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SOME ESTIMATES OF THE KOBAYASHI METRIC IN THE NORMAL DIRECTION

SIQI FU

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ABSTRACT. In this paper, we study the behavior of the Kobayashi metric in the normal direction near a Levi-pseudoconvex boundary point of a smoothly bounded domain without assuming global pseudoconvexity. As a corollary, we obtain a characterization of pseudoconvexity by the rate of the growth of the Kobayashi metric in the normal direction.

I. INTRODUCTION AND THEOREMS

Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain with smooth boundary near $z_0 \in \partial\Omega$. Let U be a neighborhood of z_0. Let $r(z)$ be a local defining function of Ω on U, i.e.,

$$\Omega \cap U = \{z \in U \mid r(z) < 0\}$$

and $r(z) \in \mathscr{C}^\infty(U)$, $\nabla r(z)|_{\partial\Omega\cap U} \equiv (\partial r(z)/\partial z_1, \partial r(z)/\partial z_2, \dots, \partial r(z)/\partial z_n) \neq 0$.

Let $\triangle$ be the unit disc and let $\triangle_\gamma \equiv \{\gamma\zeta; \quad \zeta \in \triangle\}$. The Kobayashi metric of Ω is defined by

$$F_\Omega(z, X) \equiv \inf\{1/\lambda \mid \exists f: \triangle \to \Omega \text{ is holomorphic and } f(0) = z, f'(0) = \lambda X, \lambda > 0\}.$$

For $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, set $z' = (z_1, z_2, \dots, z_{n-1})$. Let $d(z) = \operatorname{dist}(z, \partial\Omega)$ and let $\pi(z)$ be the projection to the boundary for z near z_0 such that $d(z) = |z - \pi(z)|$. Let $N_{\pi(z)}$ be the inward normal direction at $\pi(z)$. Denote

$$H_{z_0} \equiv \left\{X \in \mathbb{C}^n \mid \langle \partial r(z_0), X\rangle \equiv \sum_{i=1}^n \frac{\partial r}{\partial z_i}(z_0) X_i = 0\right\}.$$

We call $z_0 \in \partial\Omega$ a Levi-pseudoconvex point if

$$\sum_{i,j=1}^n \frac{\partial^2 r(z_0)}{\partial z_i \partial \bar{z}_j} X_i \overline{X_j} \geq 0 \quad \text{for all} \quad X \in H_{z_0}.$$

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In [K], Krantz has proven that if Ω is a bounded domain with smooth boundary near z_0, then there exists constant $C > 0$ such that

$$F_\Omega(z, \nabla r(\pi(z))) \geq C \frac{|\nabla r(\pi(z))|}{d^{3/4}(z)} \quad \text{for } z \in \Omega \text{ near } z_0.$$

In this note, we shall prove the following theorems.

Theorem A. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain with smooth boundary near $z_0 \in \partial\Omega$. Suppose that there exist $\alpha > 3/4$, $C > 0$, $X \in \mathbb{C}^n \setminus H_{z_0}$, and $\{z_k\}_{k=1}^\infty$ with $z_k \to z_0$ nontangentially (i.e., z_k stays in some cone Λ with vertex at z_0 and axis N_{z_0}) such that*

$$F_\Omega(z_k, X) \geq C \frac{|\langle \partial r(z_0), X\rangle|}{d^\alpha(z_k)} \quad \textit{for all} \quad k \in \mathbb{N}. \tag{1.1}$$

Then z_0 is a Levi-pseudoconvex point.

Theorem B. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain with smooth boundary near $z_0 \in \partial\Omega$, and let Λ be a cone with vertex at z_0 and axis N_{z_0}. Assume that z_0 is the origin. Suppose that the local defining function of Ω has the following form near z_0:*

$$r(z) = \operatorname{Re} z_n + \mathscr{O}(|z'|^m + |z_n| \cdot |z|).$$

Then there exist a neighborhood V of z_0 and a constant $C > 0$ such that

$$F_\Omega(z, X) \geq C \frac{|X_n|}{d^{1-\frac{1}{m}}(z)} \tag{1.2}$$

for $z \in \Lambda \cap \Omega \cap V$ and all $X \in \mathbb{C}^n$. Furthermore, there exists $C_1 > 0$ such that

$$F_\Omega(z, X) \geq C_1 \frac{|X_n|}{d^{1-\frac{1}{2m}}(z)} \tag{1.3}$$

for $z \in \Lambda \cap \Omega \cap V$ and $X \in \mathbb{C}^n$ with $|X| \leq K|X_n|$ (C_1 may depend on the constant K).

As corollaries of Theorem A and Theorem B, we obtain

Corollary 1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain with smooth boundary near $z_0 \in \partial\Omega$, and let Λ be a cone with vertex at z_0 and axis N_{z_0}. Then z_0 is a Levi-pseudoconvex point if and only if there exist a neighborhood V of z_0, $\alpha > 3/4$, and $C > 0$ such that*

$$F_\Omega(z, \nabla r(z_0)) \geq C \frac{|\nabla r(z_0)|}{d^\alpha(z)} \tag{1.4}$$

for all $z \in \Lambda \cap \Omega \cap V$.

Corollary 2. *Let $\Omega \subset\subset \mathbb{C}^2$ be a bounded domain with smooth pseudoconvex boundary near $z_0 \in \partial\Omega$ and let Λ be a cone with vertex at z_0 and axis N_{z_0}. Then for each $\alpha \in (0, 1)$, there exist a neighborhood V of z_0 and a constant $C > 0$ such that*

$$F_\Omega(z, X) \geq C \frac{|\langle \partial r(z_0), X\rangle|}{d^\alpha(z)} \tag{1.5}$$

for all $z \in \Lambda \cap \Omega \cap V$ and $X \in \mathbb{C}^n$.

II. Proofs of the theorems

Some ideas in the proofs of Theorem A and Theorem B come from [K]. We will also need the following lemma, which can be proven directly from Lemma 2 in [R].

Lemma. *Let Ω' be a subdomain of a bounded domain Ω with $\partial\Omega' \cap \partial\Omega \supset U \cap \partial\Omega$ for some neighborhood U of $z_0 \in \partial\Omega$. Then there exist a neighborhood $V \subset\subset U$ of z_0 and a constant $C > 0$ such that*

$$F_{\Omega'}(z, X) \le C F_{\Omega}(z, X) \tag{2.1}$$

for $z \in \Omega' \cap V$ and $X \in \mathbb{C}^n$.

We will use C to denote constants which may be different in different appearances.

Proof of Theorem A. After a translation and a unitary transformation, we may assume that z_0 is the origin and $\partial\Omega$ is locally defined by

$$r(z) = \operatorname{Re} z_n + \sum_{i,j=1}^{n} a_{ij} z_i \bar{z}_j + \mathscr{O}(|z|^3) \tag{2.2}$$

for z near z_0.

Suppose that $\partial\Omega$ is not Levi-pseudoconvex at z_0. Then the matrix

$$\left(\frac{\partial^2 r(z_0)}{\partial z_i \partial \bar{z}_j}\right)_{1 \le i, j \le n-1}$$

has at least one negative eigenvalue . Therefore, after a unitary transformation in $z' = (z_1, z_2, \dots, z_{n-1})$ and a simple change of coordinate system, we may assume that

$$r(z) = \operatorname{Re} z_n - |z_1|^2 + \sum_{i,j=2}^{n} a_{ij} z_i \bar{z}_j + \mathscr{O}(|z|^3)$$

for $z \in U$, where U is some neighborhood of z_0. Shrinking U, we have

$$r(z) \le \operatorname{Re} z_n - \frac{|z_1|^2}{2} + C \sum_{i=2}^{n} |z_i|^2, \quad \text{for} \quad z \in U. \tag{2.3}$$

Let $\Lambda = \{-\operatorname{Re} z_n > k|z|\}$ ($0 < k < 1$) be the cone. By the Implicit Function Theorem,

$$\lim_{\substack{z \to 0 \\ z \in \Lambda \cap \Omega}} \frac{-\operatorname{Re} z_n}{d(z)} = 1. \tag{2.4}$$

By the homogeneity of the Kobayashi metric, we may assume that $X = (X_1, X_2, \dots, X_{n-1}, 1)$. For $z = (z', z_n) = (z_1, z_2, \dots, z_n) \in \Lambda \cap \Omega \cap U$, let $\delta = -\operatorname{Re} z_n$. Define $\Phi_\delta(\zeta) = (\Phi_{1\delta}(\zeta), \Phi_{2\delta}(\zeta), \dots, \Phi_{n\delta}(\zeta))$ by

$$\Phi_{1\delta} = z_1 + \frac{\delta^{3/4}}{2} X_1 \zeta + 2\zeta^2;$$

$$\Phi_{k\delta} = z_k + \frac{\delta^{3/4}}{2} X_k \zeta, \quad \text{for} \quad 2 \le k \le n-1;$$

$$\Phi_{n\delta} = z_n + \frac{\delta^{3/4}}{2} \zeta.$$

Then $\Phi_\delta(0) = z$, $\Phi'_\delta(0) = \frac{\delta^{3/4}}{2} X$.

Claim. *There exists* $\gamma \in (0, 1)$ *such that for all* $\delta > 0$ *sufficiently small, we have* $\Phi_\delta(\Delta_\gamma) \subset \Omega \cap U$.

Proof of the Claim. By choosing $\gamma \in (0, 1)$ small enough, we have $\Phi_\delta(\Delta_\gamma) \subset U$. It follows from (2.3) that

$$r(\Phi_\delta(\zeta)) \leq -\delta + \frac{\delta^{3/4}}{2} \operatorname{Re} \zeta - \frac{|\Phi_{1\delta}(\zeta)|^2}{2} + C \sum_{i=2}^{n} |\Phi_{i\delta}(\zeta)|^2.$$

Since $\delta > k|z|$, we see that when δ is sufficiently small,

$$r(\Phi_\delta(\zeta)) < -\frac{3\delta}{4} + \frac{\delta^{3/4}}{2} \operatorname{Re} \zeta - \frac{|\Phi_{1\delta}(\zeta)|^2}{2}. \tag{2.5}$$

For $|\zeta| < \delta^{1/4}$, by (2.5),

$$\begin{aligned} r(\Phi_\delta(\zeta)) &< -\frac{3\delta}{4} + \frac{\delta^{3/4}}{2} \cdot \delta^{1/4} - \frac{|\Phi_{1\delta}(\zeta)|^2}{2} \\ &= -\frac{\delta}{4} - \frac{|\Phi_{1\delta}(\zeta)|^2}{2} < 0. \end{aligned}$$

For $|\zeta| \geq \delta^{1/4}$, we have

$$\begin{aligned} |\Phi_{1\delta}(\zeta)|^2 &\geq 2|\zeta|^4 - |z_1 + \frac{\delta^{3/4}}{2} X_1 \zeta|^2 \\ &\geq 2|\zeta|^4 - C\delta^{3/2}. \end{aligned}$$

Thus (2.5) implies that when δ sufficiently small,

$$\begin{aligned} r(\Phi_\delta(\zeta)) &< -\frac{3\delta}{4} + \frac{\delta^{3/4}}{2} \operatorname{Re} \zeta - |\zeta|^4 + \frac{C\delta^{3/2}}{2} \\ &\leq -\frac{3\delta}{4} + \frac{C\delta^{3/2}}{2} + \frac{|\zeta|^4}{2} - |\zeta|^4 < 0. \end{aligned}$$

This concludes the proof of the Claim.

Next, by the Claim and the definition of the Kobayashi metric,

$$F_{\Omega \cap U}(z, X) \leq \frac{C}{\delta^{3/4}}.$$

Thus, combining (2.4) and the length-decreasing property of Kobayashi metric, we obtain

$$F_\Omega(z, X) \leq \frac{C}{d^{3/4}(z)},$$

which contradicts (1.1). Therefore, $\partial\Omega$ is Levi-pseudoconvex at z_0. □

Proof of Theorem B. By the assumption, there exists a neighborhood U of z_0 such that

$$\Omega \cap U \subset \{z \in U \mid \operatorname{Re} z_n - C(|z'|^m + |z_n| \cdot |z|) < 0\}. \tag{2.6}$$

Let $\Lambda = \{-\operatorname{Re} z_n > k|z|\}$ ($k \in (0, 1)$). For $z \in \Lambda \cap \Omega \cap U$ and $X \in \mathbb{C}^n$ (by homogeneity of the Kobayashi metric, we may assume that $|X| \leq 1$), let

$\Phi(\zeta) = (\widetilde{\Phi}(\zeta), \Phi_n(\zeta)) = (\Phi_1(\zeta), \Phi_2(\zeta), \dots, \Phi_n(\zeta)): \Delta \to \Omega \cap U$ be an analytic disc satisfying

$$\Phi(0) = z, \qquad \Phi'(0) = \lambda X, \tag{2.7}$$

where $\lambda > 0$ is a constant to be estimated. By the Cauchy Integral Formula, we have

$$|\Phi_i(\zeta) - z_i| \le C|\zeta|, \qquad 1 \le i \le n, \tag{2.8}$$

and

$$|\Phi_i(\zeta) - z_i - \lambda X_i \zeta| \le C|\zeta|^2, \qquad 1 \le i \le n, \tag{2.8'}$$

for $|\zeta| < 1/2$. Also, by (2.6), we have

$$\operatorname{Re}\Phi_n(\zeta) < C\left(|\widetilde{\Phi}(\zeta)|^m + |\Phi_n(\zeta)| \cdot |\Phi(\zeta)|\right). \tag{2.9}$$

Denote $\delta = -\operatorname{Re} z_n$. By (2.8), (2.8′), and the fact that $k|z| < \delta$, we have

$$\begin{aligned} |\Phi(\zeta)| &\le C(|z| + |\zeta|) \\ &\le C\left((1/k)\delta + c\delta^{1/m}\right) \\ &\le c^{1/2}\delta^{1/m}, \qquad \text{for} \quad |\zeta| < c\delta^{1/m} \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} |\Phi(\zeta)| &\le C\left(|z| + (\lambda|X|)|\zeta| + |\zeta|^2\right) \\ &\le C\left((1/k)\delta + (\lambda|X|)c\delta^{1/2m} + c^2\delta^{1/m}\right) \\ &\le c^{1/2}\left(\delta^{1/m} + (\lambda|X|)\delta^{1/2m}\right), \qquad \text{for} \quad |\zeta| < c\delta^{1/2m} \end{aligned} \tag{2.10'}$$

when c, δ are sufficiently small.

Now, it follows from (2.10), (2.10′), and (2.9) that

$$\operatorname{Re}\Phi_n(\zeta) < \frac{\delta}{2} + \frac{1}{2}|\Phi_n(\zeta)|, \quad \text{for} \quad |\zeta| < c\delta^{1/m} \tag{2.11}$$

and

$$\operatorname{Re}\Phi_n(\zeta) < \frac{\delta + (\lambda|X|)^m\delta^{1/2}}{2} + \frac{1}{2}|\Phi_n(\zeta)|, \quad \text{for} \quad |\zeta| < c\delta^{1/2m} \tag{2.11'}$$

when c, δ are sufficiently small.

Denote

$$D_{i\delta} = \left\{ w \in \mathbb{C} \;\middle|\; \operatorname{Re} w < \frac{\delta + \varepsilon_i(\lambda|X|)^m\delta^{1/2}}{2} + \frac{1}{2}|w| \right\}, \quad \text{for} \quad i = 1, 2$$

where $\varepsilon_1 = 0$, $\varepsilon_2 = 1$. Let $g_1(\zeta) \equiv \Phi_n(c\delta^{1/m}\zeta)$ and $g_2(\zeta) \equiv \Phi_n(c\delta^{1/2m}\zeta)$. By (2.7), (2.11), and (2.11′), we have $g_i(\Delta) \subset D_{i\delta}$, $g_i(0) = z_n$ ($i = 1, 2$), $g_1'(0) = \lambda X_n c\delta^{1/m}$, and $g_2'(0) = \lambda X_n c\delta^{1/2m}$. However, it is clear that

$$D_{i\delta} \subset \widetilde{D}_{i\delta} \equiv \mathbb{C} \setminus \left\{ w \in \mathbb{C} \;\middle|\; \operatorname{Im} w = 0, \operatorname{Re} w \ge \delta + \varepsilon_i(\lambda|X|)^m\delta^{1/2} \right\}.$$

Thus $g_i(\Delta) \subset \widetilde{D}_{i\delta}$.

Since

$$F_{\widetilde{D}_{i\delta}}(z_n, 1) \geq \frac{C}{\delta + \varepsilon_i(\lambda|X|)^m \delta^{1/2}}$$

where $C > 0$ is a constant independent on δ, it follows that

$$|g_i'(0)| \leq C(\delta + \varepsilon_i(\lambda|X|)^m \delta^{1/2}).$$

Therefore,

$$\lambda|X_n|\delta^{1/m} \leq C\delta \tag{2.12}$$

and

$$\lambda|X_n|\delta^{1/2m} \leq C(\delta + (\lambda|X|)^m \delta^{1/2}). \tag{2.13}$$

Now by (2.12), $\lambda|X_n| \leq C\delta^{1-1/m}$. Thus (1.2) is valid. Furthermore, when $|X| < K|X_n|$, it follows from (2.13) that $\lambda|X_n| \leq C\delta^{1-1/2m}$ where $C > 0$ may depend on K. Therefore, (1.3) follows. □

Proof of Corollary 1. The sufficiency comes directly from Theorem A. We now prove the necessity. Suppose that z_0 is a Levi-pseudoconvex point. After a change of coordinates, we may assume that z_0 is the origin and $\partial\Omega$ is locally defined by

$$r(z) = \operatorname{Re} z_n + \sum_{i,j=1}^{n} a_{ij} z_i \bar{z}_j + \mathscr{O}(|z|^3)$$

near z_0.

Since the matrix $(a_{ij})_{1 \leq i, j \leq n-1}$ is positive semidefinite,

$$\begin{aligned} r(z) &\geq \operatorname{Re} z_n + 2\operatorname{Re}\left(\sum_{j=1}^{n-1} a_{nj} z_n \bar{z}_j\right) + a_{nn}|z_n|^2 + \mathscr{O}(|z|^3) \\ &= \operatorname{Re} z_n + \mathscr{O}(|z'|^3 + |z_n| \cdot |z|). \end{aligned}$$

Therefore, (1.4) is valid for $\alpha = 5/6$ by Theorem B. □

Proof of Corollary 2. When z_0 is of finite type, as in Property 2.1 in [M], we can construct a smoothly bounded pseudoconvex domain $\Omega' \subset \Omega$ such that $\partial\Omega' \cap \partial\Omega \supset \partial\Omega \cap U$ for some neighborhood U of z_0. By the Lemma, there exist a neighborhood V of z_0 and a constant $C > 0$ such that

$$F_\Omega(z, X) \geq C F_{\Omega'}(z, X)$$

for $z \in V \cap \Omega'$, $X \in \mathbb{C}^n$. Applying Theorem 1 in [C] to Ω', we then obtain that (1.5) is valid for $\alpha = 1$.

When z_0 is of infinite type, then for each $m \in \mathbb{N}$, after a change of coordinates, the local defining function of Ω near z_0 has the form

$$r(z) = \operatorname{Re} z_2 + \mathscr{O}(|z_1|^m + |z_2| \cdot |z|).$$

Thus, (1.5) follows from Theorem B. □

Remark. The estimates in Theorem B are sharp. Suppose that Ω is a bounded domain such that $\partial\Omega$ is locally defined by $r(z) = \operatorname{Re} z_2 - |z_1|^{2m}$ near the origin. Then for $P_\delta = (0, -\delta)$, $X = (\delta^{-1+1/2m}, 1)$, and $Y = (0, 1)$, we have

$$F_\Omega(P_\delta, X) \approx \frac{1}{\delta^{1-1/2m}}, \quad F_\Omega(P_\delta, Y) \approx \frac{1}{\delta^{1-1/4m}}.$$

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DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY, ST. LOUIS, MISSOURI 63130
E-mail address: sfu@math.wustl.edu