TOPOLOGY OF FACTORED ARRANGEMENTS OF LINES

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Abstract. A real arrangement of affine lines is a finite family \( \mathcal{A} \) of lines in \( \mathbb{R}^2 \). A real arrangement \( \mathcal{A} \) of lines is said to be factored if there exists a partition \( \Pi = (\Pi_1, \Pi_2) \) of \( \mathcal{A} \) into two disjoint subsets such that the Orlik-Solomon algebra of \( \mathcal{A} \) factors according to this partition. We prove that the complement of the complexification of a factored real arrangement of lines is a \( K(\pi, 1) \) space.

1. Introduction

Let \( K \) be a field, and let \( V \) be a vector space over \( K \). An arrangement of (affine) hyperplanes in \( V \) is a finite family \( \mathcal{A} \) of (affine) hyperplanes of \( V \). An arrangement of (affine) lines is an arrangement of hyperplanes in a 2-dimensional vector space \( V = \mathbb{K}^2 \). An arrangement \( \mathcal{A} \) of hyperplanes is said to be real (resp. complex) if \( K = \mathbb{R} \) is the field of real numbers (resp. if \( K = \mathbb{C} \) is the field of complex numbers). The complexification of a hyperplane \( H \) of \( \mathbb{R}^l \) is the hyperplane \( H_C \) of \( \mathbb{C}^l \) having the same equation as \( H \). The complexification of a real arrangement \( \mathcal{A} \) of hyperplanes in \( \mathbb{R}^l \) is the arrangement \( \mathcal{A}_C = \{H_C \mid H \in \mathcal{A} \} \) in \( \mathbb{C}^l \).

Let \( \mathcal{A} \) be a complex arrangement of hyperplanes in \( V = \mathbb{C}^l \). The complement of \( \mathcal{A} \) is the connected submanifold

\[
M(\mathcal{A}) = V - \left( \bigcup_{H \in \mathcal{A}} H \right)
\]

of \( V \). We say that \( \mathcal{A} \) is a \( K(\pi, 1) \) arrangement if \( M(\mathcal{A}) \) is a \( K(\pi, 1) \) space. We say that a real arrangement \( \mathcal{A} \) of hyperplanes is a \( K(\pi, 1) \) arrangement if its complexification \( \mathcal{A}_C \) is a \( K(\pi, 1) \) arrangement. Yet, only two classes of real \( K(\pi, 1) \) arrangements of hyperplanes are known. These are the simplicial arrangements (see [De]) and supersolvable arrangements (see [Te1]). Other examples of real \( K(\pi, 1) \) arrangements appear in [Fa] and in [JS].

Our aim in this paper is to produce a new class of real \( K(\pi, 1) \) arrangements: “factored arrangements of lines”. This class contains supersolvable arrangements of lines (see [Ja]).

We refer to [FR] for a good exposition on \( K(\pi, 1) \) arrangements.

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Let $\mathcal{A}$ be an arrangement of hyperplanes. The intersection poset of $\mathcal{A}$ is the ranked poset $\mathcal{L}(\mathcal{A})$ consisting of all nonempty intersections of elements of $\mathcal{A}$ ordered by reverse inclusion. $V = \bigcap_{H \in \mathcal{A}} H$ is assumed to be the smallest element of $\mathcal{L}(\mathcal{A})$. For $X \in \mathcal{L}(\mathcal{A})$, we set $\mathcal{A}_X = \{ H \in \mathcal{A} \mid H \supseteq X \}$.

A partition $\Pi = (\Pi_1, \ldots, \Pi_l)$ of $\mathcal{A}$ into $l$ disjoint nonempty subsets is called independent if, for any choice of hyperplanes $H_i \in \Pi_i (i = 1, \ldots, l)$, the subspace $H_1 \cap \cdots \cap H_l$ is nonempty and its rank is $l$ in $\mathcal{L}(\mathcal{A})$. If $X \in \mathcal{L}(\mathcal{A})$, then $\Pi$ induces a partition $\Pi_X$ of $\mathcal{A}_X$ whose blocks are the nonempty subsets $\Pi_i \cap \mathcal{A}_X$. A partition $\Pi = (\Pi_1, \ldots, \Pi_l)$ of $\mathcal{A}$ is a factorization (or a nice partition) if

1. $\Pi$ is independent;
2. if $X \in \mathcal{L}(\mathcal{A}) - \{V\}$, then $\Pi_X$ has at least a block which is a singleton.

If $\mathcal{A}$ is an arrangement of lines, then any factorization of $\mathcal{A}$ has to be a partition $\Pi = (\Pi_1, \Pi_2)$ of $\mathcal{A}$ into two disjoint subsets (see [Te2]). We say that an arrangement $\mathcal{A}$ of hyperplanes is factored if $\mathcal{A}$ has a factorization.

Factored arrangements have been introduced and investigated by Falk, Jambu, and Terao [FJ, Te2]. One of the main results concerning these arrangements is the following theorem due to Terao [Te2].

The homogeneous component $A^1(\mathcal{A})$ of the Orlik-Solomon algebra $A(\mathcal{A})$ of an arrangement $\mathcal{A}$ of hyperplanes can be viewed as a free $\mathbb{Z}$-module spanned by the hyperplanes of $\mathcal{A}$ (see [OS]). For $\mathcal{B} \subseteq \mathcal{A}$, we denote by $B(\mathcal{B})$ the submodule of $A^1(\mathcal{A})$ spanned by the elements of $\mathcal{B}$.

**Theorem 1** (Terao [Te2]). Let $\mathcal{A}$ be an arrangement of hyperplanes. Let $\Pi = (\Pi_1, \ldots, \Pi_l)$ be a partition of $\mathcal{A}$. The Orlik-Solomon algebra of $\mathcal{A}$, viewed as a graded $\mathbb{Z}$-module, factors as

$$A(\mathcal{A}) = \bigoplus_{\Pi_1} A^1(\mathcal{A}) \otimes \cdots \otimes \bigoplus_{\Pi_l} A^1(\mathcal{A})$$

if and only if $\Pi$ is a factorization.

Our goal in this paper is to prove the following theorem.

**Theorem 2.** If $\mathcal{A}$ is a factored real arrangement of lines, then $\mathcal{A}$ is a $K(\pi, 1)$ arrangement.

**Example.** Consider the arrangement $\mathcal{A}$ shown in Figure 1. Set $\Pi_1 = \{l_1, l_2, l_3, l_4\}$ and $\Pi_2 = \{l_5, l_6, l_7, l_8\}$. Then $\Pi = (\Pi_1, \Pi_2)$ is a factorization of $\mathcal{A}$. Note that this arrangement is neither simplicial nor supersolvable.

A direct consequence of Theorem 2 is the following corollary. Recall that an arrangement $\mathcal{A}$ of hyperplanes is said to be central if $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$.

**Corollary.** Let $\mathcal{A}$ be a real and central arrangement of hyperplanes. Assume that the rank of $\mathcal{L}(\mathcal{A})$ is 3. If $\mathcal{A}$ is a factored arrangement, then $\mathcal{A}$ is a $K(\pi, 1)$ arrangement.

**Proof.** Let $\Pi = (\Pi_1, \Pi_2, \Pi_3)$ be a factorization of $\mathcal{A}$. One may assume that $\mathcal{A}$ is an arrangement in $\mathbb{R}^3$, that $\bigcap_{H \in \mathcal{A}} H = \{0\}$, and that $\Pi_3$ is a singleton $\{H_0\}$. Let $K_0$ be an (affine) hyperplane of $\mathbb{R}^3$ parallel to $H_0$ and different
from $H_0$. Set

$$\mathfrak{A} = \{H \cap K_0 \mid H \in \mathcal{A} - \{H_0\}\},$$
$$\Pi_1 = \{H \cap K_0 \mid H \in \Pi_1\},$$
$$\Pi_2 = \{H \cap K_0 \mid H \in \Pi_2\}.$$

Then $\mathfrak{A}$ is a real arrangement of lines in $K_0$, the partition $\Pi = (\Pi_1, \Pi_2)$ is a factorization of $\mathfrak{A}$, and $M(\mathcal{A}_C)$ is homeomorphic to $C^* \times M(\mathcal{A}_C)$ (see [OT, Proposition 5.1.1]). So, $M(\mathcal{A}_C)$ is a $K(\pi, 1)$ space since, by Theorem 2, $M(\mathcal{A}_C)$ is a $K(\pi, 1)$ space. □

The proof of Theorem 2 is a direct application of Falk’s weight test for a real arrangement of lines to be $K(\pi, 1)$ (see [Fa]).

Section 2 is divided into two subsections. In §2.1 we state Falk’s weight test (Theorem 3). In §2.2 we prove Theorem 2.

2. Proof of Theorem 2

Throughout this section $\mathcal{A}$ is assumed to be an arrangement of affine lines in $V = \mathbb{R}^2$.

2.1. Falk’s weight test for $K(\pi, 1)$ arrangements. The lines of $\mathcal{A}$ subdivide $V$ into facets. The support $|f|$ of a facet $f$ is the smallest affine subspace of $V$ containing $f$. Every facet is open in its support. We denote by $\bar{f}$ the closure of $f$ in $V$. There is a partial order on the set of facets defined by $f \leq g$ if $f \subseteq g$. 0-dimensional facets are called vertices, 1-dimensional facets are called edges, and 2-dimensional facets are called faces.

Let $\Gamma(\mathcal{A})$ denote the planar 2-complex consisting of the bounded facets. We denote by $\Gamma^{(i)}(\mathcal{A})$ its $i$-skeleton ($i = 0, 1, 2$). A corner of $\Gamma(\mathcal{A})$ is a chain $(v < f)$ with $v \in \Gamma^{(0)}(\mathcal{A})$ and $f \in \Gamma^{(2)}(\mathcal{A})$. We denote by $\text{Corn}(\mathcal{A})$ the set of corners. A system of weights on $\Gamma(\mathcal{A})$ is a function $\Omega: \text{Corn}(\mathcal{A}) \to \mathbb{R}^+ = [0, +\infty[.$

Let $v \in \Gamma^{(0)}(\mathcal{A})$. The link graph of $\Gamma(\mathcal{A})$ at $v$ is the graph $\Lambda_v$ defined as follows.

(1) The vertices of $\Lambda_v$ are the chains $(v < e)$ with $e \in \Gamma^{(1)}(\mathcal{A})$. 

![Figure 1](image_url)
(2) The edges of $\Lambda_v$ are the chains (or corners) $(v < f)$ with $f \in \Gamma^{(2)}(\mathcal{A})$.

(3) An edge $(v < f)$ is incident with a vertex $(v < e)$ if $v < e < f$.

Let $\gamma = ((v < e_0), (v < e_1), \ldots, (v < e_n))$ be a path in $\Lambda_v$. For $j = 1, \ldots, n$, let $(v < f_j)$ be the edge of $\Lambda_v$ incident with $(v < e_{j-1})$ and $(v < e_j)$. For a given system of weights $\Omega : \text{Corn}(\mathcal{A}) \to \mathbb{R}^+$, we define the weight of $\gamma$ to be

$$\Omega(\gamma) = \sum_{j=1}^{n} \Omega(v < f_j).$$

A path $\gamma = ((v < e_0), (v < e_1), \ldots, (v < e_n))$ is a circuit if $e_0 = e_n$. Let $\mathcal{A}_v$ denote the set of lines of $\mathcal{A}$ which contain $v$. A circuit $\gamma = ((v < e_0), (v < e_1), \ldots, (v < e_n))$ is full if, for every $l \in \mathcal{A}_v$, there exist at least two distinct indices $1 \leq j < k \leq n$ such that $|e_j| = |e_k| = l$. A system of weights $\Omega : \text{Corn}(\mathcal{A}) \to \mathbb{R}^+$ is said to be $\mathcal{A}$-admissible if, for every $v \in \Gamma^{(0)}(\mathcal{A})$ and every full circuit $\gamma$ of $\Lambda_v$, we have $\Omega(\gamma) \geq 2\pi$.

Let $f \in \Gamma^{(2)}(\mathcal{A})$. We denote by $d(f)$ the number of vertices $v \in \Gamma^{(0)}(\mathcal{A})$ such that $v < f$. It is also the number of edges $e \in \Gamma^{(1)}(\mathcal{A})$ such that $e < f$. For a given system of weights $\Omega : \text{Corn}(\mathcal{A}) \to \mathbb{R}^+$, we define the weight of $f$ to be

$$\Omega(f) = \sum_{v < f} \Omega(v < f).$$

A system of weights $\Omega : \text{Corn}(\mathcal{A}) \to \mathbb{R}^+$ is said to be aspherical if, for every $f \in \Gamma^{(2)}(\mathcal{A})$, we have $\Omega(f) \leq (d(f) - 2)\pi$.

Theorem 3 (Falk [Fa]). If $\Gamma(\mathcal{A})$ admits a system of weights which is $\mathcal{A}$-admissible and aspherical, then $\mathcal{A}$ is a $K(\pi, 1)$ arrangement.

Remark. There is a similar criterion given in [JS] for a real arrangement of lines to be $K(\pi, 1)$.

2.2. Proof of Theorem 2. Let $\Pi = (\Pi_1, \Pi_2)$ be a factorization of $\mathcal{A}$. Let $(v < f)$ be a corner of $\Gamma(\mathcal{A})$. Let $e_1$ and $e_2$ be the two edges of $\Gamma(\mathcal{A})$ such that $v < e_i < f$ ($i = 1, 2$). We say that $(v < f)$ is coloured if, up to some permutation, $|e_1| \in \Pi_1$ and $|e_2| \in \Pi_2$. We consider the system of weights defined by

$$\Omega(v < f) = \begin{cases} \pi/2 & \text{if } (v < f) \text{ is coloured}, \\ 0 & \text{otherwise}. \end{cases}$$

Let $v \in \Gamma^{(0)}(\mathcal{A})$. Let $\gamma = ((v < e_0), (v < e_1), \ldots, (v < e_n))$ be a full circuit of $\Lambda_v$. For $j = 1, \ldots, n$, let $(v < f_j)$ be the edge of $\Lambda_v$ incident with $(v < e_{j-1})$ and $(v < e_j)$. By definition of a factorization, we may assume that $\mathcal{A}_v \cap \Pi_1$ is a singleton $\{l_0\}$. By definition of a full circuit, there exist two indices $1 \leq j < k \leq n$ such that $|e_j| = |e_k| = l_0$. Obviously, $j \neq k - 1$ and $k \neq j - 1$ (we assume that $j - 1 = n$ if $j = 1$), and $(v < f_{j-1})$, $(v < f_j)$, $(v < f_{k-1})$, and $(v < f_k)$ are coloured corners. Thus,

$$\Omega(\gamma) \geq \Omega(v < f_{j-1}) + \Omega(v < f_j) + \Omega(v < f_{k-1}) + \Omega(v < f_k) = 2\pi.$$

This shows that $\Omega : \text{Corn}(\mathcal{A}) \to \mathbb{R}^+$ is $\mathcal{A}$-admissible.

Let $f \in \Gamma^{(2)}(\mathcal{A})$. If $f$ is a triangle (i.e., $d(f) = 3$), then there exist at most two vertices $v_1, v_2 \in \Gamma^{(0)}(\mathcal{A})$ such that $v_i < f$ and the corner $(v_i < f)$ is
coloured (for \( i = 1, 2 \)). Thus,
\[
\Omega(f) \leq 2 \cdot \frac{\pi}{2} = (d(f) - 2)\pi.
\]
If \( d(f) \geq 4 \), then
\[
\Omega(f) \leq d(f) \frac{\pi}{2} \leq (d(f) - 2)\pi.
\]
This shows that \( \Omega: \text{Corn}(\mathcal{A}) \to \mathbb{R}^+ \) is aspherical. \( \square \)

**REFERENCES**


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