EXISTENCE RESULTS FOR SOME INITIAL
AND BOUNDARY VALUE PROBLEMS
WITHOUT GROWTH RESTRICTION

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ABSTRACT. In this paper, using the Schauder Fixed Point Theorem, we establish some existence results or initial and boundary value problems for differential equations withouth growth restriction on the right member.

1. INTRODUCTION

In this paper, we give existence results for initial and boundary value problems without growth restriction on the right member. The paper is divided into two sections. In §2 we examine second-order problems of the form:

$$(py')'(t) = f(t, y(t), p(t)y'(t)) \quad \text{a.e.} \quad t \in [0, 1],$$

(B)
$$a_0 y(0) - b_0 \lim_{t \to 0} p(t)y'(t) = r_0,$$

$$a_1 y(1) + b_1 \lim_{t \to 1} p(t)y'(t) = r_1,$$

where $p \in C[0, 1] \cap C^1(0, 1)$ with $p(t) > 0$ on $t \in (0, 1)$, and $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function. By a solution to (B), we mean a function $y \in C[0, 1] \cap C^1(0, 1)$ such that py' is absolutely continuous on $[0, 1]$, and which satisfies the differential equation and boundary conditions. Boundary value problems of the form (B) have been extensively treated in the literature. In most of these papers, f satisfies a growth condition in y'. The results presented in this section are obtained without any growth assumption on f, and generalize some results deduced from the location of the zeros of the nonlinearity; see, for example, [3, 5, 6, 9]. They rely on the notion of upper and lower surfaces for (B), that is, surfaces having a particular form and on which the function f has a given sign. This notion was introduced in [3] in the particular case where $p \equiv 1$.

In §3 we examine the existence of solutions for a first-order initial value

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problem of the form:

$$\text{(I)} \qquad \begin{aligned} y'(t) &= f(t, y(t)) \quad \text{a.e.} \quad t \in [0, T], \\ y(0) &= r \end{aligned}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Here, by a solution to (I), we mean an absolutely continuous function on $[0, T]$ which satisfies the differential equation and the initial condition. The existence theorems presented here generalize some known results based on the notion of upper and lower solutions for the initial value problem; see [1, 6, 8]. In this paper, we prove the existence of a solution for the problem (I) between two functions s_1 and s_0 which are not necessarily absolutely continuous or continuous, without using the derivative or the Dini derivates of those functions.

All our results are obtained via the Schauder Fixed Point Theorem.

First of all, let us establish some notation. Let $p \in C[0, 1] \cap C^1(0, 1)$ be such that $p(t) > 0$ on $t \in (0, 1)$. We define the Banach space $K^1[0, 1]$ by $\{y \in C[0, 1] \cap C^1(0, 1) \mid py' \in C[0, 1]\}$ with the norm $\|y\|_1 = \max\{\|y\|_0, \|py'\|_0\}$ where $\|y\|_0 = \max\{|y(t)| \mid t \in [0, 1]\}$, and we define $K^2[0, 1]$ by $\{y \in K^1[0, 1] \mid py' \in W^{1,1}[0, 1]\}$. We denote $K_b^1[0, 1] = \{y \in K^1[0, 1] \mid y$ satisfies boundary conditions $a_0 y(0) - b_0 \lim_{t \to 0} p(t) y'(t) = r_0, \ a_1 y(1) + b_1 \lim_{t \to 1} p(t) y'(t) = r_1\}$, $K_b^2[0, 1] = K^2[0, 1] \cap K_b^1[0, 1]$; and $C_r[0, 1] = \{y \in C[0, 1] \mid y(0) = r\}$.

For the sake of completeness, we give the following two results which will be used later. The second one is a Maximum Principle.

Lemma 1.1. *Let $x : [0, 1] \to \mathbb{R}$ be an absolutely continuous function, and $U \subset [0, 1]$ be a measurable set such that $x(U)$ is negligeable. Then $x'(t) = 0$ a.e. $t \in U$.*

Lemma 1.2. *Let $p \in C[c, d] \cap C^1(c, d)$ be such that $p(t) > 0$ for all $t \in (c, d)$; and let $y \in K^2[c, d]$ be such that $(py')'(t) \geq 0$ a.e. $t \in (c, d)$, $a_0 y(c) - b_0 \lim_{t \to c} p(t) y'(t) \leq 0$, and $a_1 y(d) + b_1 \lim_{t \to d} p(t) y'(t) \leq 0$, where $\max\{a_0, a_1\} > 0$, $a_i, b_i \geq 0$, $\max\{a_i, b_i\} > 0$, $i = 0, 1$. Then $y(t) \leq 0$ for all $t \in [c, d]$.*

Proof. Define

$$G(t) = \begin{cases} y(t)^2 & \text{if } y(t) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(py')'(t) G(t) \geq 0$ a.e. $t \in (c, d)$. Integrating from c to d and using the integration by parts formula gives:

$$G(d) \lim_{t \to d} p(t) y'(t) - G(c) \lim_{t \to c} p(t) y'(t) - \int_c^d p(t) y'(t) G'(t) \, dt \geq 0.$$

We deduce that $y(t) \leq 0$ for all $t \in [c, d]$. $\square$

2. Boundary value problems

In this section, we consider the boundary value problem (B) where $a_i, b_i \geq 0$, $\max\{a_i, b_i\} > 0$, $i = 0, 1$, $\max\{a_0, a_1\} > 0$, and p and f satisfy the following assumption:

(H) $p \in C[0, 1] \cap C^1(0, 1)$, $p(t) > 0$ for all $t \in (0, 1)$, $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, that is: (i) $t \mapsto f(t, y, q)$ is measurable for all

$(y, q) \in \mathbb{R}^2$; (ii) $(y, q) \mapsto f(t, y, q)$ is continuous for a.e. $t \in [0, 1]$; (iii) for any $r > 0$, there exists $h_r \in L^1[0, 1]$ such that $|f(t, y, q)| \leq h_r(t)$ a.e. $t \in [0, 1]$, and for all $|y| \leq r$, $|q| \leq r$. Moreover, $\int_0^1 p^{-1}(s) \int_0^s h_r(x) \, dx \, ds < \infty$ if $a_1 \neq 0$, and $\int_0^1 p^{-1}(s) \int_s^1 h_r(x) \, dx \, ds < \infty$ if $a_1 = 0$; and $p^{-1} \in L^1[0, 1]$ if $a_0 = r_0 = 0$ or $a_1 = r_1 = 0$ do not occur.

We start with a particular case in order to illustrate arguments which will be used in a more general result (Theorem 2.8).

Consider the problem

$$(py')'(t) = f(t, y(t), p(t)y'(t)) \quad \text{a.e. } t \in [0, 1],$$

(2.1)
$$\lim_{t \to 0} p(t)y'(t) = 0,$$

$$y(1) + b \lim_{t \to 1} p(t)y'(t) = 0$$

where $b \geq 0$.

Theorem 2.1. *Assume there exist $M_0 \geq 0 \geq M_1$ such that $f(t, M_0, 0) \geq 0 \geq f(t, M_1, 0)$ a.e. $t \in [0, 1]$ and there exist two functions $s_0 : [0, 1] \times [M_1, M_0] \to [0, \infty)$, $s_1 : [0, 1] \times [M_1, M_0] \to (-\infty, 0]$ such that:*

(1) *$x \mapsto s_i(t, x)$ is continuous for a.e. $t \in [0, 1]$, $i = 0, 1$;*
(2) *$s_0(t_1, x_1) \leq s_0(t_2, x_2)$ for all $t_1 \leq t_2$, $x_1 \leq x_2$; and $s_1(t_1, x_1) \leq s_1(t_2, x_2)$ for all $t_1 \geq t_2$, $x_1 \leq x_2$;*
(3) *$f(t, x, s_0(t, x)) \leq 0 \leq f(t, x, s_1(t, x))$ for a.e. $t \in [0, 1]$ and all $x \in [M_1, M_0]$.*

Then the problem (2.1) has a solution such that $M_1 \leq y(t) \leq M_0$, $s_1(t, y(t)) \leq p(t)y'(t) \leq s_0(t, y(t))$ for all $t \in [0, 1]$.

To prove this theorem, we modify the problem (2.1), and we show that this modified problem has a solution which is also a solution to (2.1).

Define the function $f_1 : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$f_1(t, y, q) = \begin{cases} f(t, M_0, 0) & \text{if } y > M_0, \\ f(t, M_1, 0) & \text{if } y < M_1, \\ f(t, y, s_0(t, y)) & \text{if } M_1 \leq y \leq M_0, \, q > s_0(t, y), \\ f(t, y, s_1(t, y)) & \text{if } M_1 \leq y \leq M_0, \, q < s_1(t, y), \\ f(t, y, q) & \text{if } M_1 \leq y \leq M_0, \, s_1(t, y) \leq q \leq s_0(t, y). \end{cases}$$

It is clear that this function may not be Carathéodory, but we have the following result:

Proposition 2.2. *Under the assumptions of the previous theorem, the operator $N_1 : K^1[0, 1] \to C_0[0, 1]$ defined by $N_1(y)(t) = \int_0^t f_1(\tau, y(\tau), p(\tau)y'(\tau)) \, d\tau$ is well defined, continuous, and compact.*

Proof. Let $R > \max\{|M_0|, |M_1|\}$ such that $|s_i(t, y)| \leq R$ for all $t \in [0, 1]$ and $y \in [M_1, M_0]$; and let $h_R \in L^1[0, 1]$ be the function given in (H)(iii). Then

(2.2) $|f_1(t, y(t), p(t)y'(t))| \leq h_R(t)$ a.e. $t \in [0, 1]$ and for all $y \in K^1[0, 1]$.

So, N_1 is well defined. Now, assume that $y_n \to y$ in $K^1[0, 1]$. By the inequality (2.2) and the Lebesgue Convergence Dominated Theorem, it suffices to show that

$$(2.3) \qquad f_1(t, y_n(t), p(t)y_n'(t)) \to f_1(t, y(t), p(t)y'(t)) \quad \text{a.e.} \ t \in [0, 1]$$

to get that $N_1 y_n \to N_1 y$ in $C_0[0, 1]$.

It is clear that (2.3) holds a.e. on $\{t \in [0, 1] \mid y(t) \neq M_0, \text{ and } y(t) \neq M_1\}$. On the other hand, by Lemma 1.1, $y'(t) = 0$ a.e. on $\{t \in [0, 1] \mid y(t) = M_0, \text{ or } y(t) = M_1\}$. So, $s_1(t, y(t)) \leq p(t)y'(t) = 0 \leq s_0(t, y(t))$ a.e. on that set. The assumptions (1) and (H)(ii) imply that (2.3) holds a.e. on that set.

The compactness of N_1 follows from (2.2) and the Arzela-Ascoli Theorem. $\square$

We consider the modified problem

$$(2.4) \qquad \begin{aligned} &(py')'(t) = f_1(t, y(t), p(t)y'(t)) \quad \text{a.e.} \ t \in [0, 1], \\ &\lim_{t \to 0} p(t)y'(t) = 0, \\ &y(1) + b \lim_{t \to 1} p(t)y'(t) = 0. \end{aligned}$$

Proposition 2.3. *Under the assumptions of Theorem 2.1, problem (2.4) has a solution.*

Proof. Set $C_0^R = \{g \in C_0[0, 1] \mid |g(t)| \leq \int_0^t h_R(s)\,ds\}$ where h_R is the function satisfying inequality (2.2). Define the continuous operator $L^{-1} : C_0^R \to K_b^1$ by

$$L^{-1}(g)(t) = -bg(1) - \int_t^1 g(s)p^{-1}(s)\,ds.$$

From (2.2), $N_1(y) \in C_0^R$ for all $y \in K_b^1$. Remark that y is a solution to (2.4) if and only if y is a fixed point of $L^{-1} \circ N_1 : K_b^1 \to K_b^1$ (see [4] for details). By Proposition 2.2, the operator $L^{-1} \circ N_1$ is compact. The Schauder Fixed Point Theorem gives the existence of a fixed point to this operator, and hence a solution to (2.4). $\square$

Proof of Theorem 2.1. Let y be a solution to (2.4) given by the previous proposition. We want to show that y is a solution to (2.1).

We claim that $M_1 \leq y(t) \leq M_0$. Indeed, let $u(t) = y(t) - M_0$. By assumption and the definition of f_1, we have

$$(pu')'(t) \geq 0 \quad \text{a.e. on} \ \{t \mid u(t) > 0\}.$$

Boundary conditions and Lemma 1.2 imply that $y(t) \leq M_0$ for all $t \in [0, 1]$. Similarly we get the other inequality.

On the other hand, assume there exists $t_1 \in [0, 1]$ such that $p(t_1)y'(t_1) > s_0(t_1, y(t_1))$. Since $\lim_{t \to 0} p(t)y'(t) = 0 \leq s_0(0, y(0))$, there exists $t_2 < t_1$ such that $s_0(t_2, y(t_2)) \geq p(t_2)y'(t_2)$ and $s_0(t, y(t)) \leq p(t)y'(t)$ for all $t \in (t_2, t_1)$. By assumption (2) and the definition of f_1, we get

$$0 < p(t_1)y'(t_1) - p(t_2)y'(t_2) = \int_{t_2}^{t_1} (py')'(t)\,dt \leq 0,$$

a contradiction. The other inequality is obtained in the same way.

Therefore, the solution y is such that $M_1 \leq y(t) \leq M_0$ and $s_1(t, y(t)) \leq p(t)y'(t) \leq s_0(t, y(t))$ for all $t \in [0, 1]$; and consequently, y is a solution to (2.1). $\square$

Now, we give a more general result. First of all, we need to introduce some definitions. The notion of upper and lower surfaces was introduced in [3] in the particular case where $p \equiv 1$.

Definition 2.4. A function $\alpha \in K^2[0, 1]$ (resp. $\beta \in K^2[0, 1]$) is called a *lower solution* (resp. *upper solution*) to (B) if $(p\alpha')'(t) \geq f(t, \alpha(t), p(t)\alpha'(t))$ a.e $t \in [0, 1]$; $a_0\alpha(0) - b_0 \lim_{t \to 0} p(t)\alpha'(t) \leq r_0$, $a_1\alpha(1) + b_1 \lim_{t \to 1} p(t)\alpha'(t) \leq r_1$ (resp. $(p\beta')'(t) \leq f(t, \beta(t), p(t)\beta'(t))$ a.e $t \in [0, 1]$; $a_0\beta(0) - b_0 \lim_{t \to 0} p(t)\beta'(t) \geq r_0$, $a_1\beta(1) + b_1 \lim_{t \to 1} p(t)\beta'(t) \geq r_1$).

Let $\alpha \leq \beta$ in $C[0, 1]$ and $D = \{(t, y) \in [0, 1] \times \mathbb{R} \mid \alpha(t) \leq y \leq \beta(t)\}$.

Definition 2.5. A surface $S \subset D \times \mathbb{R}$ is an *upper surface to D for* (B) if there exist two functions $s : D \to [0, \infty)$, $c \in K^1[0, 1]$ such that $S = \{(t, x, s(t, x)) \mid (t, x) \in D\}$, $\alpha \leq c \leq \beta$, and satisfying the following conditions:

(i) $(x - c(t))f(t, x, s(t, x)) \geq 0$ a.e. $t \in [0, 1]$, $(t, x) \in D$;
(ii) there exist $N \subset [0, 1]$ and $E \subset \mathbb{R}$, two negligeable sets such that the function $x \mapsto s(t, x)$ is continuous for all $(t, x) \in D \setminus (N \times E)$;
(iii) $p(t)c'(t) \leq s(t, c(t))$ for all $t \in [0, 1]$;
(iv) $s(t_1, x_1) \geq s(t_2, x_2)$ (resp. $s(t_1, x_1) \leq s(t_2, x_2)$) for all $t_1 \leq t_2$, $x_1 \leq x_2$ such that (t_1, x_1) and (t_2, x_2) are in the same connected part of $D^+ = \overline{\{(t, x) \mid x > c(t)\}}$ (resp. $D^- = \overline{\{(t, x) \mid x < c(t)\}}$);
(v) $a_0 c(0) \leq r_0 + b_0 s(0, \alpha(0))$, $a_1 c(1) \geq r_1 - b_1 s(1, \beta(1))$.

Definition 2.6. A surface $S \subset D \times \mathbb{R}$ is a *lower surface to D for* (B) if there exist two functions $s : D \to (-\infty, 0]$, $c \in K^1[0, 1]$ such that $S = \{(t, x, s(t, x)) \mid (t, x) \in D\}$, $\alpha \leq c \leq \beta$, and satisfying (i), (ii) of Definition 2.5 and the following conditions:

(iii) $p(t)c'(t) \geq s(t, c(t))$ for all $t \in [0, 1]$;
(iv) $s(t_1, x_1) \leq s(t_2, x_2)$ (resp. $s(t_1, x_1) \geq s(t_2, x_2)$) for all $t_1 \geq t_2$, $x_1 \leq x_2$ such that (t_1, x_1) and (t_2, x_2) are in the same connected part of $D^+ = \overline{\{(t, x) \mid x > c(t)\}}$ (resp. $D^- = \overline{\{(t, x) \mid x < c(t)\}}$);
(v) $a_0 c(0) \geq r_0 + b_0 s(0, \beta(0))$, $a_1 c(1) \leq r_1 - b_1 s(1, \alpha(1))$.

Remark 2.7. If f is a Carathéodory function and S is an upper (resp. lower) surface to D for (B), without loss of generality, we can assume that

(1) S is bounded;
(2) for $y = \alpha(t)$, $\beta(t)$, and $c(t)$, $s(t, y) = \limsup_{(t, x) \in D, x \to y} s(t, x)$ (resp. $s(t, y) = \liminf_{(t, x) \in D, x \to y} s(t, x)$).

Theorem 2.8. *Let p, f be functions satisfying* (H). *Assume there exist $\alpha \leq \beta$, respectively, lower and upper solutions to* (B). *Set $D = \{(t, y) \in [0, 1] \times \mathbb{R} \mid \alpha(t) \leq y \leq \beta(t)\}$, and suppose there exist $S_1 = S_1(s_1, c_1)$ and $S_0 = S_0(s_0, c_0)$, respectively, lower and upper surfaces to D for* (B). *Then the problem* (B) *has a*

solution such that $\alpha(t) \le y(t) \le \beta(t)$, $s_1(t, y(t)) \le p(t)y'(t) \le s_0(t, y(t))$ *for all* $t \in [0, 1]$.

To $(t, y, q) \in D \times \mathbb{R}$ we associate $\widetilde{q}$ defined by

$$\widetilde{q} = \begin{cases} s_0(t, y) & \text{if } q > s_0(t, y), \\ q & \text{if } s_1(t, y) \le q \le s_0(t, y), \\ s_1(t, y) & \text{if } q < s_1(t, y). \end{cases}$$

Lemma 2.9. *Under the assumptions of Theorem 2.8, there exist four measurable functions* $\varepsilon_{i,\alpha}$, $\varepsilon_{i,\beta} : [0, 1] \to [0, \infty)$ *such that*

$$\begin{cases} \alpha(t) < \alpha(t) + \varepsilon_{0,\alpha}(t) < c_0(t) & \text{if } \alpha(t) < c_0(t), \ s_0(t, \alpha(t)) < p(t)\alpha'(t), \\ \varepsilon_{0,\alpha}(t) = 0 & \text{otherwise}; \end{cases}$$

$$\begin{cases} \beta(t) > \beta(t) - \varepsilon_{0,\beta}(t) > c_0(t) & \text{if } \beta(t) > c_0(t), \ s_0(t, \beta(t)) < p(t)\beta'(t), \\ \varepsilon_{0,\beta}(t) = 0 & \text{otherwise}; \end{cases}$$

moreover, $q > s_0(t, y)$ *for all* $(y, q) \in B_{0,\alpha}(t) \cup B_{0,\beta}(t)$. *Similarly for* $i = 1$, *where*

$$B_{i,\alpha}(t) = \begin{cases} \{(y, q) \,|\, (t, y) \in D, \\ \quad \|(y, q) - (\alpha(t), p(t)\alpha'(t))\| < \varepsilon_{i,\alpha}(t)\} & \text{if } \varepsilon_{i,\alpha}(t) \ne 0, \\ \varnothing & \text{otherwise}; \end{cases}$$

$$B_{i,\beta}(t) = \begin{cases} \{(y, q) \,|\, (t, y) \in D, \\ \quad \|(y, q) - (\beta(t), p(t)\beta'(t))\| < \varepsilon_{i,\beta}(t)\} & \text{if } \varepsilon_{i,\beta}(t) \ne 0, \\ \varnothing & \text{otherwise}. \end{cases}$$

Proof. Let

$$A_i(t) = \overline{\mathrm{co}}\{(y, s_i(t, y)) \,|\, (t, y) \in D\},$$
$$d_{i,\alpha}(t) = \mathrm{dist}((\alpha(t), p(t)\alpha'(t)), A_i(t)).$$

Choose $\varepsilon_{i,\alpha}(t) \le \min\{c_i(t) - \alpha(t), d_{i,\alpha}(t)\}/2$. Remark that if $p(t)\widetilde{\alpha'}(t) \ne p(t)\alpha'(t)$ and $c_i(t) - \alpha(t) > 0$, then $d_{i,\alpha}(t) > 0$ by Definitions 2.5 and 2.6 and Remark 2.7(2). We define $\varepsilon_{i,\beta}$ similarly. $\square$

Define the function $f_2 : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$f_2(t, y, q) = \begin{cases} \max\{f(t, \beta(t), p(t)\beta'(t)), f(t, \beta(t), p(t)\widetilde{\beta'}(t))\} & \text{if } y > \beta(t), \\ \min\{f(t, \alpha(t), p(t)\alpha'(t)), f(t, \alpha(t), p(t)\widetilde{\alpha'}(t))\} & \text{if } y < \alpha(t), \\ (1 - \lambda_{i,\beta}(t, y, q))\max\{f(t, y, q), f(t, y, \widetilde{q})\} \\ \quad + \lambda_{i,\beta}(t, y, q)f(t, y, \widetilde{q}) & \text{if } (y, q) \in B_{i,\beta}(t), \ i = 0, 1, \\ (1 - \lambda_{i,\alpha}(t, y, q))\min\{f(t, y, q), f(t, y, \widetilde{q})\} \\ \quad + \lambda_{i,\alpha}(t, y, q)f(t, y, \widetilde{q}) & \text{if } (y, q) \in B_{i,\alpha}(t), \ i = 0, 1, \\ f(t, y, \widetilde{q}) & \text{otherwise}. \end{cases}$$

where $\lambda_{i,\beta}(t, y, q) = (\varepsilon_{i,\beta}(t))^{-1}\|(y, q) - (\beta(t), p(t)\beta'(t))\|$ if $(y, q) \in B_{i,\beta}(t)$, and $\lambda_{i,\alpha}(t, y, q) = (\varepsilon_{i,\alpha}(t))^{-1}\|(y, q) - (\alpha(t), p(t)\alpha'(t))\|$ if $(y, q) \in B_{i,\alpha}(t)$, $i = 0, 1$.

Remark 2.10. On $\{(t, y, q) \in [0, 1] \times \mathbb{R}^2 \,|\, \alpha(t) \leq y \leq \beta(t), s_1(t, y) \leq q \leq s_0(t, y)\}$, $f_2(t, y, q) = f(t, y, q)$.

For the sake of brevity, we assume that $a_1 \neq 0$; the proof of Theorem 2.8 is similar for $a_1 = 0$. Without loss of generality, fix $a_1 = 1$.

Proposition 2.11. *Under the assumptions of Theorem 2.8, the operator* $N_2 : K^1[0, 1] \to C_0[0, 1]$ *defined by*

$$N_2(y)(t) = \int_0^t f_2(\tau, y(\tau), p(\tau)y'(\tau))\,d\tau$$

is well defined, continuous, and compact.

Proof. Let $R > \max\{\|\alpha\|_0, \|\beta\|_0, \|p\alpha'\|_0, \|p\beta'\|_0\}$ such that $|s_i(t, y)| \leq R$ for all $(t, y) \in D$; and let $h_R \in L^1[0, 1]$ be the function given in (H)(iii). We have

$$(2.5) \quad |f_2(t, y(t), p(t)y'(t))| \leq h_R(t) \quad \text{a.e. } t \in [0, 1] \text{ and all } y \in K^1[0, 1].$$

Hence, N_2 is well defined and the compactness of N_2 follows from (2.5) and the Arzela-Ascoli Theorem.

To show that N_2 is continuous, according to the inequality (2.5) and the Lebesgue Dominated Convergence Theorem, it suffices to show that

$$(2.6) \quad f_2(t, y_n(t), p(t)y_n'(t)) \to f_2(t, y(t), p(t)y'(t)) \quad \text{a.e. } t \in [0, 1]$$

when $y_n \to y$ in $K^1[0, 1]$.

It is clear that the relation (2.6) holds a.e. on $\{t \,|\, \alpha(t) \neq y(t) \text{ and } \beta(t) \neq y(t)\}$. On the other hand, on $\{t \,|\, \alpha(t) = y(t), \ B_{i,\alpha}(t) \neq \varnothing\}$, $p(t)\alpha'(t) = p(t)y'(t)$ a.e. and

$$f_2(t, y(t), p(t)y'(t)) = \min\{f(t, \alpha(t), p(t)\alpha'(t)), \ \widetilde{f(t, \alpha(t), p(t)\alpha'(t))}\}.$$

If $y_n(t) \geq \alpha(t)$, then for n sufficiently large, we have $(y_n(t), p(t)y_n'(t)) \in B_{i,\alpha}(t)$ and $\lambda_{i,\alpha}(t, y_n(t), p(t)y_n'(t)) \to 0$. Hence (2.6) holds a.e. on that set. On $\{t \,|\, \alpha(t) = y(t), \ B_{0,\alpha}(t) = B_{1,\alpha}(t) = \varnothing\}$, $\widetilde{p(t)\alpha'(t)} = p(t)\alpha'(t) = p(t)y'(t)$ a.e. and (2.6) holds a.e. Similarly, the relation (2.6) holds a.e. on $\{t \,|\, \beta(t) = y(t)\}$ and the proof is complete. $\square$

Consider the modified problem

$$(2.7) \quad \begin{aligned} (py')'(t) &= f_2(t, y(t), p(t)y'(t)) \quad \text{a.e. } t \in [0, 1], \\ a_0 y(0) - b_0 \lim_{t \to 0} p(t)y'(t) &= r_0, \\ y(1) + b_1 \lim_{t \to 1} p(t)y'(t) &= r_1. \end{aligned}$$

Proposition 2.12. *Under the assumptions of Theorem 2.8, problem (2.7) has a solution.*

Proof. To prove this result, argue as for Proposition 2.3 with

$$L^{-1}(g)(t) = \begin{cases} r_1 + r_0 b_1 b_0^{-1} - b_1 g(1) + \int_t^1 (r_0 b_0^{-1} - g(s))p^{-1}(s)\,ds & \text{if } a_0 = 0, \\ (r_0 + b_0 A)a_0^{-1} + \int_0^t (A + g(s))p^{-1}(s)\,ds & \text{if } a_0 \neq 0 \end{cases}$$

where

$$A = \frac{r_1 - r_0 a_0^{-1} - b_1 g(1) - \int_0^1 g(s)p^{-1}(s)\,ds}{b_1 + b_0 a_0^{-1} + \int_0^1 p^{-1}(s)\,ds}. \quad \square$$

Proof of Theorem 2.8. Let y be a solution to (2.7) given by Proposition 2.12. According to Remark 2.10, we must show that $\alpha(t) \leq y(t) \leq \beta(t)$, $s_1(t, y(t)) \leq p(t)y'(t) \leq s_0(t, y(t))$ for all $t \in [0, 1]$.

By assumption and the definition of f_2, we have

$$(py')'(t) \geq (p\beta')'(t) \quad \text{a.e. on } \{t \mid y(t) > \beta(t)\}$$

and

$$(py')'(t) \leq (p\alpha')'(t) \quad \text{a.e. on } \{t \mid y(t) > \alpha(t)\}.$$

By boundary conditions and the Maximum Principle, Lemma 1.2, we deduce that $\alpha(t) \leq y(t) \leq \beta(t)$.

On the other hand, assume that $p(t)y'(t) \not\leq s_0(t, y(t))$. By Definition 2.5 and boundary conditions, one of the following cases holds:

  (a) there exist $t_1 < t_2 \in [0, 1]$ such that $y(t) \geq c_0(t)$, $p(t)y'(t) \geq s_0(t, y(t))$ a.e. $t \in (t_1, t_2)$, $p(t_1)y'(t_1) > s_0(t_1, y(t_1))$, $p(t_2)y'(t_2) \leq s_0(t_2, y(t_2))$;
  (b) there exist $t_1 < t_2 \in [0, 1]$ such that $y(t) \leq c_0(t)$, $p(t)y'(t) \geq s_0(t, y(t))$ a.e. $t \in (t_1, t_2)$, $p(t_1)y'(t_1) \leq s_0(t_1, y(t_1))$, $p(t_2)y'(t_2) > s_0(t_2, y(t_2))$.

Without loss of generality, assume (a) holds. Definition 2.5(i) and the definition of f_2 imply that $(py')'(t) \geq 0$ a.e. $t \in (t_1, t_2)$. Therefore, using the monotonicity condition on s_0 (Definition 2.5(iv)), we get

$$0 > p(t_2)y'(t_2) - p(t_1)y'(t_1) = \int_{t_1}^{t_2} (py')'(t)\, dt \geq 0,$$

a contradiction. Similarly, we can show that $s_1(t, y(t)) \leq p(t)y'(t)$. $\square$

3. INITIAL VALUE PROBLEMS

In this section, we consider the initial value problem (I). The following theorem gives the existence of a solution to (I) under an assumption which generalizes the usual notion of upper and lower solutions for a first-order problem; see [1, 6, 8].

Theorem 3.1. *Let f be a Carathéodory function. Assume there exist two functions s_0, s_1 such that:*

  (i) *$s_i = b_i + c_i$ with $c_i \in C[0, T]$ and b_i a function of bounded variation on $[0, T]$, $i = 0, 1$;*
  (ii) *$s_1(0^+) \leq r \leq s_0(0^+)$, $s_1(t) \leq s_0(t)$ for all $t \in [0, T]$;*
  (iii) *$\int_a^b f(t, s_0(t))\, dt \leq s_0(b^-) - s_0(a^+)$, $\int_a^b f(t, s_1(t))\, dt \geq s_1(b^-) - s_1(a^+)$ for all $a < b \in [0, T]$.*

Then problem (I) has a solution such that $s_1(t) \leq y(t) \leq s_0(t)$ for a.e. $t \in [0, T]$.

Proof. Define $f_3 : [0, T] \times \mathbb{R} \to \mathbb{R}$ by

$$f_3(t, y) = \begin{cases} f(t, s_0(t)) & \text{if } y > s_0(t), \\ f(t, y) & \text{if } s_1(t) \leq y \leq s_0(t), \\ f(t, s_1(t)) & \text{if } y < s_1(t); \end{cases}$$

and consider the associated problem

(3.1)
$$\begin{aligned} y'(t) &= f_3(t, y(t)) \quad \text{a.e. } t \in [0, T], \\ y(0) &= r. \end{aligned}$$

A solution to (3.1) is a fixed point to the operator $N_3 : C_r[0, T] \to C_r[0, T]$ defined by

$$N_3(y)(t) = r + \int_0^t f_3(\tau, y(\tau)) \, d\tau.$$

Observe that the operator N_3 is well defined since f is Carathéodory and s_0, s_1 are measurable and bounded, and that there exists a function $h \in L^1[0, T]$ such that

(3.2) $|f_3(t, y(t))| \le h(t)$ a.e. $t \in [0, T]$ and all $y \in C_r[0, T]$.

This inequality and the Arzela-Ascoli Theorem imply that N_3 is compact. Moreover, the continuity of N_3 follows from the inequality (3.2), the Lebesgue Dominated Convergence Theorem, and the following relation:

$$f_3(t, y_n(t)) \to f_3(t, y(t))   \text{a.e. } t \in [0, T]$$

when $y_n \to y$ in $C_r[0, T]$.

Therefore, the Schauder Fixed Point Theorem gives the existence of a fixed point to N_3; that is, a solution to y to (3.1).

Now, we want to show that y satisfies $s_1(t) \le y(t) \le s_0(t)$ for a.e. $t \in [0, T]$, and consequently, is a solution to (I). Assume $y(t) \not\le s_0(t)$ a.e. $t \in [0, T]$. Since $y(0) = r \le s_0(0^+)$, there exist $t_1 < t_2 \in [0, T]$ such that $y(t_1) \le s_0(t_1^+)$, $y(t_2) > s_0(t_2^-)$, and $y(t) \ge s_0(t)$ a.e. $t \in (t_1, t_2)$. By the definition of f_3 and assumption (iii), we have

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} f(t, s_0(t)) \, dt \le s_0(t_2^-) - s_0(t_1^+).$$

This leads to a contradiction. Similarly, we show the other inequality.   □

We get as a corollary a result concerning upper and lower solutions for a first-order problem; see [1, 8].

Corllary 3.2. *Let f be a Carathéodory function. Assume there exist two absolutely continuous functions $\alpha \le \beta$ such that $\alpha(0) \le r \le \beta(0)$, $f(t, \beta(t)) \le \beta'(t)$, $\alpha'(t) \le f(t, \alpha(t))$ a.e. $t \in [0, T]$. Then problem (I) has a solution such that $\alpha(t) \le y(t) \le \beta(t)$ for all $t \in [0, T]$.*

Corollary 3.3. *Let f be a Carathéodory function. Assume there exist two non-decreasing functions $-s_1$, s_0 such that $s_1 \le s_0$, $s_1(0^+) \le r \le s_0(0^+)$, and $f(t, s_0(t)) \le 0 \le f(t, s_1(t))$ a.e. $t \in [0, T]$. Then problem (I) has a solution such that $s_1(t) \le y(t) \le s_0(t)$ for all $t \in [0, T]$.*

The next result generalizes a result of O'Farrell and O'Regan [7].

Corollary 3.4. *Let $\psi, \phi : [0, T] \to [0, \infty)$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $\psi > 0$ on $(0, T]$ and $\phi > 0$ on $(0, T)$, $\phi/\psi \in L^1[0, T]$. Assume there exist two constants $s_1 \le r \le s_0$ such that $g(s_1) \ge 0 \ge g(s_0)$. Then the problem*

$$\psi(t)y'(t) = \phi(t)g(y(t))   \text{a.e. } t \in [0, T],$$
$$y(0) = r$$

has a solution $y \in C[0, T] \cap C^1(0, T]$.

In Theorem 3.1, we can reverse the inequalities in assumption (iii) if assumption (ii) is replaced by a stronger one.

Theorem 3.5. *Let f be a Carathéodory function. Assume there exist two functions s_0, s_1 such that:*

   (i) $s_i = b_i + c_i$ *with* $c_i \in C[0, T]$ *and* b_i *a function of bounded variation on* $[0, T]$, $i = 0, 1$;

   (ii) $s_1(0^+) = r = s_0(0^+)$, $s_1(t) \leq s_0(t)$ *for all* $t \in [0, T]$;

   (iii) $\int_a^b f(t, s_0(t))\, dt \geq s_0(b^-) - s_0(a^+)$, $\int_a^b f(t, s_1(t))\, dt \leq s_1(b^-) - s_1(a^+)$ *for all* $a < b \in [0, T]$.

Then problem (I) *has a solution such that* $s_1(t) \leq y(t) \leq s_0(t)$ *for a.e.* $t \in [0, T]$.

Proof. Let $\hat{r} \in [s_1(1^-), s_0(1^-)]$. Consider the problem

$$
(3.3) \qquad \begin{aligned} y'(t) &= f(t, y(t)) \quad \text{a.e. } t \in [0, T], \\ y(1) &= \hat{r}. \end{aligned}
$$

By making the change of variable $\tau = 1 - t$, problem (3.3) can be written as a problem of form (I) on which we can apply Theorem 3.1. Therefore, problem (3.3) has a solution such that $s_1(t) \leq y(t) \leq s_0(t)$ for a.e. $t \in [0, T]$. By assumption (ii), we deduce that y is a solution to (I). $\square$

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