FIXED-POINT SETS OF AUTOHOMEOMORPHISMS
OF COMPACT $F$-SPACES

K. P. HART AND J. VERMEER

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ABSTRACT. We investigate fixed-point sets of autohomeomorphisms of compact $F$-spaces. If the space in question is finite dimensional (in the sense of covering dimension), then the fixed-point set is a $P$-set; on the other hand there is an infinite-dimensional compact $F$-space with an involution whose fixed-point set is not a $P$-set.

In addition we show that under CH a closed subset of $\omega^*$ is a $P$-set iff it is the fixed-point set of an autohomeomorphism.

INTRODUCTION

In this note we investigate the fixed-point sets of autohomeomorphisms of compact $F$-spaces. In Vermeer [6, 7] the second author studied fixed-point sets of continuous self-maps of extremally and basically disconnected spaces. It was proved that whenever $X$ is a compact $\kappa$-basically disconnected space (i.e., the Stone space of a $\kappa$-complete Boolean algebra) and $\phi: X \to X$ is injective and continuous, the fixed-point set of $\phi$ is a $P_{\kappa}$-set of $X$. In particular for a basically disconnected (i.e., $\omega_1$-basically disconnected) space the fixed-point set of a self-embedding is always a $P$-set.

The methods used to obtain the above-mentioned result do not readily generalize to the natural extension of the class of basically disconnected spaces: the class of $F$-spaces. The point is that these methods relied heavily on the fact that a countable increasing union of clopen sets in a basically disconnected space has a clopen closure and this last property hardly ever holds nontrivially in general $F$-spaces.

Here we use results about fixed-point free extensions of fixed-point free maps to obtain the result that the fixed-point set of an autohomeomorphism of a finite-dimensional compact $F$-space is a $P$-set of that space. This seems to be new, even for the space $\omega^*$.

If we assume the Continuum Hypothesis, then we can even show that a closed subset of $\omega^*$ is a $P$-set iff it is the fixed-point set of an autohomeomorphism.
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(even an involution) of \( \omega^* \). This gives an external characterization of the \( P \)-sets in \( \omega^* \) and is a partial answer to Problem 218 of Hart and van Mill [4]. We finish the paper with an example of an infinite-dimensional compact \( F \)-space and an involution on it whose fixed-point set is not a \( P \)-set.

1. Preliminaries

By convention all spaces under consideration are completely regular. We call—as usual—a space \( X \) an \( F \)-space if every cozero set in it is \( C^* \)-embedded, i.e., if \( M \) is a cozero set of \( X \) and \( f : M \to \mathbb{R} \) is a bounded continuous function, then \( f \) can be extended to a bounded continuous function from \( X \) to \( \mathbb{R} \). For compact spaces this takes the following convenient form: A compact space \( X \) is an \( F \)-space iff for every \( F_\sigma \)-subset \( F \) of \( X \) the equality \( \cl F = \beta F \) holds. A rich supply of compact \( F \)-spaces can be gotten from the well-known fact that \( \beta X \setminus X \) is an \( F \)-space whenever \( X \) is \( \sigma \)-compact and locally compact.

We also need the characterization of \( \omega^* \) given by Parovičenko in [5]. This characterization is valid under the assumption of the Continuum Hypothesis (CH).

**Theorem 1.1 (CH).** A compact space \( X \) is homeomorphic to \( \omega^* \) if and only if it is a compact, zero-dimensional \( F \)-space of weight \( \mathfrak{c} \) without isolated points in which nonempty \( G_\delta \)-sets have nonempty interiors.

This theorem is particularly useful when one works with \( P \)-sets in \( \omega^* \); we recall that a subset of a space is a \( P \)-set if every \( G_\delta \)-set containing it is a neighbourhood of it or, equivalently, a set \( A \) is a \( P \)-set if for every \( F_\sigma \)-set \( F \) disjoint from it one has \( A \cap \cl F = \emptyset \).

For example, in the proof of Lemma 1.3 below we use the fact that \( \omega^* \setminus \Int A \) is homeomorphic to \( \omega^* \) whenever \( A \) is a \( P \)-set of \( \omega^* \). A second application occurs in the proof of Theorem 2.2.

From van Douwen and van Mill [2] we quote the following theorem, the homeomorphism extension theorem for nowhere dense \( P \)-sets.

**Theorem 1.2 (CH).** Let \( A \) and \( B \) be nowhere dense \( P \)-sets of \( \omega^* \) and \( h : A \to B \) a homeomorphism. Then there is an autohomeomorphism \( \hat{h} \) of \( \omega^* \) that extends \( h \).

We shall need the following mild extension of this theorem.

**Lemma 1.3 (CH).** Let \( A \) and \( B \) be proper \( P \)-subsets of \( \omega^* \), and let \( h : A \to B \) be a homeomorphism that maps the interior of \( A \) onto the interior of \( B \). Then there is an autohomeomorphism \( \hat{h} \) of \( \omega^* \) that extends \( h \).

**Proof.** Consider \( \omega^* \setminus \Int A \) and \( \omega^* \setminus \Int B \). As noted above both spaces are homeomorphic to \( \omega^* \) because \( A \) and \( B \) are \( P \)-sets.

The homeomorphism extension theorem for nowhere dense \( P \)-sets now gives us an extension \( h' : \omega^* \setminus \Int A \to \omega^* \setminus \Int B \) of the restriction \( h \mid \Fr A \). To finish we let \( \hat{h} = h \cup h' \).

The final result that we need is from van Douwen [1]. We use the term ‘finite-dimensional’ in the sense of the covering dimension \( \dim \).
**Theorem 1.4.** Let $X$ be a finite-dimensional paracompact space and $f : X \to X$ a closed self-map for which there is a natural number $k$ such that $|f^{-1}(x)| \leq k$ for all $x \in X$. Then $f$ has a fixed point if and only if $\beta f$ has a fixed point.

2. **Finite-dimensional spaces**

We get our first result by a judicious application of van Douwen's theorem.

**Theorem 2.1.** Let $X$ be a compact finite-dimensional $F$-space and $\phi : X \to X$ a continuous and injective map. The fixed-point set $F$ of $\phi$ is a $P$-set of $X$.

**Proof.** Let $K$ be an $F_\sigma$-subset of $X$ that is disjoint from $F$. We must show that $\text{cl} K$ is disjoint from $F$. To this end we take the set $L = \bigcup_{k \in \mathbb{Z}} \phi^k[K]$. Observe that $L$ is also an $F_\sigma$-set that is disjoint from $F$; that $L$ is an $F_\sigma$-set is clear. To see that $L$ contains no fixed points of $\phi$ combine the facts that $K$ contains none and that $\phi$ is injective. It is also clear that $\phi[L] \subseteq L$. Finally we observe that $\phi \upharpoonright L$ is closed: use the fact that $\phi^{-1}[L] = L$.

Now, because $X$ is an $F$-space, we have $\text{cl} L = \beta L$. Then van Douwen's theorem implies that $\text{cl} L$ contains no fixed points of $\phi$ either. It follows that $\text{cl} L \cap F = \emptyset$, so certainly $\text{cl} K \cap F = \emptyset$. \(\square\)

For the space $\omega^*$ we can reverse the implication, provided we assume CH.

**Theorem 2.2 (CH).** A closed subset $A$ of $\omega^*$ is a $P$-set iff it is the fixed-point set of some autohomeomorphism of $\omega^*$.

**Proof.** Let $A$ be a $P$-set of $\omega^*$. We shall find an autohomeomorphism $\phi$ of $\omega^*$ of which $A$ is the fixed-point set; indeed, $\phi$ will be an involution, i.e., $\phi^2$ is the identity.

Consider $\omega^* \times 2$ and identify, for every $x \in A$, the points $(x, 0)$ and $(x, 1)$ (we glue the two copies of $\omega^*$ together along the copies of $A$). Because $A$ is a $P$-set, the resulting quotient space $Q$ is homeomorphic to $\omega^*$: it satisfies the conditions from Parovičenko's theorem.

Define an autohomeomorphism $\psi$ of $Q$ by sending $(x, 0)$ to $(x, 1 - i)$ for every $x$. Clearly $\psi^2$ is the identity and the copy $A_Q$ of $A$ in $Q$ is the fixed-point set of $\psi$.

It remains to turn $\psi$ into an autohomeomorphism of $\omega^*$ whose fixed-point set is $A$ itself.

The identity $\text{Id} : A_Q \to A$ is a homeomorphism that maps the interior of $A_Q$ onto the interior of $A$ and so by Lemma 1.3 it may be extended to a homeomorphism $h : Q \to \omega^*$. In the end we take $\phi = h \circ \psi \circ h^{-1}$ of course. \(\square\)

3. **Infinite-dimensional spaces**

In this section we give an example of compact infinite-dimensional $F$-space $X$ and an autohomeomorphism $\phi$ of $X$ whose fixed-point set is not a $P$-set. Again $\phi$ can be taken to be an involution.

Our starting point is the following example, considered by van Douwen in [1]. Let $S = \bigoplus_n S^n$, where $S^n$ is the standard $n$-sphere. Next let $\phi : S \to S$ be the sum of the antipodal mappings. Now $\phi$ has no fixed points, yet $\beta \phi$ does have fixed points; this can be seen as follows: if $\beta \phi$ would have no fixed points, then there would be a finite closed cover $\{F_1, \ldots, F_n\}$ of $\beta S$ such
that $\beta\phi[F_i] \cap F_i = \emptyset$ for all $i$. However, the Lusternik-Schnirelman-Borsuk Theorem (Dugundji and Granas [3, Theorem 4.4]) implies that $\phi[F_i] \cap F_i \cap S^n \neq \emptyset$ for some $i$.

To begin we take for every $n$ the closed $n$-ball $B^n$. Remove the origin and call the result $X_n$. The antipodal map $e_n$ on $X_n$ has no fixed points and, as $\text{dim } X_n = n$, neither do $\beta e_n$ and $e_n^* = \beta e_n \mid X_n^*$ (apply Theorem 1.4). Also note that $X_n^*$ is an $F$-space.

Write $X = \bigoplus_n \beta X_n$ and $e = \bigoplus_n \beta e_n$. The map $e$ has no fixed points but $\beta e$ has many of them: for any sequence $(S_n)_n$ of spheres centered at the origins of the $B^n$ we get fixed points of $\beta e$ in the closure of $\bigoplus_n S_n$.

Now take any neighbourhood of $(\bigoplus_n X_n^*)^*$ in $\beta X$; it contains a tail of a sequence of spheres as in the preceding paragraph and hence a fixed point of $\beta e$. But then $(\bigoplus_n X_n^*)^*$ contains fixed points of $\beta e$ as well.

Our example is the closure of $\bigoplus_n X_n^*$ in $\beta X$, and the map $\phi$ is the restriction of $\beta e$. It is clearly an $F$-space, and the (nonempty) fixed-point set of $\phi$ is contained in the nowhere dense $G_\delta$-set $(\bigoplus_n X_n^*)^*$.

References