ON THE INTEGRALITY OF SOME GALOIS REPRESENTATIONS

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Abstract. We find an appropriate topology to put on $K$, the fraction field of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T]]$, so that compact subgroups of $K^\times$ are in fact contained in $\Lambda^\times$. This ensures that Galois representations to $K^\times$ have image in $\Lambda^\times$.

Let $\Lambda = \mathbb{Z}_p[[T]]$ be the Iwasawa algebra. $\Lambda$ is a unique factorization domain. The $p$-adic Weierstrass Preparation Theorem says that elements of $\Lambda$ may be represented as $uf$, where $f$ is a polynomial and $u$ is a unit.

Let $M = (p, T)$ be the maximal ideal of $\Lambda$. Topologize $\Lambda$ so that a base of neighborhoods of 0 is given by powers of $M$, and define neighborhoods of other elements of $\Lambda$ by translation.

Let $K$ be the field of fractions of $\Lambda$. The first question to consider is how to topologize $K$. One somewhat obvious approach is to say that a set $U \subset K$ is open in $K$ precisely when $kU \cap \Lambda$ is an open subset of $\Lambda$ for all $k \in K^\times$. This definition makes addition and multiplication continuous. Topologized in this way, a compact subset of $\text{GL}_n(K)$ which is also a subgroup is conjugate to a subset of $\text{GL}_n(\Lambda)$. Unfortunately, there is one major drawback to this topology.

Proposition. The function $f(x) = x^{-1}$ is not continuous in this topology.

Proof. There are many ways to see this. Perhaps the simplest is to observe that the sequence $a_n = p + T^n$ converges to $p$. However, the sequence $a_n^{-1}$ is closed, since for a fixed $k \in K^\times$, $ka_n^{-1}$ will be an element of $\Lambda$ for only finitely many $n$. Hence, $a_n^{-1}$ cannot converge to $p^{-1}$.

We therefore need a different topology on $K$, and fortunately there is an obvious candidate. If $\lambda \in \Lambda$, we can define $v(\lambda) = n$ if $\lambda \in M^n$ and $\lambda \notin M^{n+1}$ and $v(0) = \infty$. Krull's Theorem [1] implies that $\bigcap M^n = \{0\}$, and so the function $v$ is well defined.

Lemma. $v$ is a valuation on $\Lambda$.

Proof. Let $f, g \in \Lambda$. Set $v(f) = m$ and $v(g) = n$. Obviously, $v(f + g) \geq \min(v(f), v(g))$, so we need only show that $v(fg) = v(f) + v(g)$.

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Use the Weierstrass Preparation Theorem to write \( f = u_1 f' \), \( g = u_2 g' \), where \( u_1 \) and \( u_2 \) are units and \( f' \) and \( g' \) are polynomials. Write \( f' = \sum a_i T^i \) and \( g' = \sum b_j T^j \). Let \( v_p \) be the usual \( p \)-adic valuation on \( \mathbb{Z}_p \). Of those terms in \( \sum a_i T^i \) with \( v_p(a_i T^i) = m \), let \( a_k T^k \) be the term so that \( v_p(a_k) \) is minimal. (It is easy to see that there is a unique minimum, because if \( v_p(a_i T^i) = m \), then \( v_p(a_i) = m-i \).) Similarly, let \( b_j T^j \) be the term in the second sum minimizing \( v_p(b_j) \) subject to \( v_p(b_j T^j) = n \).

If we now consider the coefficient \( c_{k+l} \) of \( T^{k+l} \) in the product \( fg = u_1 u_2 f' g' \), we see that \( v_p(c_{k+l}) = v_p(a_k) + v_p(b_l) \). Therefore, \( v_p(c_{k+l} T^{k+l}) = m + n \), and we finally have \( v_p(fg) = m + n \).

This lemma in fact is true in considerably greater generality, but the statement does not seem to appear in the literature in this form.

Because \( \Lambda \) is a unique factorization domain, we can extend \( v \) to \( K \) by defining \( v(f/g) = v(f) - v(g) \), and the valuation still is well defined. Let

\[
R = \{ k \in K : v(k) \geq 0 \}
\]

and

\[
P = \{ k \in K : v(k) > 0 \}.
\]

Notice that \( R \) is a discrete valuation ring and \( P \) is the unique maximal ideal. In fact, \( P \) is principal, and we choose \( p \) as a generator.

**Proposition.** \( R/P \cong \mathbb{F}_p(t) \).

**Proof.** Though this fact appears to be well known to valuation theorists, there is no statement of it in the number-theoretic literature, so we sketch a proof.

Let \( k \in K \) be an element with \( v(k) = 0 \). We can write \( k = \frac{f}{g} u \), where \( u \in \Lambda^\times \) and \( f, g \in \mathbb{Z}_p[T] \). Let \( v(f) = v(g) = n \). Then \( f/p^n \) and \( g/p^n \) are elements of \( \mathbb{Z}_p[\frac{T}{p}] \). The reduction modulo \( P \) now sends \( \frac{T}{p} \) to \( t \), \( u \) to its constant term, and \( \mathbb{Z}_p \) to \( \mathbb{F}_p \).

**Corollary.** \( R \) is neither compact nor locally compact.

**Proof.** Because \( R/P \) is infinite, we can cover \( R \) with an infinite cover of the form \( a + P \) with no finite subcover. Similarly, any neighborhood of \( 0 \) contains \( P^k \) for some \( k \), and \( p^k/p^{k+1} \) is an infinite group.

If we now consider a continuous Galois representation \( \rho \) with image in \( K \), a priori, such a representation must have image in \( R^\times \) because the image must be compact. However, the preceding proposition gives us reason to hope that we can do considerably better.

**Proposition.** Compact subgroups of \( K^\times \) are subgroups of \( \Lambda^\times \).

**Proof.** Let \( G \) be a compact subgroup of \( K^\times \). Let \( a \in G \). The closure of the set \( \{a^n : n \in \mathbb{Z} \} \) must be compact, which means that \( v(a) = 0 \). If we now reduce \( \{a^n\} \) modulo \( P \), we get an image that is a compact subgroup of \( \mathbb{F}_p(t) \). Since \( \mathbb{F}_p(t) \) has the discrete topology, the reduction of \( \{a^n\} \) must be a finite subgroup. Hence, the reduction maps not just to \( \mathbb{F}_p(t) \), but to \( \mathbb{F}_p \). Let \( b = a^{p-1} \), and then \( b \equiv 1 \pmod{P} \).

Because \( P \) is principal, we can write \( b = 1 + pr \), where \( r \in R \). Using the fact that \( v_p((p^m/m)) = k - v_p(m) \) for \( 1 \leq m \leq p^k \), it is simple to show that

\[
\lim_{n \to \infty} b^{p^n} = 1.
\]
Let $c$ be any element of $\mathbb{Z}_p$, and write $c = \lim c_n$, where $c_n \in \mathbb{Z}$. The set $\{b^{c_n}\}$ is contained in $G$, and hence must have a convergent subsequence; however, since $c = \lim c_n$, ($\ast$) means that all subsequences must converge to the same limit, which we might as well denote by $b^c$.

In particular, we may let $c = (1 + p^i)^{-1}$, for any positive integer $i$. The preceding discussion shows that $b^{1/(1+p^i)}$ is an element of $K$ for any positive integer $i$. Write $b = \frac{f}{g}$, where $f$ and $g$ are relatively prime, and then we may conclude that $(1+p^i)|v(g)$ for all positive integers $i$. That, in turn, forces $v(g) = 0$, and then $b$ must be an element of $\Lambda$. Since this argument applies to $f$ as well, $b$ is a unit in $\Lambda$. Because $\Lambda$ is integrally closed in $K$, we see that $a \in \Lambda^\times$ as well.

Notice that a key feature of this argument is that $K$ is not complete, though $\Lambda$ is.

**Corollary.** Suppose that $\rho : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to K^\times$ is a continuous Galois representation. Then the image of $\rho$ is contained in $\Lambda^\times$.

Though the above result is already of considerable interest in Hida theory, representations to $\text{GL}_2(K)$ are of much more interest. It is tempting to

**Conjecture.** Suppose that $\rho : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \text{GL}_n(K)$ is a continuous Galois representation. Then the image of $\rho$ is conjugate to a subgroup of $\text{GL}_n(\Lambda)$.

Unfortunately, the above methods do not suffice to prove this conjecture. A generalization, using a localization argument, proves only that eigenvalues of a matrix in the image of $\rho$ are units in the integral closure of $\Lambda$ in an extension of $K$.

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**REFERENCES**