THE WEAK STABILITY OF THE POSITIVE FACE IN $L^1$

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Abstract. Let $F$ be the positive face of the unit ball of $L^1[0, 1]$. We show that $F$ is weakly stable in the sense that the midpoint map $\Phi_{1/2}: F \times F \to F$, with $\Phi_{1/2}(f, g) = \frac{1}{2}(f + g)$, is open with respect to the weak topology. This weak stability of the set $F$ is the reason behind the fact that the notions of “huskable” and “strongly regular” operators coincide for operators from $L^1[0, 1]$ to a Banach space $X$. We prove this stability by showing that if $f_1, f_2 \in F$, $\lambda \in (0, 1)$, $\epsilon > 0$, and $\delta \geq \max\{2\epsilon/\lambda, 2\epsilon/(1 - \lambda)\}$, then

$$\lambda V_{P, \delta}(f_1) + (1 - \lambda)V_{P, \delta}(f_2) \supset V_{P, \epsilon}(\lambda f_1 + (1 - \lambda)f_2),$$

where $P = \{A_1, \ldots, A_n\}$ is a finite positive partition of $[0, 1]$ and

$$V_{P, \epsilon}(f) = \left\{ g \in F : \left| \frac{1}{n} \sum_{i=1}^{n} \int_{A_i} (f - g)(t) \, d\mu(t) \right| \leq \epsilon \right\}$$

for any $f$ in $F$. We construct an example showing that for any $0 < \lambda < 1$ there are functions $f_1$ and $f_2$ in $F$ such that if $0 < \epsilon < 2\min\{\lambda, 1 - \lambda\}$ and $0 \leq \delta < \max\{\epsilon/\lambda, \epsilon/(1 - \lambda)\}$, then

$$\lambda V_{P, \delta}(f_1) + (1 - \lambda)V_{P, \delta}(f_2) \supset V_{P, \epsilon}(\lambda f_1 + (1 - \lambda)f_2).$$

Thus the “formula” that $\lambda V_{P, \epsilon}(f_1) + (1 - \lambda)V_{P, \epsilon}(f_2) = V_{P, \epsilon}(\lambda f_1 + (1 - \lambda)f_2)$ given by Ghoussoub et al. in Mem. Amer. Math. Soc., vol. 70, no. 378, which is used there to establish the weak stability of $F$, is false.

Let $C$ be a convex subset of some topological vector space $X$. $C$ is said to be stable if the midpoint map $\Phi_{1/2}: C \times C \to C$, with $\Phi_{1/2}(x, y) = \frac{1}{2}(x + y)$, is open. If $X$ is a Banach space and $C$ is stable with respect to the weak topology, then we say that $C$ is weakly stable. It was proved in [1, Proposition 1.1] that if $C$ is stable and $X$ is locally convex, then for any $\lambda$ in $[0, 1]$ the map $\Phi_\lambda: C \times C \to C$, with $\Phi_\lambda(x, y) = \lambda x + (1 + \lambda)y$, is also open. Note that the conclusion holds without assuming $X$ to be locally convex. Hence convex combinations of nonempty relatively open subsets of a stable set are relatively open.

Throughout, the triple $(\Omega, \Sigma, \mu)$ will denote the Lebesgue measure space on $[0, 1]$ and $L^1$ will be the Banach space of all (equivalence classes of) Lebesgue integrable functions on $[0, 1]$ equipped with the norm $\|f\|_1 = \int_\Omega |f(t)| \, d\mu(t)$. 

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We shall denote by $F$ the positive face of the unit ball of $L^1$, i.e.,

$$F = \{ f \in L^1 : f \geq 0 \text{ and } \|f\|_1 = 1 \}.$$

Let $P = \{ A_1, \ldots, A_n \}$ be a finite positive partition of $\Omega$ and $\varepsilon \geq 0$. Define, for $f \in F$,

$$V_{P, \varepsilon}(f) = \left\{ g \in F : \sum_{i=1}^{n} \left| \int_{A_i} (f - g) \, d\mu \right| \leq \varepsilon \right\}$$

and

$$V_{P, \varepsilon}^0(f) = \left\{ g \in F : \sum_{i=1}^{n} \left| \int_{A_i} (f - g) \, d\mu \right| < \varepsilon \right\}.$$

As pointed out in [2] the sets $V_{P, \varepsilon}^0(f)$ form a relative weak neighborhood base of $f$ in $F$ when $P$ runs through the finite positive partitions of $\Omega$ and $\varepsilon$ runs through $(0, 1]$.

The weak stability of the face $F$ is the reason behind the fact that the notions of "huskable" and "strongly regular" operators coincide for operators from $L^1[0, 1]$ to a Banach space $X$ (cf. [2, Theorem IV.10]). In [2], the weak stability of $F$ is established using the following lemma.

**Lemma 1** [2, Lemma IV.4]. If $f_1, f_2 \in F$ and $\lambda \in [0, 1]$, then

$$\lambda V_{P, \varepsilon}(f_1) + (1 - \lambda)V_{P, \varepsilon}(f_2) = V_{P, \varepsilon}(\lambda f_1 + (1 - \lambda)f_2).$$

However, the formula in this lemma need not hold as seen by the following counterexample (Example 3). Consequently, it is important to establish that $F$ is weakly stable without using this formula. Theorem 4 gives a correct variant of the above lemma which is strong enough to conclude the desired weak stability result. We will use the following lemma in Example 3.

**Lemma 2** [2, Lemma IV.3]. For $f \in F$ and $\varepsilon \geq 0$, the following holds true:

$$V_{P, \varepsilon}(f) = [V_{P, \varepsilon}^0(f) + \varepsilon B_{L^1}] \cap F,$$

where $B_{L^1}$ is the closed unit ball of $L^1$.

**Example 3.** Fix $\lambda \in (0, 1)$. Let $A_1 = [0, \lambda)$, $A_2 = [\lambda, 1]$, $f_1 = \frac{1}{2} \chi_{A_1}$, $f_2 = \frac{1 - \lambda}{2} \chi_{A_2}$, and $P = \{ A_1, A_2 \}$. Fix $0 < \varepsilon < 2 \min\{\lambda, 1 - \lambda\}$, and let $g = \frac{\varepsilon}{2(1 - \lambda)} \chi_{A_1} - \frac{\varepsilon}{2(1 - \lambda)} \chi_{A_2}$ and $g' = -g$. It is clear that $f_1, f_2$ are in $F$ and that $\lambda f_1 + (1 - \lambda)f_2 + g$ and $\lambda f_1 + (1 - \lambda)f_2 + g'$ are in $V_{P, \varepsilon}(\lambda f_1 + (1 - \lambda)f_2)$. Let $\rho = \max\{\frac{1}{2}, \frac{1}{1 - \lambda}\}$. The proof below shows that if $0 < \delta < \rho \varepsilon$ then $\lambda f_1 + (1 - \lambda)f_2 + g$ or $\lambda f_1 + (1 - \lambda)f_2 + g'$ does not belong to $\lambda V_{P, \delta}(f_1) + (1 - \lambda)V_{P, \delta}(f_2)$. In particular, if $0 \leq \delta < \rho \varepsilon$ then

$$\lambda V_{P, \delta}(f_1) + (1 - \lambda)V_{P, \delta}(f_2) \not\subseteq V_{P, \varepsilon}(\lambda f_1 + (1 - \lambda)f_2).$$

Thus by taking $\delta = \varepsilon$ we see that [2, Lemma IV.4] need not hold.

**Proof.** Suppose that for some $\delta \geq 0$ both $\lambda f_1 + (1 - \lambda)f_2 + g$ and $\lambda f_1 + (1 - \lambda)f_2 + g'$ are in $\lambda V_{P, \delta}(f_1) + (1 - \lambda)V_{P, \delta}(f_2)$. We will show that $\delta \geq \rho \varepsilon$.

Since $\lambda f_1 + (1 - \lambda)f_2 + g \in \lambda V_{P, \delta}(f_1) + (1 - \lambda)V_{P, \delta}(f_2)$, by Lemma 2 there are functions $h_i \in V_{P, \delta}(f_i)$ and $g_i \in \delta B_{L^1}$ such that $h_i + g_i \in V_{P, \delta}(f_i)$ and

$$\lambda f_1 + (1 - \lambda)f_2 + g = \lambda(h_1 + g_1) + (1 - \lambda)(h_2 + g_2).$$
Note that $h_i$ and $h_i + g_i$ are in $F$ and that $\int_{A_j} h_i \, d\mu = \int_{A_j} f_i \, d\mu = \delta_{ij}$ for $i, j \in \{1, 2\}$. Thus $\int_{A} g_i \, d\mu = 0$. Furthermore, almost everywhere on $A_2$, we have that $h_1 = 0$ and so $g_1 \geq 0$. It follows that

$$
(1 - \lambda) - \frac{\varepsilon}{2} = \int_{A_2} [\lambda f_1 + (1 - \lambda) f_2 + g] \, d\mu \\
= \int_{A_2} [\lambda (h_1 + g_1) + (1 - \lambda) (h_2 + g_2)] \, d\mu \\
= (1 - \lambda) + \lambda \int_{A_2} g_1 \, d\mu + (1 - \lambda) \int_{A_2} g_2 \, d\mu
$$

and so

$$
- \int_{A_1} g_2 \, d\mu = \int_{A_2} g_2 \, d\mu = -\frac{\varepsilon}{2(1 - \lambda)} - \frac{\lambda}{1 - \lambda} \int_{A_2} g_1 \, d\mu \leq -\frac{\varepsilon}{2(1 - \lambda)}.
$$

Thus $\frac{\varepsilon}{1 - \lambda} \leq g_2$. Similarly $\frac{\varepsilon}{1 - \lambda} \leq g_1$. Therefore, $\frac{\varepsilon}{1 - \lambda}$. Hence, one can still conclude the weak stability of $F$ from the following variant of [2, Lemma IV.4].

**Theorem 4.** Let $f_1, f_2$ be in $F$. If $\lambda \in (0, 1), \varepsilon > 0$, and $\delta \geq \max\{\frac{\varepsilon}{\lambda^2}, \frac{\varepsilon}{1 - \lambda}\}$, then

$$
\lambda V_{P, \varepsilon} (f_1) + (1 - \lambda) V_{P, \varepsilon} (f_2) \supset V_{P, \varepsilon} (\lambda f_1 + (1 - \lambda) f_2).
$$

**Proof.** Let $\{A_1, \ldots, A_n\}$ be the sets of the partition $P$. Let $a_i = \int_{A_i} f_1 \, d\mu$ and $b_i = \int_{A_i} f_2 \, d\mu$ for $1 \leq i \leq n$. Let $g$ be any function in $V_{P, \varepsilon} (\lambda f_1 + (1 - \lambda) f_2)$. Put

$$
\alpha_i = \frac{a_i}{\lambda a_i + (1 - \lambda) b_i} \quad \text{and} \quad \beta_i = \frac{b_i}{\lambda a_i + (1 - \lambda) b_i},
$$

observing the convention that $0 \frac{a_i}{b_i}$ is 1. Note that $0 \leq \alpha_i \leq \frac{1}{\lambda}$ and $0 \leq \beta_i \leq \frac{1}{1 - \lambda}$. Let

$$
h_1(\cdot) = \sum_{i=1}^{n} \alpha_i g(\cdot) \chi_{A_i} \quad \text{and} \quad h_2(\cdot) = \sum_{i=1}^{n} \beta_i g(\cdot) \chi_{A_i}.
$$

Clearly $h_i \geq 0$ and $\lambda h_1 + (1 - \lambda) h_2 = g$. Thus, without loss of generality, $\|h_i\|_1 \geq 1$. Let

$$
g_1 = \frac{h_1}{\|h_1\|_1} \quad \text{and} \quad g_2 = \frac{1}{1 - \lambda} (g - \frac{\lambda}{1 - \lambda} g_1).
$$

Clearly $g_1 \in F$ and $h_1 \geq g_1 \geq 0$ and $\lambda g_1 + (1 - \lambda) g_2 = g$. Hence $\lambda (h_1 - g_1) = (1 - \lambda) (g_2 - h_2)$ and so $g_2 \geq h_2 \geq 0$, thus $g_2 \in F$. To complete the proof we need only to show that $\sum_{i=1}^{n} \left| \int_{A_i} (f_j - g_j) \, d\mu \right| \leq \delta$ for $j = 1$ and 2.

Towards this, first note that

$$
\sum_{i=1}^{n} \left| \int_{A_i} (f_1 - h_1) \, d\mu \right| = \sum_{i=1}^{n} \left| a_i - \alpha_i \int_{A_i} g \, d\mu \right| = \sum_{i=1}^{n} \alpha_i \left| \lambda a_i + (1 - \lambda) b_i - \int_{A_i} g \, d\mu \right|
$$

$$
= \sum_{i=1}^{n} \alpha_i \left| \int_{A_i} [\lambda f_1 + (1 - \lambda) f_2 - g] \, d\mu \right| \leq \frac{\varepsilon}{\lambda}.
$$
Likewise, $\sum_{i=1}^{n} \left| \int_{A_i} (f_2 - h_2) \, d\mu \right| \leq \frac{\varepsilon}{1 - \lambda}$. Thus

$$\sum_{i=1}^{n} \left| \int_{A_i} (h_1 - g_1) \, d\mu \right| = \sum_{i=1}^{n} \left| \int_{A_i} h_1 \left( 1 - \frac{1}{\|h_1\|_1} \right) \, d\mu \right| = \left| 1 - \frac{1}{\|h_1\|_1} \right| \|h_1\|_1$$

$$= \|h_1\|_1 - 1 = \left| \int_{\Omega} h_1 \, d\mu - \int_{\Omega} f_1 \, d\mu \right|$$

$$\leq \sum_{i=1}^{n} \left| \int_{A_i} (f_1 - h_1)(t) \, d\mu(t) \right| \leq \frac{\varepsilon}{\lambda},$$

and so

$$\sum_{i=1}^{n} \left| \int_{A_i} (h_2 - g_2) \, d\mu \right| = \frac{\lambda}{1 - \lambda} \sum_{i=1}^{n} \left| \int_{A_i} (h_1 - g_1) \, d\mu \right| \leq \frac{\varepsilon}{1 - \lambda}.$$

Hence $\sum_{i=1}^{n} \left| \int_{A_i} (f_j - g_j) \, d\mu \right| \leq \delta$ for $j = 1$ and $2$, as needed, and the proof is complete.

**Corollary 5.** The positive face $F$ is weakly stable.

**Proof.** Suppose $V_1$ and $V_2$ are two nonempty relatively weakly open subsets of $F$. Let $f_i$ be any point in $V_i$. Since the set $V_{P,\varepsilon}^0(f)$ forms a relative weak neighborhood base of $f$ in $F$, when $P$ runs through the finite positive partitions of $\Omega$ and $\varepsilon$ runs through $(0, 1]$, there exist partitions $P_1$ and $P_2$ and positive numbers $\delta_1$ and $\delta_2$ such that $V_{P_i,\delta_i}^0(f_i) \subset V_i$. Let $0 < \delta < \min\{\delta_1, \delta_2\}$, $0 < \varepsilon < \frac{1}{4} \delta$, and let $P$ be a finer partition of $\Omega$ than $P_1$ and $P_2$. By Theorem 4, we have

$$\frac{1}{2} V_{P,\varepsilon}^0(f_1) + \frac{1}{2} V_{P,\varepsilon}^0(f_2) \supset V_{P,\varepsilon}^0(\frac{1}{2} f_1 + \frac{1}{2} f_2).$$

Since

$$\frac{1}{2} V_1 + \frac{1}{2} V_2 \supset \frac{1}{2} V_{P_1,\delta_1}^0(f_1) + \frac{1}{2} V_{P_2,\delta_2}^0(f_2) \supset \frac{1}{2} V_{P,\varepsilon}^0(f_1) + \frac{1}{2} V_{P,\varepsilon}^0(f_2),$$

$$\supset V_{P,\varepsilon}^0(\frac{1}{2} f_1 + \frac{1}{2} f_2) \supset V_{P,\varepsilon}^0(\frac{1}{2} f_1 + \frac{1}{2} f_2),$$

the set $\frac{1}{2} V_1 + \frac{1}{2} V_2$ is a weak neighborhood of $\frac{1}{2} f_1 + \frac{1}{2} f_2$ and so $\frac{1}{2} V_1 + \frac{1}{2} V_2$ is weakly open. Therefore, $F$ is weakly stable, and the proof is complete.

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**References**


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