

## SOME PROJECTIVE PLANES OF LENZ-BARLOTTI CLASS I

JOHN T. BALDWIN

(Communicated by Andreas R. Blass)

**ABSTRACT.** We construct an infinite projective plane with Lenz-Barlotti class I. Moreover, the plane is almost strongly minimal in a very strong sense: each automorphism of each line extends uniquely to an automorphism of the plane.

In [1] we modified an idea of Hrushovski [4] to construct a family of almost strongly minimal non-Desarguesian projective planes. In this note we determine the position of these planes in the Lenz-Barlotti classification [7]. We further make a minor variant on the construction to build an almost strongly minimal projective plane whose automorphism group is isomorphic to the automorphism group of any line in the plane.

By a projective plane we mean a structure for a language with a unary predicate for lines, a unary predicate for points, and an incidence relation that satisfies the usual axioms for a projective plane.

We work with a collection  $K$  of finite graphs and an embedding relation  $\leq$  (often called strong embedding) among these graphs. Each of these projective planes  $(M, P, L, I)$  is constructed from a  $(K, \leq)$ -homogeneous universal graph  $(M^*, R)$  as in [1] and [4]. This paper depends heavily on the methods of those two papers and on considerable technical notation introduced in [1]. Each plane  $M$  is derived from a graph  $M^* = (M^*, R)$ . Formally this transformation sets  $P = M^* \times \{0\}$  and  $L = M^* \times \{1\}$ , and  $\langle m, 0 \rangle$  is on  $\langle n, 1 \rangle$  if and only if  $R(m, n)$ . Thus, if  $G^*$  denotes the automorphism group of  $M^*$ , there is a natural injection of  $G^*$  into the automorphism (collineation) group  $G$  of  $M$ . There is a natural polarity  $\rho$  of  $M(\rho(\langle m, i \rangle) = \langle m, 1 - i \rangle$  for  $i = 0, 1)$ . Then  $G^*$  is naturally regarded as the subgroup of  $G$  consisting of those collineations that are automorphisms of the structure  $(M, P, L, I, \rho)$ .

It should be noticed that this self-duality plays an important role in the construction of the homogenous-universal graph. If one attempts to avoid this by constructing a homogeneous universal bipartite graph, it is soon observed that the amalgamation property fails. (Construct  $B$  and  $C$  extending  $A$  and an element  $x$  of  $A$  such that  $B$  thinks  $x$  is a point and  $C$  thinks  $x$  is a line.)

The almost strong minimality of  $M$  is established in [1] by showing any line is strongly minimal and noting that the plane is in the algebraic closure of any

---

Received by the editors January 4, 1992 and, in revised form, April 9, 1993.  
1991 *Mathematics Subject Classification*. Primary 51A35; Secondary 03C45.  
Partially supported by NSF grant 90000139.

single line and two points off the line. After making a minor variant in the construction we build here a plane which is in the definable closure of any line (without additional parameters!).

We can control certain aspects of the behavior of  $G^*$  by varying the class  $\mathbf{K}$ . In order to actually determine, e.g., the transitivity of  $G$  we must vary the construction further.

1. *Notation.* Let  $A$  be a subset of  $N$ .

- (i) We write  $stb_N(A)$  for the group of automorphisms of  $N$  that fix  $A$  pointwise.
- (ii) We write  $sstb_N(A)$  for the group of automorphisms of  $N$  that fix  $A$  as a set.

Note that if  $l$  is a line, the notation  $stb_M(l)$  implicitly demands that we think of  $l$  as a subset of  $P$ , not an element of  $L$ .

2. **Definition.** (i) For any point  $p$  and line  $l$  a  $(p, l)$ -collineation is a collineation that fixes  $l$  pointwise and  $p$  linewise.

(ii) A projective plane  $M$  is  $(p, l)$ -transitive if the subgroup of  $(p, l)$ -collineations acts 1-transitively on  $l' - \{p, q_{l'}\}$ , where  $q_{l'}$  is the intersection of  $l$  and  $l'$ , for each line  $l'$  through  $p$  (except  $l$  if  $p$  is on  $l$ ).

Note that any collineation  $\alpha$  that fixes a line  $l$  pointwise also fixes a (unique) point  $p$  linewise [5, Theorem 4.9]. This point is called the *center* of the collineation.  $\alpha$  is referred to as either a  $(p, l)$ -central collineation or a  $(p, l)$ -perspectivity. Then,  $stb_M(l)$  is the collection of all  $(p, l)$ -perspectivities as  $p$  varies.

The Lenz-Barlotti classification describes a projective plane  $M$  by the set  $F_M = \{(p, l) : M \text{ is } (p, l)\text{-transitive}\}$ . Various conditions on the class  $F_M$  correspond to properties of the ternary ring that coordinatizes  $P$ . While there are 16 different categories in the classification, we deal here with only class  $I_1: F_M = \emptyset$ .

3. **Definition.** (i) A flag in the projective plane  $M$  is a pair  $(p, l)$  with  $p$  on  $l$ .

(ii) The collineation group of  $M$  is *flag transitive* if it acts 1-transitively on flags.

In [1] each of a certain family of functions  $\mu$  from  $\omega$  to  $\omega$  determines a different projective plane. All of these planes behave in the same way for the properties discussed in this paper, so we do not keep track of the function  $\mu$ . It is not clear if the collineation group of any of the planes  $M_0$  constructed in [1] is either flag transitive or acts transitively on pairs  $(p, l)$  with  $p$  not on  $l$ . However, by modifying the construction to define  $M_1$  we can guarantee increased transitivity by ensuring that  $G_1^*$  acts with a sufficient required degree of transitivity on  $M_1^*$ .

To see that  $G_0^*$  does not act flag transitively on  $M_0$  note that by slightly extending the construction in [1] (or see [2]) of a 14-element graph  $A$  with  $y(A) = 3$ , it is possible to construct a square-free graph  $B$  with 17 elements and  $y(B) = 2$ . Now if  $C = \langle a, b \rangle$  is contained in  $B$  with  $aRb$  and  $B$  is strongly embedded in  $M_0^*$ , there is no automorphism of  $M_0^*$  taking  $C$  to a  $C' \leq M_0^*$  since  $C$  is not a strong submodel of  $M_0^*$ . But  $C$  and  $C'$  both

represent flags of  $M$ . But this leaves open the possibility that  $G_0$  could still act flag transitively.

As in [1] we fix the dimension function  $y(A) = 2|A| - e(A)$  where  $e(A)$  is the number of edges of the graph  $A$ .

4. **Definition.**  $K$  is the collection of all finite graphs  $B$  such that:

- (i) for every nonempty  $B' \subseteq B$ ,  $y(B') > 1$ ;
- (ii) there is no square (4-cycle) embedded in  $B$ ;
- (iii) the adjacency relation is symmetric and irreflexive.

Fix a function  $\mu$  as in [1] and recall from there such notions as minimally 0-simply algebraic.

5. **Definition.** For  $n = 2$  or  $3$ , let  $K_n^*$  be the class of finite structures  $M \in K$  satisfying the following additional conditions:

- (i) If  $A \subseteq M$  and  $|A| > n$  then  $y(A) > n$ .
- (ii) For each pair  $\langle A, B \rangle$  with  $B$  minimally 0-simply algebraic over  $A$  there are at most  $\mu(A, B)$  disjoint copies  $B_i$  of  $B$  over  $A$  in  $M$ .

Now slightly varying the proof in [1] yields the following

6. **Lemma.** For  $n = 2$  or  $3$ , the class  $K_n^*$  has the amalgamation property.

*Proof.* We present in detail the case of  $K_3^*$ . Recall from [1] that the proof that there is an amalgam of the strong extensions  $B_1$  and  $B_2$  over  $B_0$  proceeds by induction on  $|\widehat{B}_1| + |\widehat{B}_2|$ . ( $\widehat{B}_i$  denotes  $B_i - B_0$ .) The proof reduces to checking two cases. In the first  $B_1 = B_0b$  and there is one relation between  $b$  and  $B_0$ . In the second  $B_1$  is 0-simply algebraic over  $B_0$ . In the first case  $B_1 \otimes_{B_0} B_2$  is the amalgam. In the second case, either  $B_1$  can be embedded in  $B_2$  over  $B_0$  or  $B_1 \otimes_{B_0} B_2$  is the amalgam. Thus, to extend the result to our situation we simply have to show that  $B_1 \otimes_{B_0} B_2$  does not contain a  $D$  with  $|D| > 3$  and  $y(D) \leq 3$  if neither  $B_1$  nor  $B_2$  does. For this, note that  $3 \geq y(D) = y(\widehat{D}_2/D_1) + y(D_1)$  and  $y(\widehat{D}_2/D_1) \geq y(\widehat{D}_2/B_0) \geq 0$ , so  $y(D_1) \leq 3$ . Since  $B_1 \in K_3$ , this implies  $|D_1| \leq 3$ . Similarly,  $|D_2| \leq 3$ . Thus,  $|D| \leq 6$ . It can easily be checked by inspection that there are no square-free graphs  $D$  of cardinality 4, 5, or 6 with  $y(D) \leq 3$ . (See [2] for more general results as well as this fact.) The possibility that  $|D| \leq 3$  is easily eliminated to complete the proof.

The case of  $K_2^*$  is even easier as the reduction is to graphs of size 3 or 4.

7. **Theorem.** (i) *The almost strongly minimal plane associated with  $K_2^*$  is flag transitive.*

(ii) *The almost strongly minimal plane  $M_3$  associated with  $K_3^*$  is flag transitive. There are at most two orbits of pairs  $(p, l)$  where  $p$  does not lie on  $l$ . For each  $l$ ,  $sstb_{M_3}(l)$  is infinite.*

*Proof.* We write out only the more complicated second argument. The choice of  $K_3$  shows that any embedding of a pair into the homogeneous-universal model  $M_3^*$  is actually a strong embedding. If  $p$  lies on  $l$ , then  $p = \langle a, 0 \rangle$  and  $l = \langle b, 1 \rangle$  for some  $a, b$  in  $M_3^*$  with  $aRb$ . All such pairs lie in the same orbit under  $G^*$ ; this establishes flag transitivity. Now suppose  $p = \langle a, 0 \rangle$  and  $l = \langle b, 1 \rangle$  for  $a, b$  in  $M_3^*$  but that  $p$  does not lie on  $l$ . There are two cases. In the first,  $a \neq b$  and point line pairs of this kind are automorphic since  $G^*$

acts transitively on pairs  $a, b$  with  $\neg aRb$ . In the second,  $a = b$  and point line pairs of this kind are automorphic since  $G^*$  acts 1-transitively on  $M_3^*$ .

Now fix any  $l$ . There is a point  $p$  not on  $l$  such that  $\langle p, l \rangle$  is of the second type. For any other  $p', p'', \langle p', l \rangle$  and  $\langle p'', l \rangle$  are of the first type, so there is a collineation fixing the line  $l$  (setwise) and mapping  $p'$  to  $p''$ . Thus  $stb_{M_3}(l)$  is infinite.

8. **Corollary.**  $M_3$  satisfies one of the following two conditions:

- (i)  $G$  acts transitively on pairs  $(p, l)$  with  $p$  not on  $l$ .
- (ii)  $\rho$  is definable in  $(M_3, P, L, I)$ .

*Proof.* We know from Theorem 7(ii) that there are at most two orbits of non-incident point-line pairs. One of them contains all pairs of the form  $(\langle a, 0 \rangle, \langle a, 1 \rangle)$ , which is the graph of  $\rho$ . All pairs not in this orbit realize the same type. If there are really two distinct orbits, since  $M_3$  is homogeneous, there must be a formula which holds exactly of the graph of  $\rho$  as required.

The following result is just a special case of Zilber's general definition of a linking group as reported in Theorem 2.20 of [6].

9. **Lemma.** Let  $M$  be an  $\aleph_1$ -categorical projective plane.

- (i) For any line  $l$ ,  $stb_M(l)$  is a definable group and its action on  $M$  is definable.
- (ii) For any line  $l$  and point  $p$  the group of  $(p, l)$ -collineations is definable.

*Proof.* Here is a sketch of the proof of (i). This situation is somewhat simpler than the general case. Fix two points  $p_1, p_2$  that are not on  $l$ . As  $l$  is an infinite definable subset of  $M$  and  $Th(M)$  admits no two cardinal models,  $M$  is prime over  $l$ , so the orbit  $X$  of  $\langle p_1, p_2 \rangle$  under  $G = stb_M(l)$  is definable by some formula  $\gamma$ . If  $\alpha \in G$ ,  $\alpha$  is determined by  $\langle \alpha p_1, \alpha p_2 \rangle$ , and this determines a 1-1 correspondence between  $G$  and  $X$ . Thus we regard  $X$  as the universe of our definable copy of  $G$ . For any  $x \in M$ , that is not on  $p_1 p_2$ , call the intersection of  $x p_1$  and  $x p_2$  with  $l$  the  $\langle p_1, p_2 \rangle$  coordinate of  $x$ . Suppose  $\langle q_1, q_2 \rangle, \langle q'_1, q'_2 \rangle$ , and  $\langle q''_1, q''_2 \rangle$  are in  $X$ . The product of  $\langle q_1, q_2 \rangle$  and  $\langle q'_1, q'_2 \rangle$  is  $\langle q''_1, q''_2 \rangle$  just if the  $\langle q''_1, q''_2 \rangle$  coordinates of  $\alpha' \circ \alpha(x)$  are the  $\langle p_1, p_2 \rangle$  coordinates of  $x$ . We have shown how to define the value of the composition on any point  $x$  not on  $p_1 p_2$ . To extend this to points on  $p_1 p_2$  note that the entire argument can be repeated by replacing  $p_1$  by any  $p'_1$  not on  $p_1 p_2$  to determine the action on  $x$ . Moreover, this construction does not depend on the choice of  $p'_1$ . Since  $\alpha' \circ \alpha(x)$  is definably computed from  $x, \langle q_1, q_2 \rangle$ , the multiplication is definable. In particular, we have defined the action of each member of  $stb_M(l)$ .

(ii) If  $p$  is not on  $l$ , the  $(p, l)$ -collineations are just the subgroup of  $stb_M(l)$  that fix  $p$ . If  $p$  is on  $l$ , it is the subgroup of those elements of  $stb_M(l)$  that fix (setwise) each line  $l'$  through  $p$ .

10. **Lemma.** If there is no infinite group definable in the projective plane  $M$  and lines in  $M$  are infinite then  $F_M = \emptyset$ . Thus all the planes constructed in [1] are of Lenz-Barlotti class  $I_1$ .

*Proof.* By Lemma 9(ii), for any  $p$  and  $l$  the group of  $(p, l)$  collineations is definable. If  $M$  is  $(p, l)$  transitive, it is infinite. But this is impossible, so  $F_M = \emptyset$  as required.

Note that applying this line of reasoning in the other direction and using Theorem 7(ii) we conclude that for each line  $l$ ,  $stb_{M_3}(l)$  is not definable.

A ternary ring is definable in any projective plane. In any ternary ring addition and multiplication are defined by  $a + b = T(a, 1, b)$  and  $a \cdot b = T(a, b, 0)$ . The ring is *linear* if  $T(x, a, b) = x \cdot a + b$ . The basic properties of the Lenz-Barlotti classification as expounded in, e.g., [3] show that for each plane of Lenz-Barlotti class  $I_1$  the associated planar ternary ring is not even linear. So not only is neither the additive nor the multiplicative structure of the ternary ring defined in our projective plane associative, but it is actually impossible to split the ternary operation into two binary operations.

We can extend this "rigidity" even further.

**11. Proposition.** *If the automorphism group of the plane  $M$  acts both flag transitively and transitively on pairs  $(p, l)$  with  $p$  not on  $l$  and  $stb_M(l)$  is nontrivial then  $stb_M(l)$  is infinite.*

*Proof.* Choose  $\beta \in stb_M(l)$  which is not the identity. Let  $p_\beta$  be the center of  $\beta$ . If  $p_\beta$  is on  $l$  then by flag transitivity for each point  $p_i$  on  $l$  there is an automorphism  $\alpha_i$  mapping  $p_\beta$  to  $p - i$ . If  $p_\beta$  is not on  $l$ , let  $\{p_i : i < \omega\}$  be an infinite set of points that are not on  $l$ . Choose for each  $i$ , by the second transitivity hypothesis, an automorphism  $\alpha_i \in sstb_M(l)$  with  $\alpha_i p_\beta = p_i$ . In either case  $\beta^{\alpha_i}$  is a perspectivity with axis  $l$  and center  $\alpha_i p_\beta = p_i$ , so  $stb_M(l)$  is infinite.

**12. Theorem.** *The almost-strongly minimal plane  $M_3$  associated with  $K_3^*$  is in the definable closure of any line. That is,  $M_3$  admits no perspectivities.*

*Proof.* By Corollary 8 either  $\rho$  is definable or  $G$  acts transitively on nonincident point-line pairs. In the first case let  $a_0, a_1, a_2$  lie on  $l$ . Then at least two of the three pairwise intersections of the  $\rho(a_i)$  are not on  $l$ . Thus two points off  $l$  and a fortiori all points in  $P$  are in  $dcl(l)$ . In the second case, it suffices to show that  $stb_M(l)$  is the identity for any  $l$ . By Lemma 9  $stb_M(l)$  is definable and by Proposition 11 and Theorem 7, if  $stb_M(l)$  is nontrivial then it is infinite. Thus, for any  $l$ ,  $stb_M(l)$  is trivial.

Three questions arise.

- (i) What is a geometric explanation of the phenomenon of Theorem 12?
- (ii) Is  $\rho$  always definable?
- (iii) Does  $\text{Aut}(M_3)$  contain any involutions?

We can essentially rephrase (ii) by asking whether  $M$  admits any automorphisms that are not induced in the obvious way by automorphisms of  $M^*$ .

We show now that  $\text{Aut}(M_3)$  has no definable involutions. By a subplane of  $M$  we mean a subset  $M_0$  of points and lines that, with respect to the same incidence relation, form a projective plane.  $M_0$  is a Baer subplane if each point (line) in  $M$  lies on (contains) a line (point) from  $M_0$ .

**13. Corollary.** *Any involution of  $M_3$  fixes a subplane pointwise.*

*Proof.* By Theorems 4.3 and 4.4 of [5], the fixed set of any involution of a projective plane is a Baer subplane unless the involution is a perspectivity, but we know there are no perspectivities.

**14. Lemma.** *If  $\alpha$  is an involution of  $M_3$  and  $l$  is any line fixed by  $\alpha$ , then  $stb_M(\alpha) \cap l$  is infinite and coinfinite.*

*Proof.* Let  $F = \text{stb}_{M_3}(\alpha)$ ,  $F_P$  be the points fixed by  $\alpha$ ,  $F_L$  be the lines fixed by  $\alpha$ , and  $X = F_P \cap l$ . Clearly the intersection of  $F_P$  with any line in  $F_L$  is infinite. Choose  $l_1 \in F_L$  with  $l_1 \cap l = b \in F_P$ . Let  $a \in l_1 - F_P$ . Fix  $l_2 \in F_L$  but not equal to  $l$  or  $l_1$ . Now, for any of the infinitely many points  $y$  on  $l_2 \cap F_P$ , the line determined by  $a$  and  $y$  intersects  $l$  in a point not in  $F_P$ . So  $l - F_P$  is infinite.

**15. Corollary.**  $M_3$  has no definable involutions.

*Proof.* If  $\alpha$  were a definable automorphism of  $M$ , its stabilizer  $F$  would be defined and, for any  $l$  in  $F$ ,  $F \cap l$  would be definable. But then Lemma 14 contradicts the strong minimality of each line in  $M$ .

This leaves open the question of whether this structure admits any involutions that are not definable.

#### REFERENCES

1. J. T. Baldwin, *An almost strongly minimal non-desarguesian projective plane*, Trans. Amer. Math. Soc. (to appear).
2. C. R. J. Clapham, A. Flockhart, and J. Sheehan, *Graphs without four-cycles*, J. Graph Theory 13 (1987), 29–47.
3. Peter Dembowski, *Finite geometries*, Springer-Verlag, New York, 1977.
4. E. Hrushovski, *Construction of a strongly minimal set*, preprint, 1988.
5. D. R. Hughes and F. C. Piper, *Projective planes*, Springer-Verlag, New York, 1973.
6. Bruno Poizat, *Sous-groupes définissables d'un groupe stable*, J. Symbolic Logic 46 (1981), 137–146.
7. J. Yaqub, *The Lenz-Barlotti classification*, Proceedings of the Projective Geometry Conference, 1967 (R. Sandler, ed.), University of Illinois at Chicago Circle, 1967.

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, UNIVERSITY OF ILLINOIS  
AT CHICAGO, CHICAGO, ILLINOIS 60680  
E-mail address: u12800@uicvm