

CONTINUITY OF RANDOM DERIVATIONS

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ABSTRACT. In this paper we extend the well-known Johnson-Sinclair theorem in a stochastic sense showing that stochastically derivative linear random operators on semisimple Banach algebras are stochastically continuous. Besides, we prove that probably derivative linear random operators on semisimple Banach algebras are probably continuous.

1. INTRODUCTION

A classical topic in the automatic continuity problem of linear operators is the problem of the automatic continuity of derivations. Many authors have been interested in this topic (see, e.g., [2–4, 6–9, 11, 15, 16]) since, twenty-five years ago, Johnson proved (in [10]) that *derivations on complex commutative semisimple Banach algebras are continuous*. For Banach algebras, the Johnson-Sinclair theorem, which establishes that *every derivation on a semisimple Banach algebra (real or complex) is continuous* [11], is perhaps the most outstanding result in this line.

Our goal in this paper is to randomize the Johnson-Sinclair theorem. For that, we say that a linear random operator, D , on a Banach algebra A , is *stochastically derivative* when

$$\mathbb{P}[D(ab) = D(a)b + aD(b)] = 1, \quad \forall a, b \in A,$$

and we show (Theorem 3.3.1) that *stochastically derivative linear random operators, on semisimple Banach algebras, are stochastically continuous*.

Now, following the pattern of [17], it is logical to weaken the condition of being stochastically derivative in the following way.

We say that a linear random operator, D , on a Banach algebra A is *probably derivative* if there exists $0 < \delta < 1$ such that

$$\mathbb{P}[D(ab) = aD(b) + D(a)b] \geq \delta, \quad \forall a, b \in A.$$

In the last section of this paper our purpose is to inquire into the stochastic continuity of such operators.

For a probably derivative linear random operator, D , we define the number

$$\gamma(D) := \inf\{\mathbb{P}[D(ab) = aD(b) + D(a)b] : a, b \in A\}$$

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(which in some sense can be considered as *the probability of D being stochastically derivative*), and, in Corollary 4.3, we show that for a *probably derivative linear random operator*, D , on a semisimple Banach algebra, A , a measurable set, Ω_0 , with $\mathbb{P}[\Omega_0] \geq \gamma(D)$, can be got such that the linear random operator on A , $D_{/\Omega_0}$, defined by $D_{/\Omega_0}(a) = D(a)_{/\Omega_0}$ being stochastically continuous.

2. SOME TOPICS ABOUT THE CONTINUITY OF LINEAR RANDOM OPERATORS

Throughout this paper, we consider a fixed probability space $(\Omega, \Sigma, \mathbb{P})$ and, given a Banach space, Y , we denote by $\mathcal{L}_0(\Omega, Y)$ the linear space of all Y -valued Bochner random variables on $(\Omega, \Sigma, \mathbb{P})$, which, with the almost surely (abbr. a.s.) identification, is an F -space with the convergence in probability topology [13].

A linear random operator from a Banach space X to $\mathcal{L}_0(\Omega, Y)$ is called *linear random operator* from X to Y .

It is said that a linear random operator, T , from X to Y , is *stochastically continuous* when

$$\{x_n\} \rightarrow x, \text{ in } X \Rightarrow \{T(x_n)\} \rightarrow T(x), \text{ in probability.}$$

This property can be characterized as follows:

for every $0 < \epsilon < 1$, $\exists M > 0$ such that $\mathbb{P}[\|T(x)\| \leq M\|x\|] \geq \epsilon$, $\forall x \in X$.

Besides, if we denote the *separating subspace* for T by $\mathcal{S}(T)$, i.e.,

$$\mathcal{S}(T) = \{y \in \mathcal{L}_0(\Omega, Y) : \exists x_n \rightarrow 0 \text{ with } T(x_n) \rightarrow y \text{ in probability}\},$$

we have that, by the closed graph theorem,

$$T \text{ is stochastically continuous} \Leftrightarrow \mathbb{P}[y = 0] = 1, \quad \forall y \in \mathcal{S}(T).$$

For a linear random operator, T , we define the real number

$$\alpha(T) := \inf\{\mathbb{P}[y = 0] : y \in \mathcal{S}(T)\}.$$

Since $\alpha(T)$ is indeed a minimum (see [17]), it is clear that T is stochastically continuous if and only if $\alpha(T) = 1$.

Now, we consider linear random operators which behave as stochastically continuous ones on some measurable subset of Ω . For that, let Ω_0 be a measurable set, with $\mathbb{P}[\Omega_0] > 0$, and let T be a linear random operator from X to Y . Considering Ω_0 as a new probability space (with the inherited structure from Ω), we can define the operator $T_{/\Omega_0} : X \rightarrow \mathcal{L}_0(\Omega, Y)$ by $T_{/\Omega_0}(x) = T(x)_{/\Omega_0}$, and this linear random operator $T_{/\Omega_0}$ is said to be a *conditional operator* of T .

We say that a linear random operator, T , is *probably continuous* when some of its conditional operators are stochastically continuous. Besides, if $T_{/\Omega_0}$ is stochastically continuous then $\alpha(T) \geq \mathbb{P}[\Omega_0]$, trivially. Reciprocally, probably continuous linear random operators are these which have stochastically small separating subspace. To be exact, as we proved in [17, Theorem 7], *if T is probably continuous then, there exists a measurable set, Ω_0 , with $\mathbb{P}[\Omega_0] = \alpha(T)$, such that $T_{/\Omega_0}$ is stochastically continuous. Moreover, the set*

$$\{\mathbb{P}[\Omega_0] : T_{/\Omega_0} \text{ is stochastically continuous}\}$$

has a maximum which coincides with $\alpha(T)$. (In consequence, $\alpha(T)$ can be considered, in this sense, as the probability of T being stochastically continuous.)

We conclude this section observing that by [17, Corollary 8], if T is a non-stochastically continuous linear random operator and δ is in $] \alpha(T), 1[$ then, for every positive constant M , there exists x_M in X , such that

$$\mathbb{P} [\|T(x_M)\| \geq M\|x_M\|] > 1 - \delta,$$

so, the next result follows.

Theorem 2.1. *If T is not stochastically continuous then, for every δ in $]0, 1 - \alpha(T)[$ and every positive random variable Z , we can find an element, $x_{\delta, z}$ in X , which can be chosen with arbitrary small norm, satisfying that*

$$\mathbb{P} [\|T(x_{\delta, z})\| \geq Z] \geq \delta.$$

3. STOCHASTIC CONTINUITY OF STOCHASTICALLY DERIVATIVE LINEAR RANDOM OPERATORS

The following lemma is essential for the conclusion of the basic results of this paper. Given a set, J , we denote by $\mathcal{F}(J)$ the family of all finite subsets of J .

Lemma 3.1. *Let $\{Q_j : j \in J\}$ be a family of closed sets in a Banach space Y . Then, for every y in $\mathcal{L}_0(\Omega, Y)$,*

$$\mathbb{P} \left[y \in \bigcap_{j \in J} Q_j \right] = \inf_{F \in \mathcal{F}(J)} \left\{ \mathbb{P} \left[y \in \bigcap_{j \in F} Q_j \right] \right\}.$$

Proof. Let y be in $\mathcal{L}_0(\Omega, Y)$. By [14, Theorem 1.1], there exists a separable Banach space, L , (depending on y) in which y take its values almost surely, so

$$\mathbb{P} \left[y \in \bigcap_{j \in J} Q_j \right] = \mathbb{P} \left[y \in \left(\bigcap_{j \in J} Q_j \right) \cap L \right].$$

But, by [12, Theorem 13.1.3], there exists a sequence $\{j_n\}_{n \in \mathbb{N}}$, in J , such that

$$L \setminus \bigcap_{j \in J} Q_j = \bigcup_{j \in J} (L \setminus Q_j) = \bigcup_{n \in \mathbb{N}} (L \setminus Q_{j_n}) = L \setminus \left(\bigcap_{n \in \mathbb{N}} Q_{j_n} \right),$$

and, therefore,

$$\begin{aligned} \mathbb{P} \left[y \in \bigcap_{j \in J} Q_j \right] &= \mathbb{P} \left[y \in \left(\bigcap_{j \in J} Q_j \right) \cap L \right] = \mathbb{P} \left[y \in \left(\bigcap_{n \in \mathbb{N}} Q_{j_n} \right) \cap L \right] \\ &= \mathbb{P} \left[y \in \bigcap_{n \in \mathbb{N}} Q_{j_n} \right] = \lim_{n \rightarrow \infty} \mathbb{P} \left[y \in \bigcap_{k=1}^n Q_{j_k} \right]. \end{aligned}$$

Moreover,

$$\mathbb{P} \left[y \in \bigcap_{j \in J} Q_j \right] \leq \mathbb{P} \left[y \in \bigcap_{j \in F} Q_j \right], \quad \text{for all } F \text{ in } \mathcal{F}(J),$$

so the result follows. \square

Throughout this section let D be a stochastically derivative linear random operator on a Banach algebra, A . Moreover, the family of all primitive ideals of A is denoted by $\{P_i\}_{i \in I}$, and we define the sets

$$I_1 := \{i \in I: P_i \text{ has infinite codimension}\}$$

and

$$I_2 := \{i \in I: P_i \text{ has finite codimension}\}.$$

3.1. Working with the primitive ideals of infinite codimension. The following lemma is a refinement of [11, Lemma 2.1].

Lemma 3.1.1. *Let X be an infinite-dimensional Banach space such that there exists a strictly irreducible representation of A on X . Let \mathfrak{D} be the centralizer of A on X . If the vectors $x_0, x_1, \dots \in X$ are linearly independent over \mathfrak{D} then, given y in $X \setminus \{0\}$, there exists a in A satisfying that*

$$ax_0 = 0, \quad ax_1 = y, \quad \text{and} \quad ax_1, ax_2, \dots \text{ are } \mathfrak{D}\text{-independent.}$$

Proof. By the Jacobson density theorem [5, Theorem 24.10], there exists b_1 in A such that

$$b_1x_0 = 0 \quad \text{and} \quad b_1x_1 = y.$$

Since X is infinite-dimensional over \mathfrak{D} , we can find a vector, y_2 , in X , which is not a linear combination (over \mathfrak{D}) of the vectors b_1x_1, b_1x_2 . Again by the Jacobson density theorem, there exists c_2 in A , such that

$$c_2x_1 = c_2x_0 = 0 \quad \text{and} \quad c_2x_2 = y_2,$$

so, multiplying c_2 by a convenient constant, we obtain b_2 , in A , satisfying that

$$\|b_2\| \leq \frac{1}{2}, \quad b_2x_1 = b_2x_0 = 0,$$

and

$$b_2x_2 \text{ is not a linear combination (over } \mathfrak{D}\text{) of } b_1x_1, \text{ and } b_1x_2.$$

An analogous argument shows that, for all $n \geq 2$, we can get an element, b_n , in A , such that

$$\|b_n\| \leq \frac{1}{2^n}, \quad b_nx_{n-1} = \dots = b_nx_0 = 0,$$

and

$$b_nx_n \text{ is not a linear combination (over } \mathfrak{D}\text{) of } \left(\sum_{i=1}^{n-1} b_i\right)x_1, \dots, \left(\sum_{i=1}^{n-1} b_i\right)x_n.$$

Thus, defining $a := \sum_{n=1}^{\infty} b_n$ we have that $ax_0 = 0$ and $ax_1 = y$, obviously. Moreover, if $m < n$,

$$ax_n = b_nx_n + \left(\sum_{i=1}^{n-1} b_i\right)x_n, \quad ax_m = \left(\sum_{i=1}^{n-1} b_i\right)x_m \text{ are } \mathfrak{D}\text{-independents. } \square$$

Let X be a Banach A -module. Given x in X , we denote by D_x the linear random operator, from A to X , defined by $D_x(a) = D(a)x$, for all a in A , (where $(D(a)x)(\omega) = D(a)(\omega)x$, $\forall \omega \in \Omega$).

Lemma 3.1.2. *If X is an infinite dimensional irreducible Banach A -module then, for all x in X , the application $D_x: a \rightarrow D(a)x$ is stochastically continuous.*

Proof. We assume that y in X is such that D_y is not stochastically continuous, and we shall deduce a contradiction from this assumption.

We denote the centralizer of A on X by \mathfrak{D} . Since X is infinite dimensional over \mathfrak{D} , there exists a countable set $\{x_0, x_1, \dots\}$ of \mathfrak{D} -independent vectors in X such that $\|x_n\| = 1$. Applying inductively the last lemma, we obtain a sequence $\{a_n\}_{n \in \mathbb{N}}$ in A satisfying that, for every n in \mathbb{N} ,

$$(1.1) \quad a_n \cdots a_1 x_{n-1} = 0,$$

$$(1.2) \quad a_n \cdots a_1 x_n = y,$$

and

$$\{a_n \cdots a_1 x_j, j \geq n\} \text{ is } \mathfrak{D}\text{-independent.}$$

Now, we can assume that

$$(1.3) \quad \|a_n\| \leq \frac{1}{2^n}, \quad \forall n \in \mathbb{N},$$

changing (1.2) by

$$(1.4) \quad a_n \cdots a_1 x_n = \tau_n y$$

where τ_n is a convenient positive constant.

Since D_y is not stochastically continuous, by Theorem 2.1, there exists $0 < \delta < 1$ such that we can obtain, inductively, a sequence, $\{b_n\}_{n \in \mathbb{N}}$, in A , satisfying that

$$(1.5) \quad \|b_{n+1}\| \leq \|b_n\| \leq 1, \quad \forall n \in \mathbb{N},$$

$$(1.6) \quad \mathbb{P}[\|b_n\| \leq (1 + \|D(a_j \cdots a_1)\|)^{-1}] \geq 1 - \frac{1}{n}, \quad 1 \leq j \leq n,$$

and

$$(1.7) \quad \mathbb{P}\left[\|D(b_n)\tau_n y\| \geq n + \sum_{j=1}^{n-1} \|D(b_j a_j \cdots a_1)\|\right] \geq \delta, \quad \forall n \geq 2.$$

We consider the elements

$$c = \sum_{j=1}^{\infty} b_j a_j \cdots a_1$$

and

$$c_n = b_{n+1} + \sum_{j=n+2}^{\infty} b_j a_j \cdots a_{n+2}$$

(given by series which are absolutely convergent).

Since

$$D(c) = \sum_{j=1}^{n-1} D(b_j a_j \cdots a_1) + D(b_n) a_n \cdots a_1 + b_n D(a_n \cdots a_1) \\ + D(c_n) a_{n+1} \cdots a_1 + c_n D(a_{n+1} \cdots a_1) \quad (\text{a.s.}),$$

we have, by (1.1) and (1.4), that

$$D(c)x_n = \sum_{j=1}^{n-1} D(b_j a_j \cdots a_1)x_n + D(b_n)\tau_n y \\ + b_n D(a_n \cdots a_1)x_n + c_n D(a_{n+1} \cdots a_1)x_n \quad (\text{a.s.}).$$

Therefore,

$$\|D(c)x_n\| \geq \|D(b_n)\tau_n y\| - \sum_{j=1}^{n-1} \|D(b_j a_j \cdots a_1)\| \\ - \|b_n\| \|D(a_n \cdots a_1)\| - \|c_n\| \|D(a_{n+1} \cdots a_1)\| \quad (\text{a.s.}),$$

so that

$$\begin{aligned} \mathbb{P}[\|D(c)\| \geq n-3] &\geq \mathbb{P}[\|D(c)x_n\| \geq n-3] \\ &\geq \mathbb{P} \left[\|D(b_n)\tau_n y\| \geq n + \sum_{j=1}^{n-1} \|D(b_j a_j \cdots a_1)\|, \right. \\ (1.8) \quad &\left. \|b_n\| \|D(a_n \cdots a_1)\| \leq 1, \|c_n\| \|D(a_{n+1} \cdots a_1)\| \leq 2 \right] \\ &\geq \mathbb{P} \left[\|D(b_n)\tau_n y\| \geq n + \sum_{j=1}^{n-1} \|D(b_j a_j \cdots a_1)\| \right] \\ &\quad + \mathbb{P}[\|b_n\| \|D(a_n \cdots a_1)\| \leq 1] + \mathbb{P}[\|c_n\| \|D(a_{n+1} \cdots a_1)\| \leq 2] - 2. \end{aligned}$$

But, we observe that

$$\mathbb{P}[\|b_n\| \|D(a_n \cdots a_1)\| \leq 1] \geq \mathbb{P}[\|b_n\| \leq (1 + \|D(a_n \cdots a_1)\|)^{-1}],$$

so, by (1.6),

$$(1.9) \quad \mathbb{P}[\|b_n\| \|D(a_n \cdots a_1)\| \leq 1] \geq 1 - \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

On the other hand, by (1.3) and (1.5), we have that

$$\|c_n\| \leq 2\|b_{n+1}\|,$$

so

$$\mathbb{P}[\|c_n\| \|D(a_{n+1} \cdots a_1)\| \leq 2] \geq \mathbb{P}[\|b_{n+1}\| \|D(a_{n+1} \cdots a_1)\| \leq 1]$$

and, by (1.9),

$$(1.10) \quad \mathbb{P}[\|D(c_n)\| \|D(a_{n+1} \cdots a_1)\| \leq 2] \leq \left(1 - \frac{1}{n+1}\right) \geq \left(1 - \frac{1}{n}\right).$$

In consequence it follows from (1.8), applying (1.7), (1.9), and (1.10), that

$$\mathbb{P}[\|D(c)\| \geq n-3] \geq \delta + \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right) - 2 = \delta - \frac{2}{n}.$$

This contradiction proves the result. \square

Lemma 3.1.3. *For all \mathbf{b} in $\mathcal{S}(D)$ the following is satisfied:*

$$\mathbb{P} \left[\mathbf{b} \in \bigcap_{i \in I_1} P_i \right] = 1.$$

Proof. Let P be an infinite-codimensional primitive ideal of A , and let X be the left A -module corresponding to an irreducible representation of A whose kernel is P (see [5, Proposition 24.12]). Since X is an infinite-dimensional irreducible Banach A -module, the previous lemma shows that the operator $D_x: a \rightarrow D(a)x$ is stochastically continuous, for all x in X . Thus, if \mathbf{b} is in $\mathcal{S}(D)$, it is satisfied that

$$\mathbb{P}[\mathbf{b}x = 0] = 1, \quad \forall x \in X$$

(because $\mathbf{b}x \in \mathcal{S}(D_x)$), so denoting

$$\ker x := \{a \in A: ax = 0\},$$

we have a family, $\{\ker x: x \in X\}$, of closed subsets of A , such that

$$\mathbb{P} \left[\mathbf{b} \in \bigcap_{x \in F} \ker x \right] = 1,$$

for every finite subset, F , of X . Therefore, Lemma 3.1 proves that

$$\mathbb{P} \left[\mathbf{b} \in \bigcap_{x \in X} \ker x \right] = 1,$$

that is,

$$\mathbb{P}[\mathbf{b} \in P] = 1.$$

Now, applying Lemma 3.1 to the family $\{P_i: i \in I_1\}$, the result follows. \square

3.2. Working with the primitive ideals of finite codimension.

Lemma 3.2.1. *Let $\{Q_n\}_{n \in \mathbb{N}}$ be a sequence of ideals of A such that:*

- (i) Q_n has finite codimension, $\forall n \in \mathbb{N}$,
- (ii) A/Q_n has a unit element, $\forall n \in \mathbb{N}$,
- (iii) $Q_n + Q_m = A$, for all $n \neq m$.

Then, there exists a sequence, $\{a_n\}_{n \in \mathbb{N}}$, in A satisfying that, for every k in \mathbb{N} :

- (i) $a_k + Q_n = 0$, $n = 1, \dots, k-1$,
- (ii) $a_k + Q_n$ is invertible, $\forall n \geq k$,
- (iii) $\|a_k\| \leq 1$.

Proof. In a first step, we show that if n is in \mathbb{N} , then $Q_m + \bigcap_{j=1}^n Q_j = A$, $\forall m > n$.

In effect, let $n := \max \{k \in \{1, \dots, m-1\}: Q_m + \bigcap_{j=1}^k Q_j = A\}$. If $n < m-1$, then, for every a in A , we have that there exists a_1 in Q_m and b_1 in $\bigcap_{j=1}^n Q_j$ such that $a = a_1 + b_1$. Let u be a modular unit of Q_m (which exists because A/Q_m has a unit element). Since $A = Q_m + Q_{n+1}$, we can find a_2 in Q_m and b_2 in Q_{n+1} such that $u = a_2 + b_2$, so that $au = (a_1(a_2 + b_2) + b_1 a_2) + b_1 b_2$ lies in $Q_m + \bigcap_{j=1}^{n+1} Q_j$. Therefore, $a = (a - au) + au$ belongs to $Q_m + \bigcap_{j=1}^{n+1} Q_j$, which contradicts the definition of n .

Now, proceeding as in the proof of [18, Lemma 2.2] from the first step, it follows that *the homomorphism* $a \rightarrow (a + Q_1, \dots, a + Q_n)$, *from* A *to* $\bigoplus_{j=1}^n A/Q_j$ *is onto, for all* n *in* \mathbb{N} . In consequence, the same argument used in [18, Lemma 2.6] proves the result. \square

Lemma 3.2.2. *Let* \mathbf{b} *be in* $\mathcal{S}(D)$. *Then, for every* $0 < \epsilon < 1$, *there exists a finite subset,* F_ϵ , *of* I_2 *such that*

$$\mathbb{P} \left[\mathbf{b} \in \bigcap_{i \in I_2 \setminus F_\epsilon} P_i \right] \geq \epsilon.$$

Proof. We assume to the contrary that there exists $0 < \epsilon_0 < 1$ such that

$$\mathbb{P} \left[\mathbf{b} \in \bigcap_{i \in I_2 \setminus F} P_i \right] < \epsilon_0, \quad \text{for all finite subsets } F \text{ of } I_2.$$

Since $\mathbb{P} \left[\mathbf{b} \in \bigcap_{i \in I_2} P_i \right] < \epsilon_0$, Lemma 3.1 shows that there exists a finite subset, F_1 , of I_2 such that

$$\mathbb{P} \left[\mathbf{b} \in \bigcap_{i \in I_2} P_i \right] \leq \mathbb{P} \left[\mathbf{b} \in \bigcap_{i \in F_1} P_i \right] < \epsilon_0.$$

Thus, we obtain inductively, applying Lemma 3.1, a sequence of finite subsets of I_2 , denoted by $\{F_n\}_{n \in \mathbb{N}}$, satisfying that,

$$\mathbb{P} \left[\mathbf{b} \in \bigcap_{I_2 \setminus (F_1 \cup \dots \cup F_{n-1})} P_i \right] \leq \mathbb{P} \left[\mathbf{b} \in \bigcap_{i \in F_n} P_i \right] < \epsilon_0$$

and

$$F_n \cap (F_1 \cup \dots \cup F_{n-1}) = \emptyset, \quad \forall n > 1.$$

Now we define

$$Q_n = \bigcap_{i \in F_n} P_i, \quad \forall n \in \mathbb{N}.$$

Since A/Q_n is isomorphic to $\bigoplus_{i \in F_n} (A/P_i)$ (see [11, Lemma 3.1]), the codimension of Q_n is finite. On the other hand A/P is a primitive Banach algebra, for every primitive ideal, P [5, Proposition 26.9], so it is semisimple [5, Proposition 24.14]. If, in addition, the codimension of P is finite, then A/P has a unit element [1, Theorem 3.1]. Thus, we also deduce that A/Q_n has a unit element.

Since $(P_n + P_m)/P_n$ and $(P_n + P_m)/P_m$ are two-sided ideals of the simple algebras A/P_n and A/P_m , respectively, we have that $P_n + P_m = A$, $\forall n \neq m$. Therefore, Lemma 2.1 of [18] proves that if m is not in F_n then $P_k + Q_m = A$, $\forall k \in F_n$, and, applying the same lemma again, we deduce that

$$Q_n + Q_m = A, \quad \forall n \neq m.$$

We just show that Lemma 3.2.1 can be applied, so there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$, in A satisfying that

$$(2.1) \quad \begin{aligned} a_k + Q_n &= 0, & \forall n = 1, \dots, k-1, \\ a_k + Q_n &\text{ is invertible, } & \forall n \geq k, \end{aligned}$$

and

$$(2.2) \quad \|a_k\| \leq 1.$$

Now, let δ be in $]0, 1 - \epsilon_0[$. Since, for every natural number, n , the linear random operator from A to A/Q_n , given by $a \rightarrow D(a) + Q_n$, and denoted by $\Pi_{Q_n} \circ D$, satisfies that

$$\alpha(\Pi_{Q_n} \circ D) = \inf\{\mathbb{P}[y = 0] : y \in \mathcal{S}(\Pi_{Q_n} \circ D)\} \leq \mathbb{P}[\mathbf{b} \in Q_n] < \epsilon_0,$$

we can get, by Theorem 2.1, inductively, a sequence $\{b_n\}_{n \in \mathbb{N}}$, in A such that

$$(2.3) \quad \begin{aligned} & \|b_n\| \leq \frac{1}{2^n}, \quad \forall n \in \mathbb{N}, \\ & \mathbb{P} \left[\|b_n\| \leq \left(1 + \sum_{j=1}^n \|D(a_j)\| \right)^{-1} \right] \geq 1 - \frac{1}{n}, \quad \forall n \in \mathbb{N}, \end{aligned}$$

and

$$(2.4) \quad \begin{aligned} & \mathbb{P} \left[\|D(b_n) + Q_n\| \geq \prod_{j=1}^n \|(a_j + Q_n)^{-1}\| \left(n + \sum_{j=1}^{n-1} \|D(b_j a_j \cdots a_1)\| \right) \right] \\ & > \delta, \quad \forall n \geq 2. \end{aligned}$$

We define

$$c := \sum_{j=1}^{\infty} b_j a_j \cdots a_1.$$

Then, by (2.1),

$$\begin{aligned} D(c) + Q_n &= \sum_{j=1}^{n-1} (D(b_j a_j \cdots a_1) + Q_n) + (D(b_n) + Q_n)(a_n + Q_n) \cdots (a_1 + Q_n) \\ &+ (b_n + Q_n)(D(a_n) + Q_n) \cdots (a_1 + Q_n) \\ &+ \cdots + (b_n + Q_n)(a_n + Q_n) \cdots (D(a_1) + Q_n) \\ &+ (b_{n+1} + Q_n)(D(a_{n+1}) + Q_n)(a_n + Q_n) \cdots (a_1 + Q_n) \quad (\text{a.s.}), \end{aligned}$$

and, since

$$\|D(b_n) + Q_n\| \leq \|(D(b_n) + Q_n)(a_n + Q_n) \cdots (a_1 + Q_n)\| \prod_{j=1}^n \|(a_j + Q_n)^{-1}\|,$$

we have that, applying (2.2),

$$\begin{aligned} \|D(c)\| &\geq \|D(c) + Q_n\| \\ &\geq \|D(b_n) + Q_n\| \prod_{j=1}^n \|(a_j + Q_n)^{-1}\|^{-1} - \sum_{j=1}^{n-1} \|D(b_j a_j \cdots a_1)\| \\ &- \|b_n\| \sum_{j=1}^n \|D(a_j)\| - \|b_{n+1}\| \|D(a_{n+1})\| \quad (\text{a.s.}), \end{aligned}$$

and thus

$$\begin{aligned}
 & \mathbb{P}[\|D(c)\| \geq n-2] \\
 & \geq \mathbb{P}[\|D(c) + Q_n\| \geq n-2] \\
 & \geq \mathbb{P}\left[\|D(b_n) + Q_n\| \prod_{j=1}^n \|(a_j + Q_n)^{-1}\|^{-1} - \sum_{j=1}^{n-1} \|D(b_j a_j \cdots a_1)\| \geq n, \right. \\
 (2.5) \quad & \left. \|b_n\| \sum_{j=1}^n \|D(a_j)\| \leq 1, \|b_{n+1}\| \|D(a_{n+1})\| \leq 1\right] \\
 & \geq \mathbb{P}\left[\|D(b_n) + Q_n\| \geq \prod_{j=1}^n \|(a_j + Q_n)^{-1}\| \left(n + \sum_{j=1}^{n-1} \|D(b_j a_j \cdots a_1)\|\right)\right] \\
 & + \mathbb{P}\left[\|b_n\| \sum_{j=1}^n \|D(a_j)\| \leq 1\right] + \mathbb{P}[\|b_{n+1}\| \|D(a_{n+1})\| \leq 1] - 2.
 \end{aligned}$$

Since

$$\mathbb{P}\left[\|b_n\| \sum_{j=1}^n \|D(a_j)\| \leq 1\right] \geq \mathbb{P}\left[\|b_n\| \leq \left(1 + \sum_{j=1}^n \|D(a_j)\|\right)^{-1}\right],$$

we have, by (2.3), that

$$(2.6) \quad \mathbb{P}\left[\|b_n\| \sum_{j=1}^n \|D(a_j)\| \leq 1\right] \geq 1 - \frac{1}{n}.$$

We also observe that

$$\mathbb{P}[\|b_{n+1}\| \|D(a_{n+1})\| \leq 1] \geq \mathbb{P}\left[\|b_{n+1}\| \leq \left(1 + \sum_{j=1}^{n+1} \|D(a_j)\|\right)^{-1}\right],$$

so by (2.3),

$$(2.7) \quad \mathbb{P}[\|b_{n+1}\| \|D(a_{n+1})\| \leq 1] \geq 1 - \frac{1}{n+1} \geq 1 - \frac{1}{n}.$$

Finally, applying (2.6) and (2.7), it follows from (2.5) that

$$\mathbb{P}[\|D(c)\| \geq n-2] \geq \delta + 2\left(1 - \frac{1}{n}\right) - 2 = \delta - \frac{2}{n}, \quad \forall n \in \mathbb{N},$$

which is impossible. \square

3.3. The main result.

Theorem 3.3.1. *Stochastically derivative linear random operators, on semisimple Banach algebras, are stochastically continuous.*

Proof. Let D be such an operator, and let \mathbf{b} be in $\mathcal{S}(D)$. Given $0 < \epsilon < 1$, by Lemma 3.2.2, there exists a finite set, F_ϵ , of I_2 , such that

$$\mathbb{P}\left[\mathbf{b} \in \bigcap_{i \in I_2 \setminus F_\epsilon} P_i\right] \geq \epsilon,$$

and, by Lemma 3.1.3,

$$\mathbb{P} \left[\mathbf{b} \in \bigcap_{i \in I_1} P_i \right] = 1,$$

so, if $(I_1 \cup I_2) \setminus F_\epsilon$ is denoted by I_0 ,

$$(3.1) \quad \mathbb{P} \left[\mathbf{b} \in \bigcap_{i \in I_0} P_i \right] \geq \epsilon.$$

Since A is semisimple, the mapping $a \rightarrow \bigoplus_{i \in F_\epsilon} (a + P_i)$, from $\bigcap_{i \in I_0} P_i$ to $\bigoplus_{i \in F_\epsilon} (A/P_i)$ is injective [5, Proposition 24.14]. Therefore, $\bigcap_{i \in I_0} P_i$ is a (semisimple) finite-dimensional subalgebra of A , so by [1, Theorem 3.1.1], it has a unit element, which is denoted by e . Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence in A converging to zero and such that $\{D(a_n)\}_{n \in \mathbb{N}}$ converges to \mathbf{b} , in probability. Then $\{D(ea_n)\}_{n \in \mathbb{N}}$ converges to $e\mathbf{b}$, in probability, so $e\mathbf{b}$ lies in the separating subspace for the restriction of D to $\bigcap_{i \in I_0} P_i$, that is stochastically continuous because $\bigcap_{i \in I_0} P_i$ is finite dimensional. Therefore, $e\mathbf{b} = 0$ (a.s.), so that,

$$\mathbb{P}[\mathbf{b} = 0] = \mathbb{P} \left[e\mathbf{b} = 0, \mathbf{b} \in \bigcap_{i \in I_0} P_i \right] = \mathbb{P} \left[\mathbf{b} \in \bigcap_{i \in I_0} P_i \right].$$

Thus, by (3.1), $\mathbb{P}[\mathbf{b} = 0] \geq \epsilon$. Since ϵ is arbitrary, this proves that $\mathbb{P}[\mathbf{b} = 0] = 1$ (i.e., D is stochastically continuous). \square

4. PROBABLY CONTINUITY OF PROBABLY DERIVATIVE LINEAR RANDOM OPERATORS

Given a probably derivative linear random operator, D , with $\gamma(D) = \delta$, we know that, for every a, b in A , there exists a measurable set, $\Omega_{a,b}$, with $\mathbb{P}[\Omega_{a,b}] \geq \delta$, such that

$$D(ab) = D(a)b + aD(b) \quad (\text{a.s.}) \text{ on } \Omega_{a,b}.$$

The difficulty of working with probably derivative linear random operators arises because we do not know the relationship between $\Omega_{a,b}$ and $\Omega_{c,d}$, for a, b, c, d in A . The main result of this section shows that a measurable set Ω_0 , with $\mathbb{P}[\Omega_0] = \gamma(D)$, can be got, such that

$$D(ab) = D(a)b + aD(b) \quad (\text{a.s.}) \text{ on } \Omega_0, \quad \forall a, b \in A.$$

This result is a straightforward consequence of the following theorem.

Multilinear uniform randomization principle 4.1. *Let X_1, \dots, X_n, Y be Banach spaces and T a multilinear random operator, from $X_1 \times \dots \times X_n$ to Y .*

(i) *If there exists $0 < \delta < 1$ such that*

$$\mathbb{P}[T(x_1, \dots, x_n) = 0] \geq \delta, \quad \forall (x_1, \dots, x_n) \in X_1 \times \dots \times X_n,$$

then there exists a measurable set Ω_0 , with $\mathbb{P}[\Omega_0] \geq \delta$, such that $T|_{\Omega_0} = 0$ (a.s.).

(ii) *The set*

$$\{\mathbb{P}[\Omega'] : \Omega' \text{ is measurable and } T|_{\Omega'} = 0 \text{ (a.s.)}\}$$

has a maximum, the set

$$\mathbb{P}[T(x_1, \dots, x_n) = 0]: (x_1, \dots, x_n) \in X_1 \times \dots \times X_n$$

has a minimum, and these values coincide.

Proof. For simplicity, we assume that T is bilinear. Now, we consider the number

$$\gamma := \inf\{\mathbb{P}[T(x, y) = 0]: x \in X_1, y \in X_2\},$$

and we prove, by induction, that

$$(4.1) \quad \mathbb{P}[T(x_1, y_1) = 0, \dots, T(x_n, y_n) = 0] \geq \gamma, \\ \forall x_1, \dots, x_n \in X_1, \forall y_1, \dots, y_n \in X_2.$$

For $n = 1$ the result is true. We assume that the previous equality is satisfied for a natural number, n . Then, if x_1, \dots, x_{n+1} are in X_1 and y_1, \dots, y_{n+1} are in X_2 , we have that for λ, β in \mathbb{R}^+ ,

$$\begin{aligned} \gamma \leq & \mathbb{P}[T(x_1, y_1) = 0, \dots, T(x_{n-1}, y_{n-1}) = 0, T(x_n - \beta x_{n+1}, y_n - \lambda y_{n+1}) = 0] \\ & = \mathbb{P}[T(x_1, y_1) = 0, \dots, T(x_n - \beta x_{n+1}, y_n) = T(x_n - \beta x_{n+1}, y_{n+1}) = 0] \\ & \quad + \mathbb{P}[T(x_1, y_1) = 0, \dots, T(x_n - \beta x_{n+1}, y_n) = \lambda T(x_n - \beta x_{n+1}, y_{n+1}), \\ & \quad \quad \quad T(x_n - \beta x_{n+1}, y_{n+1}) \neq 0], \end{aligned}$$

so, if $\lambda \rightarrow \infty$, we obtain that

$$\gamma \leq \mathbb{P}[T(x_1, y_1) = 0, \dots, T(x_n - \beta x_{n+1}, y_n) = 0, T(x_n - \beta x_{n+1}, y_{n+1}) = 0].$$

But

$$\begin{aligned} & \mathbb{P}[T(x_1, y_1) = 0, \dots, T(x_n - \beta x_{n+1}, y_n) = 0, T(x_n - \beta x_{n+1}, y_{n+1}) = 0] \\ & \leq \mathbb{P}[T(x_1, y_1) = 0, \dots, T(x_n, y_n) = T(x_{n+1}, y_n) = 0, \\ & \quad \quad \quad T(x_n, y_{n+1}) = T(x_{n+1}, y_{n+1}) = 0] \\ & \quad + \mathbb{P}[T(x_1, y_1) = 0, \dots, T(x_n, y_n) = \beta T(x_{n+1}, y_n), T(x_{n+1}, y_n) \neq 0] \\ & \quad + \mathbb{P}[T(x_1, y_1) = 0, \dots, T(x_n, y_{n+1}) = \beta T(x_{n+1}, y_{n+1}), T(x_{n+1}, y_{n+1}) = 0], \end{aligned}$$

so, if $\beta \rightarrow \infty$, we have that

$$\begin{aligned} \gamma \leq & \mathbb{P}[T(x_1, y_1) = 0, \dots, T(x_n, y_n) = 0, T(x_{n+1}, y_n) = 0, \\ & \quad \quad \quad T(x_n, y_{n+1}) = 0, T(x_{n+1}, y_{n+1}) = 0], \end{aligned}$$

and, particularly,

$$\gamma \leq \mathbb{P}[T(x_1, y_1) = 0, \dots, T(x_n, y_n) = 0, T(x_{n+1}, y_{n+1}) = 0].$$

Now, let $\{(u_n, v_n)\}$ be a sequence in $X_1 \times X_2$ such that

$$(4.2) \quad \gamma \leq \mathbb{P}[T(u_n, v_n) = 0] < \gamma + \frac{1}{n}.$$

We define the measurable set

$$\Omega_0 = \bigcap_{n=1}^{\infty} [T(u_1, v_1) = 0, \dots, T(u_n, v_n) = 0],$$

and we observe that, by (4.1),

$$\gamma \leq \mathbb{P}[T(u_1, v_1) = 0, \dots, T(u_n, v_n) = 0]$$

and, by (4.2),

$$\mathbb{P}[T(u_1, v_1) = 0, \dots, T(u_n, v_n) = 0] < \gamma + \frac{1}{n},$$

so

$$\mathbb{P}[\Omega_0] = \lim_{n \rightarrow \infty} \mathbb{P}[T(u_1, v_1) = 0, \dots, T(u_n, v_n) = 0] = \gamma \geq \delta.$$

Finally, if the equality $T_{/\Omega_0}(x, y) = 0$ (a.s.) is not held, for some (x, y) in $X_1 \times X_2$, then

$$\mathbb{P}[T(x, y) = 0, \Omega_0] < \mathbb{P}[\Omega_0] = \gamma.$$

But

$$\begin{aligned} & \mathbb{P}[T(x, y) = 0, \Omega_0] \\ &= \mathbb{P}\left[\bigcap_{n=1}^{\infty} [T(x, y) = 0, T(u_1, v_1) = 0, \dots, T(u_n, v_n) = 0]\right] \\ &= \lim_{n \rightarrow \infty} \mathbb{P}[T(x, y) = 0, T(u_1, v_1) = 0, \dots, T(u_n, v_n) = 0], \end{aligned}$$

so that, for some natural n ,

$$\mathbb{P}[T(x, y) = 0, T(u_1, v_1) = 0, \dots, T(u_n, v_n) = 0] < \gamma,$$

which contradicts (4.1). Therefore,

$$\inf\{\mathbb{P}[T(x, y) = 0]: x \in X_1, y \in X_2\} = \sup\{\mathbb{P}[\Omega']: T_{/\Omega'} = 0 \text{ (a.s.)}\} = \mathbb{P}[\Omega_0],$$

and the result is proved. \square

The multilinear uniform randomization principle, applied to the bilinear random operator $T(a, b) := D(ab) - D(a)b - aD(b)$, for a probably derivative linear random operator, D , proves that we can obtain a measurable set, Ω_0 , with $\mathbb{P}[\Omega_0] = \gamma(D)$, such that the conditional operator $D_{/\Omega_0}$ is stochastically derivative. Therefore, by Theorem 3.3.1, $D_{/\Omega_0}$ is stochastically continuous, and thus “the probability of D being stochastically continuous” ($\alpha(D)$) is at least “the probability of D being stochastically derivative” ($\gamma(D)$).

Corollary 4.2. *Every probably derivative linear random operator, D , on a semi-simple Banach algebra, is probably continuous. Moreover,*

$$\alpha(D) \geq \gamma(D).$$

We observe that, if in the above corollary $\gamma(D) = 1$ (i.e., D is stochastically derivative), then Theorem 3.3.1 is obtained.

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