \quad F_\Omega(z, X) \geq C \frac{|(\partial r(z_0), X)|}{d^{\alpha}(z)}
\]

for all \( z \in \Lambda \cap \Omega \cap V \) and \( X \in \mathbb{C}^n \).
II. Proofs of the theorems

Some ideas in the proofs of Theorem A and Theorem B come from [K]. We will also need the following lemma, which can be proven directly from Lemma 2 in [R].

Lemma. Let \( \Omega' \) be a subdomain of a bounded domain \( \Omega \) with \( \partial \Omega' \cap \partial \Omega \supset U \cap \partial \Omega \) for some neighborhood \( U \) of \( z_0 \in \partial \Omega \). Then there exist a neighborhood \( V \subset U \) of \( z_0 \) and a constant \( C > 0 \) such that

\[
F_{\Omega'}(z, X) \leq CF_{\Omega}(z, X)
\]

for \( z \in \Omega' \cap V \) and \( X \in \mathbb{C}^n \).

We will use \( C \) to denote constants which may be different in different appearances.

Proof of Theorem A. After a translation and a unitary transformation, we may assume that \( z_0 = 0 \) and \( \partial \Omega \) is locally defined by

\[
r(z) = \text{Re} z_n + \sum_{i,j=1}^{n} a_{ij} z_i \bar{z}_j + \mathcal{O}(|z|^3)
\]

for \( z \) near \( z_0 \).

Suppose that \( \partial \Omega \) is not Levi-pseudoconvex at \( z_0 \). Then the matrix

\[
\left( \frac{\partial^2 r(z_0)}{\partial z_i \partial \bar{z}_j} \right)_{1 \leq i,j \leq n-1}
\]

has at least one negative eigenvalue. Therefore, after a unitary transformation in \( z' = (z_1, z_2, \ldots, z_{n-1}) \) and a simple change of coordinate system, we may assume that

\[
r(z) = \text{Re} z_n - |z_1|^2 + \sum_{i,j=2}^{n} a_{ij} z_i \bar{z}_j + \mathcal{O}(|z|^3)
\]

for \( z \in U \), where \( U \) is some neighborhood of \( z_0 \). Shrinking \( U \), we have

\[
r(z) \leq \text{Re} z_n - \frac{|z_1|^2}{2} + C \sum_{i=2}^{n} |z_i|^2, \quad \text{for} \quad z \in U.
\]

Let \( \Lambda = \{ -\text{Re} z_n > k|z| \} \) (\( 0 < k < 1 \)) be the cone. By the Implicit Function Theorem,

\[
\lim_{z \to 0 \atop z \in \Lambda \cap \Omega} \frac{-\text{Re} z_n}{d(z)} = 1.
\]

By the homogeneity of the Kobayashi metric, we may assume that \( X = (X_1, X_2, \ldots, X_{n-1}, 1) \). For \( z = (z', z_n) = (z_1, z_2, \ldots, z_n) \in \Lambda \cap \Omega \cap U \), let \( \delta = -\text{Re} z_n \). Define \( \Phi_\delta(\zeta) = (\Phi_1(\zeta), \Phi_2(\zeta), \ldots, \Phi_n(\zeta)) \) by

\[
\Phi_1(\zeta) = z_1 + \frac{\delta^{3/4}}{2} X_1 \zeta + 2 \zeta^2;
\]

\[
\Phi_k(\zeta) = z_k + \frac{\delta^{3/4}}{2} X_k \zeta, \quad \text{for} \quad 2 \leq k \leq n-1;
\]

\[
\Phi_n(\zeta) = z_n + \frac{\delta^{3/4}}{2} \zeta.
\]

Then \( \Phi_\delta(0) = z, \quad \Phi'_\delta(0) = \frac{\delta^{3/4}}{2} X \).
Claim. There exists $\gamma \in (0, 1)$ such that for all $\delta > 0$ sufficiently small, we have $\Phi_\delta(\Delta_\gamma) \subset \Omega \cap U$.

Proof of the Claim. By choosing $\gamma \in (0, 1)$ small enough, we have $\Phi_\delta(\Delta_\gamma) \subset U$. It follows from (2.3) that

$$r(\Phi_\delta(\zeta)) \leq -\delta + \frac{3\delta^4}{2} \Re \zeta - \frac{1}{2} |\Phi_{1\delta}(\zeta)|^2 + C \sum_{i=2}^n |\Phi_i(\zeta)|^2.$$

Since $\delta > k|z|$, we see that when $\delta$ is sufficiently small,

(2.5) \hspace{1cm} r(\Phi_\delta(\zeta)) < -\frac{3\delta}{4} + \frac{3\delta^4}{2} \cdot \delta^{1/4} - \frac{1}{2} |\Phi_{1\delta}(\zeta)|^2

For $|\zeta| < \delta^{1/4}$, by (2.5),

$$r(\Phi_\delta(\zeta)) < -\frac{3\delta}{4} + \frac{3\delta^4}{2} \cdot \delta^{1/4} - \frac{1}{2} |\Phi_{1\delta}(\zeta)|^2
= -\frac{\delta}{4} - \frac{1}{2} |\Phi_{1\delta}(\zeta)|^2 < 0.$$

For $|\zeta| \geq \delta^{1/4}$, we have

$$|\Phi_{1\delta}(\zeta)|^2 \geq 2|\zeta|^4 - |z_1 + \frac{3\delta^4}{2} - X_1 \zeta|^2
\geq 2|\zeta|^4 - C\delta^{3/2}. $$

Thus (2.5) implies that when $\delta$ sufficiently small,

$$r(\Phi_\delta(\zeta)) < -\frac{3\delta}{4} + \frac{3\delta^4}{2} \Re \zeta - |\zeta|^4 + \frac{C\delta^{3/2}}{2}
\leq -\frac{3\delta}{4} + \frac{C\delta^{3/2}}{2} + \frac{|\zeta|^4}{2} - |\zeta|^4 < 0.$$

This concludes the proof of the Claim.

Next, by the Claim and the definition of the Kobayashi metric,

$$F_{\Omega \cap U}(z, X) \leq \frac{C}{\delta^{3/4}}.$$

Thus, combining (2.4) and the length-decreasing property of Kobayashi metric, we obtain

$$F_{\Omega}(z, X) \leq \frac{C}{d^{3/4}(z)},$$

which contradicts (1.1). Therefore, $\partial \Omega$ is Levi-pseudoconvex at $z_0$. $\Box$

Proof of Theorem B. By the assumption, there exists a neighborhood $U$ of $z_0$ such that

(2.6) \hspace{1cm} \Omega \cap U \subset \{z \in U \mid \Re z_n - C(|z'|^m + |z_n| \cdot |z|) < 0\}.

Let $\Lambda = \{-\Re z_n > k|z|\}$ ($k \in (0, 1)$). For $z \in \Lambda \cap \Omega \cap U$ and $X \in \mathbb{C}^n$ (by homogeneity of the Kobayashi metric, we may assume that $|X| \leq 1$), let
SOME ESTIMATES OF THE KOBAYASHI METRIC IN THE NORMAL DIRECTION

\[ \Phi(\zeta) = (\Phi(\zeta), \Phi_n(\zeta)) = (\Phi_1(\zeta), \Phi_2(\zeta), \ldots, \Phi_n(\zeta)): \Delta \to \Omega \cap U \] be an analytic disc satisfying

\[ (2.7) \quad \Phi(0) = z, \quad \Phi'(0) = \lambda \nu, \]

where \( \lambda > 0 \) is a constant to be estimated. By the Cauchy Integral Formula, we have

\[ (2.8) \quad |\Phi_i(\zeta) - z_i| \leq C|\zeta|, \quad 1 \leq i \leq n, \]

and

\[ (2.8') \quad |\Phi_i(\zeta) - z_i - \lambda X_i \zeta| \leq C|\zeta|^2, \quad 1 \leq i \leq n, \]

for \( |\zeta| < 1/2 \). Also, by (2.6), we have

\[ (2.9) \quad \text{Re} \Phi_n(\zeta) < C \left( |\Phi(\zeta)|^m + |\Phi_n(\zeta)| \cdot |\Phi(\zeta)| \right). \]

Denote \( \delta = -\text{Re} \ z_n \). By (2.8), (2.8'), and the fact that \( k|z| < \delta \), we have

\[ (2.10) \quad |\Phi(\zeta)| \leq C(|z| + |\zeta|) \]

\[ \leq C \left( (1/k) \delta + c\delta^{1/m} \right) \]

\[ \leq C^{1/2} \delta^{1/m}, \quad \text{for} \quad |\zeta| < c\delta^{1/m} \]

and

\[ (2.10') \quad |\Phi(\zeta)| \leq C \left( |z| + (\lambda|X|)|\zeta| + |\zeta|^2 \right) \]

\[ \leq C \left( (1/k) \delta + (\lambda|X|)c\delta^{1/2m} + c^2\delta^{1/m} \right) \]

\[ \leq C^{1/2} \left( \delta^{1/m} + (\lambda|X|)\delta^{1/2m} \right), \quad \text{for} \quad |\zeta| < c\delta^{1/2m} \]

when \( c, \delta \) are sufficiently small.

Now, it follows from (2.10), (2.10'), and (2.9) that

\[ (2.11) \quad \text{Re} \Phi_n(\zeta) < \frac{\delta}{2} + \frac{1}{2} |\Phi_n(\zeta)|, \quad \text{for} \quad |\zeta| < c\delta^{1/m} \]

and

\[ (2.11') \quad \text{Re} \Phi_n(\zeta) < \frac{\delta + (\lambda|X|)^m \delta^{1/2}}{2} + \frac{1}{2} |\Phi_n(\zeta)|, \quad \text{for} \quad |\zeta| < c\delta^{1/2m} \]

when \( c, \delta \) are sufficiently small.

Denote

\[ D_{\delta} = \left\{ w \in \mathbb{C} \quad \text{Re} \ w < \frac{\delta + \epsilon_1 (\lambda|X|)^m \delta^{1/2}}{2} + \frac{1}{2} |w| \right\}, \quad \text{for} \quad i = 1, 2 \]

where \( \epsilon_1 = 0, \epsilon_2 = 1 \). Let \( g_1(\zeta) = \Phi_n(c\delta^{1/m} \zeta) \) and \( g_2(\zeta) = \Phi_n(c\delta^{1/2m} \zeta) \).

By (2.7), (2.11), and (2.11'), we have \( g_i(\Delta) \subset D_{\delta} \), \( g_i(0) = z_n \) (\( i = 1, 2 \)), \( g'_1(0) = \lambda X_n c\delta^{1/m} \), and \( g'_2(0) = \lambda X_n c\delta^{1/2m} \). However, it is clear that

\[ D_{\delta} \subset \tilde{D}_{\delta} \equiv \mathbb{C} \setminus \left\{ w \in \mathbb{C} \quad \text{Im} \ w = 0, \ \text{Re} \ w \geq \delta + \epsilon_1 (\lambda|X|)^m \delta^{1/2} \right\}. \]

Thus \( g_i(\Delta) \subset \tilde{D}_{\delta} \).
Since
\[ F_{D,\delta}(z_n, 1) \geq \frac{C}{\delta + \varepsilon_i(\lambda|X|)^m\delta^{1/2}} \]
where \( C > 0 \) is a constant independent on \( \delta \), it follows that
\[ |g_i'(0)| \leq C(\delta + \varepsilon_i(\lambda|X|)^m\delta^{1/2}). \]
Therefore,
\[ \lambda|X_n|\delta^{1/m} \leq C\delta \]
and
\[ \lambda|X_n|\delta^{1/2m} \leq C(\delta + (\lambda|X|)^m\delta^{1/2}). \]

Now by (2.12), \( \lambda|X_n| \leq C\delta^{1-1/m} \). Thus (1.2) is valid. Furthermore, when \( |X| < K|X_n| \), it follows from (2.13) that \( \lambda|X_n| \leq C\delta^{1-1/2m} \) where \( C > 0 \) may depend on \( K \). Therefore, (1.3) follows. \( \square \)

**Proof of Corollary 1.** The sufficiency comes directly from Theorem A. We now prove the necessity. Suppose that \( z_0 \) is a Levi-pseudoconvex point. After a change of coordinates, we may assume that \( z_0 \) is the origin and \( \partial \Omega \) is locally defined by
\[ r(z) = \Re z_n + \sum_{i,j=1}^{n} a_{ij}z_i z_j + O(|z|^3) \]
near \( z_0 \).

Since the matrix \((a_{ij})_{1 \leq i,j \leq n-1}\) is positive semidefinite,
\[ r(z) \geq \Re z_n + 2 \Re \left( \sum_{j=1}^{n-1} a_{nj}z_n z_j \right) + a_{nn}|z_n|^2 + O(|z|^3) \]
\[ = \Re z_n + O(|z'|^3 + |z_n| \cdot |z|). \]
Therefore, (1.4) is valid for \( \alpha = 5/6 \) by Theorem B. \( \square \)

**Proof of Corollary 2.** When \( z_0 \) is of finite type, as in Property 2.1 in [M], we can construct a smoothly bounded pseudoconvex domain \( \Omega' \subset \Omega \) such that \( \partial \Omega' \cap \partial \Omega \supset \partial \Omega \cap U \) for some neighborhood \( U \) of \( z_0 \). By the Lemma, there exist a neighborhood \( V \) of \( z_0 \) and a constant \( C > 0 \) such that
\[ F_{\Omega}(z, X) \geq CF_{\Omega}(z, X) \]
for \( z \in V \cap \Omega' \), \( X \in \mathbb{C}^n \). Applying Theorem 1 in [C] to \( \Omega' \), we then obtain that (1.5) is valid for \( \alpha = 1 \).

When \( z_0 \) is of infinite type, then for each \( m \in \mathbb{N} \), after a change of coordinates, the local defining function of \( \Omega \) near \( z_0 \) has the form
\[ r(z) = \Re z_2 + O(|z_1|^m + |z_2| \cdot |z|). \]
Thus, (1.5) follows from Theorem B. \( \square \)

**Remark.** The estimates in Theorem B are sharp. Suppose that \( \Omega \) is a bounded domain such that \( \partial \Omega \) is locally defined by \( r(z) = \Re z_2 - |z_1|^{2m} \) near the origin. Then for \( P_\delta = (0, -\delta) \), \( X = (\delta^{-1} - 1/2m, 1) \), and \( Y = (0, 1) \), we have
\[ F_{\Omega}(P_\delta, X) \approx \frac{1}{\delta^{1-1/2m}}, \quad F_{\Omega}(P_\delta, Y) \approx \frac{1}{\delta^{1-1/4m}}. \]
ACKNOWLEDGMENT

The author thanks Professor Steven Krantz for his advice and encouragement.

REFERENCES


Department of Mathematics, Washington University, St. Louis, Missouri 63130
E-mail address: sfu@math.wustl.edu