ON CROSSED PRODUCTS OF HOPF ALGEBRAS

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Abstract. Let $B = A \#_a H$ denote a crossed product of the associative algebra $A$ with the Hopf algebra $H$. We investigate the weak dimension and the global dimension of $B$ and show that $\text{wdim } B \leq \text{wdim } H + \text{wdim } A$ and $\text{l.gldim } B \leq \text{r.gldim } H + \text{l.gldim } A$.

1. Introduction

Let $B = A \#_a H$ denote a crossed product of the associative algebra $A$ with the Hopf algebra $H$. We establish the following estimates for the weak dimension and the global dimension of $B$ in terms of the corresponding data for $H$ and $A$:

$$\text{wdim } B \leq \text{wdim } H + \text{wdim } A \quad \text{and} \quad \text{l.gldim } B \leq \text{r.gldim } H + \text{l.gldim } A.$$ 

The first of these estimates is a consequence of a suitable spectral sequence

$$E^2_{p,q} = \text{Tor}^H_p(k, \text{Tor}^A_q(V, W)) \Rightarrow \text{Tor}^B_p(V, W),$$

where $k$ is the trivial $H$-module (i.e., $H$ acts via the counit) and $V_B$ and $W_B$ are arbitrary $B$-modules. This spectral sequence will be constructed in Section 2.3 along with an analogous spectral sequence for Ext which yields the estimate for global dimension. Since a ring is von Neumann regular precisely if its weak dimension is 0, we conclude in particular that if $H$ and $A$ are both von Neumann regular, then $B$ is likewise. Specializing to the case of global dimension 0, we also deduce the known fact that if $H$ and $A$ are both semisimple, then so is $B$ (cf. [Mont], Theorem 7.4.2). Finally, we briefly discuss relative projectivity of $B$ with respect to $A$.

Notation and basic facts. Our reference for general material about Hopf algebras are the standard texts [Abe] and [Sw]. For crossed products in particular we follow the notes [Mont]. Throughout this article, we will keep the following notation:

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k denotes a commutative field;

H will be a Hopf algebra over k, with counit \( \epsilon \); the H-module k will always be the trivial H-module;

A denotes an associative k-algebra with identity 1 so that there is a weak H-action on A, denoted \((h, a) \mapsto h \cdot a\) \((h \in H, a \in A)\);

\(B = A \#_\sigma H\) will denote a crossed product, with cocycle \(\sigma : H \otimes_k H \to A\).

Thus \(B\) is an associative algebra such that there is an isomorphism of left A-modules

\[ A \otimes_k H \cong B, \quad a \otimes h \mapsto a#h. \]

The map \(a \mapsto a#1\) identifies \(A\) with a subalgebra of \(B\). Defining a k-linear map \(\gamma : H \to B\) by

\[ \gamma(h) = 1#h \quad (h \in H), \]

we have \(a\gamma(h) = a#h\) for \(a \in A, h \in H\). It is known (cf. [Mont], Chapter 7) that \(\gamma\) is convolution invertible and satisfies the following identities, for \(h, k \in H\) and \(a \in A\),

\[ \sigma(h,k) = \sum \gamma(h_1)\gamma(k_1)\gamma^{-1}(h_2k_2), \]

\[ \gamma(h)\gamma(k) = \sum \sigma(h_1,k_1)\gamma(h_2k_2), \]

\[ \gamma^{-1}(hk) = \sum \gamma^{-1}(k_1)\gamma^{-1}(h_1)\sigma(h_2,k_2), \]

\[ \gamma(h)a = \sum (h_1 \cdot a)\gamma(h_2). \]

2. Proofs

2.1. Action of \(H\) on homomorphisms. Let \(B V\) and \(B W\) be left \(B\)-modules. For each \(\phi \in \text{Hom}_A(V, W)\) and \(h \in H\) define \(\phi h : V \to W\) by

\[ (\phi h)(v) = \sum \gamma^{-1}(h_1)\phi(\gamma(h_2)v) \quad (v \in V). \]

Then we have the following

**Lemma.** The above definition makes \(\text{Hom}_A(V, W)\) a right \(H\)-module. There is a canonical k-linear isomorphism

\[ \text{Hom}_H(k, \text{Hom}_A(V, W)) \cong \text{Hom}_B(V, W). \]

Furthermore,

\[ \text{Hom}_A(B, W) \cong \text{Hom}_k(H, W) \]

as right \(H\)-modules (where \(H\) acts on the right-hand side by \((\psi h)(k) = \psi(hk)\) for \(\psi \in \text{Hom}_k(H, W)\) and \(h, k \in H\)). Finally, if \(f : V \to V'\) and \(g : W \to W'\) are \(B\)-module maps, then \(g \circ f^* : \text{Hom}_A(V', W) \to \text{Hom}_A(V, W')\) is an \(H\)-module map.

**Proof.** The fact that \(\phi h : V \to W\) is \(A\)-linear is proved exactly as in [Mont], proof of Theorem 7.4.2. Furthermore, the map \(\text{Hom}_A(V, W) \times H \to \text{Hom}_A(V, W), (\phi, h) \mapsto \phi h\) is clearly k-bilinear. Using the identities (1a)
and (1b) we compute, for \( h, k \in H \) and \( v \in V \),

\[
[\phi(hk)](v) = \sum \gamma^{-1}(h_1k_1)\phi(\gamma(h_2k_2)v) \\
= \sum \gamma^{-1}(k_1)\gamma^{-1}(h_1)\sigma(h_2, k_2)\phi(\gamma(h_3k_3)v) \\
= \sum \gamma^{-1}(k_1)\gamma^{-1}(h_1)\phi[\gamma(h_2)\gamma(k_2)v] \\
= [(\phi k)v] \ .
\]

Thus \( \text{Hom}_H(V, W) \) is a right \( H \)-module.

In order to establish the first isomorphism, we first note that there is a canonical isomorphism of \( \text{Hom}_H(k, \text{Hom}_H(V, W)) \) with the \( k \)-space of \( H \)-invariants in \( \text{Hom}_H(V, W) \), that is, with

\[ \text{Hom}_H(V, W)^H = \{ \phi \in \text{Hom}_H(V, W) \mid \phi h = \varepsilon(h)\phi \text{ for all } h \in H \} \ . \]

Thus it suffices to show that \( \text{Hom}_H(V, W)^H = \text{Hom}_B(V, W) \). Let \( \phi \in \text{Hom}_H(V, W) \), \( h \in H \) and \( v \in V \). Then

\[ (\phi h)(v) = \varepsilon(h)\phi(v) \Leftrightarrow \sum \gamma^{-1}(h_1)\phi(\gamma(h_2)v) = \sum \gamma^{-1}(h_1)\gamma(h_2)\phi(v) \]

\[ \Leftrightarrow \phi(\gamma(h)v) = \gamma(h)\phi(v) \ . \]

Since \( B = A\gamma(H) \), the last condition is equivalent with \( \phi \in \text{Hom}_B(V, W) \). This proves the first isomorphism.

Now consider the map \( f : \text{Hom}_H(B, W) \rightarrow \text{Hom}_k(H, W) \) that is defined by

\[ f(\phi)(h) = (\phi h)(1) = \sum \gamma^{-1}(h_1)\phi(\gamma(h_2)) \]

for \( \phi \in \text{Hom}_H(B, W) \) and \( h \in H \). Then \( f \) is right \( H \)-linear. Define a map \( g : \text{Hom}_k(H, W) \rightarrow \text{Hom}_H(B, W) \) by

\[ g(\psi)(\gamma(h)) = \sum \gamma(h_1)\psi(h_2) \]

for \( \psi \in \text{Hom}_k(H, W) \) and \( h \in H \). Note that \( g(\psi) \) is well defined because \( B \cong A \otimes_k \gamma(H) \) as left \( A \)-modules. One readily checks that \( f \) and \( g \) are inverse to each other, whence the second isomorphism follows.

Finally, the last assertion is trivial and so the lemma is proved. \( \square \)

2.2. Action of \( H \) on tensors. Let \( V_B \) and \( B^W \) be \( B \)-modules. For \( v \otimes w \in V \otimes_A W \) and \( h \in H \) define \( h(v \otimes w) \in V \otimes_A W \) by

\[ h(v \otimes w) = \sum v_1 \gamma^{-1}(h_1) \otimes \gamma(h_2)w \ . \]

Using identity (2), one easily checks that this is well defined, i.e., that \( h(va \otimes w) = h(v \otimes aw) \) holds for all \( v \in V \), \( w \in W \), \( h \in H \), and \( a \in A \).

**Lemma.** The above definition makes \( V \otimes_A W \) a left \( H \)-module. There is a canonical \( k \)-linear isomorphism

\[ k \otimes_H (V \otimes_A W) \cong V \otimes_B W \ . \]

Furthermore,

\[ V \otimes_A B \cong H \otimes_k V \]
as left $H$-modules (where the $H$-action on the right-hand side is via the action on the factor $H$). So $H \otimes_k V \cong H^{\dim_k V}$). Finally, if $f : V \to V'$ and $g : W \to W'$ are $B$-module maps, then $g \otimes f : V \otimes_A W \to V' \otimes_A W'$ is an $H$-module map.

**Proof.** The module properties again follow readily from the identities (1a) and (1b). For the first isomorphism, note that

$$k \otimes_H (V \otimes_A W) \cong V \otimes_A W/(\text{Ker } \epsilon)(V \otimes_A W).$$

Now $(\text{Ker } \epsilon)(V \otimes_A W)$ is the $k$-subspace of $V \otimes_A W$ that is generated by the elements of the form $h(v \otimes w) - \epsilon(h)v \otimes w$ for $h \in H$, $v \in V$, $w \in W$. But

$$h(v \otimes w) - \epsilon(h)v \otimes w = \sum [v \gamma^{-1}(h_1) \otimes \gamma(h_2)w - v \gamma^{-1}(h_1)\gamma(h_2) \otimes w],$$

and hence $(\text{Ker } \epsilon)(V \otimes_A W)$ equals the $k$-subspace of $V \otimes_A W$ that is generated by the elements of the form $v \gamma(h) \otimes w - v \otimes \gamma(h)w$. Since $B = A \otimes_k \gamma(H)$, this proves the first isomorphism.

For the second isomorphism, define $f : H \otimes_k V \to V \otimes_A B$ by

$$f(h \otimes v) = h(v \otimes 1) = \sum v \gamma^{-1}(h_1) \otimes \gamma(h_2).$$

Then $f$ is clearly $H$-linear. Furthermore, since $B \cong A \otimes_k \gamma(H)$ as left $A$-modules, we can define $g : V \otimes_A B \to H \otimes_k V$ by

$$g(v \otimes \gamma(h)) = \sum h_2 \otimes v \gamma(h_1).$$

One easily checks that $f$ and $g$ are inverse to each other, and hence $f$ is an isomorphism.

The last assertion is again clear and so the lemma is proved. \qed

2.3. \textbf{Ext and Tor.} The $H$-actions in Sections 2.1 and 2.2 extend to $H$-actions on Ext and Tor. We explain this for Ext, the case of Tor being entirely analogous. So let $B V$ and $B W$ be left $B$-modules and let

$$P : \ldots \to P_{n+1} \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} 0$$

be a projective resolution of $V$, so $H_n(P) = 0$ for $n \neq 0$ and $H_0(P) \cong V$. Since $B$ is projective (in fact, free) as a left $A$-module, the restriction of $P$ to $A$ is a projective resolution of $A V$ and so we have $\text{Ext}_A(V, W) \cong H^*(\text{Hom}_A(P, W))$. By Section 2.1, the components of the complex $\text{Hom}_A(P, W)$ are right $H$-modules and the differential $(f_n^*)$ is $H$-linear. Thus the cohomology $H^*(\text{Hom}_A(P, W))$ is a right $H$-module and hence so is $\text{Ext}_A^*(V, W)$.

**Proposition.** (a) Let $B V$ and $B W$ be left $B$-modules. Then there is a third quadrant spectral sequence

$$E_2^{p,q} = \text{Ext}_{B}^p(k, \text{Ext}_{A}^q(V, W)) \Rightarrow \text{Ext}_{B}^{p+q}(V, W).$$

(b) Let $V_B$ and $B W$ be $B$-modules. Then there is a first quadrant spectral sequence

$$E_2^{p,q} = \text{Tor}^B_{p}(k, \text{Tor}_{q}^{A}(V, W)) \Rightarrow \text{Tor}_{B}^{p+q}(V, W).$$

**Proof.** Both spectral sequences can be obtained as applications of the Grothendieck spectral sequence (cf. [Rot], Chapter 11). We let $\mathcal{B}$ denote the category
of left $B$-modules and similarly for the other algebras under consideration and for right modules.

(a) Let $b W$ be a given left $B$-module. Define functors

$$G: B\mathcal{M} \to \mathcal{M}_H, \quad G(V) = \text{Hom}_A(V, W)$$

and

$$F: \mathcal{M}_H \to \mathcal{M}_k, \quad F(X) = \text{Hom}_H(k, X).$$

By Lemma 2.1, $FG$ is equivalent with the functor $\text{Hom}_B(., W)$ and so the right derived functors $R^n(FG)$ are equivalent with $\text{Ext}^n_B(., W)$. Moreover, if $P \in B\mathcal{M}$ is projective, then $(R^nF)(G(P)) = \text{Ext}^n_H(k, G(P)) = 0$ for all $n > 0$ and so $G(P)$ is right $F$-acyclic. Indeed, it suffices to check this equality for $P = B$. In this case, Lemma 2.1 and [Rot], Theorem 11.56, together imply that

$$\text{Ext}^n_H(k, G(B)) = \text{Ext}^n_H(k, \text{Hom}_k(H, W))$$

$$\cong \text{Ext}^n_k(k \otimes_H H, W)$$

$$= \text{Ext}^n_k(k, W)$$

$$= 0 \quad (n > 0).$$

The required spectral sequence now follows from [Rot], Theorem 11.38.

(b) Let $V_B$ be a given right $B$-module. Define functors

$$G: B\mathcal{M} \to H\mathcal{M}, \quad G(W) = V \otimes_A W$$

and

$$F: H\mathcal{M} \to k\mathcal{M}, \quad F(X) = k \otimes_H X.$$

By Lemma 2.2, $FG$ is equivalent with the functor $V \otimes_B (.)$ and so the left derived functors $L^n(FG)$ are equivalent with $\text{Tor}^B_n(V, .)$. Furthermore, Lemma 2.2 implies that $G$ maps projective $A$-modules to projective $B$-modules. Since projective $B$-modules are left $F$-acyclic, the required spectral sequence follows from [Rot], Theorem 11.39.

2.4. Homological dimension. The above proposition directly implies the following estimates for the flat dimension and the projective dimension of modules, denoted $\text{fdim}$ and $\text{pdim}$, respectively.

**Corollary.** (a) Let $b V$ be a $B$-module. Then $\text{pdim}_B V \leq \text{pdim}_H k + \text{pdim}_A V$. Consequently, $\text{l.gl.dim} B \leq \text{r.gl.dim} H + \text{l.gl.dim} A$. In particular, if $A$ and $H$ are both semisimple ($\text{gl.dim} 0$), then so is $B$ (cf. [Mont], Theorem 7.4.2).

(b) Let $V_B$ be a $B$-module. Then $\text{fdim}_B V_B \leq \text{fdim}_H k + \text{fdim}_A V_A$. Therefore, $\text{w.dim} B \leq \text{w.dim} H + \text{w.dim} A$. In particular, if $A$ and $H$ are both von Neumann regular (w.dim 0), then so is $B$.

We note that

$$\text{r.gl.dim} H = \text{pdim}_H k \quad \text{and} \quad \text{w.dim} H = \text{fdim}_H k.$$
2.5. Relative projectivity. Recall that if \( R \subseteq S \) is a pair of rings, then \( S \) is called \textit{projective relative to} \( R \) (or \( R \)-projective) if the following holds: Given \( S \)-modules \( SW \subseteq SV \) so that \( W \) is a direct summand of \( V \) as \( R \)-modules, then \( W \) is a direct summand of \( V \) as \( S \)-modules. The following result is identical with [Mont], Theorem 7.4.2(1), but the proof below is a nice application of the techniques of Section 2.1.

\textbf{Corollary.} If \( H \) is semisimple, then \( B \) is projective relative to \( A \).

\textit{Proof.} First note that if \( f : X \to Y \) is an epimorphism in \( \mathcal{M}_H \) then \( f(X^H) = Y^H \), because \( f \) splits by assumption on \( H \). Now let \( BW \subseteq BV \) so that \( W \) is a direct summand of \( V \) as \( A \)-modules. Then the canonical epimorphism \( \pi : V \to V/W \) splits in \( \mathcal{M}_A \) and so the map \( \pi_* : \text{Hom}_A(V/W, V) \to \text{End}_A(V/W) \) is surjective. Since \( \pi_* \) is a map in \( \mathcal{M}_H \), we deduce from the foregoing that \( \pi_* (\text{Hom}_A(V/W, V)^H) = \text{End}_A(V/W)^H \). By Section 2.1, \( \text{Hom}_A(V/W, V)^H = \text{Hom}_B(V/W, V) \) and \( \text{End}_A(V/W)^H = \text{End}_B(V/W) \). Thus there exists \( \mu \in \text{Hom}_B(V/W, V) \) with \( \pi_*(\mu) = \pi \circ \mu = \text{Id}_{V/W} \) and so \( W \) is a direct summand of \( V \) as \( B \)-modules, as required. □

\textbf{References}


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