ON THE APPROXIMATION OF FIXED POINTS FOR LOCALLY PSEUDO-CONTRACTIVE MAPPINGS

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Abstract. Let $X$ and its dual $X^*$ be uniformly convex Banach spaces, $D$ an open and bounded subset of $X$, $T$ a continuous and pseudo-contractive mapping defined on $\text{cl}(D)$ and taking values in $X$. If $T$ satisfies the following condition: there exists $z \in D$ such that $\|z - Tz\| < \|x - Tx\|$ for all $x$ on the boundary of $D$, then the trajectory $t \mapsto z_t \in D$, $t \in [0, 1)$, defined by $z_t = tT(z_t) + (1 - t)z$ is continuous and converges strongly to a fixed point of $T$ as $t \to 1^-$.

1. Introduction

Let $X$ be a real Banach space, and let $D$ be a subset of $X$. An operator $T : D \to X$ is said to be $k$-pseudo-contractive ($k > 0$) (see [9]) if for each $x, y \in D$ and $\lambda > k$

$$\lambda - k \|x - y\| \leq \|\lambda(x - y) - (Tx - Ty)\|.$$  

For $k = 1$ ($k < 1$) such mappings are said to be pseudo-contractive (respectively, strongly pseudo-contractive). By letting $r = 1/(\lambda - 1)$ and $k = 1$ in (1), we may derive the original definition of pseudo-contractive mappings, due to Browder [1], as follows:

$$\|x - y\| \leq \|(1 + r)(x - y) - r(Tx - Ty)\|$$

holds for all $x, y \in D$ and all $r > 0$. However, by taking a semi-inner product approach (see also Kato [6]) we may describe (2) by

$$\langle Tx - Ty, j \rangle \leq \|x - y\|^2$$

for some $j \in J(x - y)$. The mapping $J : X \to 2^{X^*}$ is called the normalized duality mapping which is defined by

$$J(x) = \{j \in X^* : \langle x, j \rangle = \|x\|^2, \|j\| = \|x\|\}.$$  

Here $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We should mention that this latter family of mappings is intimately related to the so-called accretive operators, which play an important role in the theory of evolution equations.
Furthermore, if condition (1) holds locally, i.e., if each point $x \in D$ has a neighborhood $U$ such that the restriction of $T$ to $U$ is $k$-pseudo-contractive with (uniform) constant $k$, then $T$ is said to be locally $k$-pseudo-contractive.

The purpose of this paper is to continue the discussion concerning the strong convergence of the path $t \to z_t$, $0 \leq t < 1$, defined by (4). In fact, we prove for locally pseudo-contractive mappings under condition (3) that the strong $\lim_{t \to 1^-} z_t$ exists and is a fixed point of $T$. We should also mention that this result appears to be new even in Hilbert spaces. The first results of this nature were established by Browder [2] and Browder and Petryshyn [3], and more recently Bruck et al. [4] proved Theorem 1 for locally nonexpansive mappings. Another result, perhaps more revealing, is Proposition 2 where we prove that the mapping $2I - T$ is globally one-to-one. This fact, by itself, appears to have a significant connotation in the theory of locally pseudo-contractive mappings.

To fix our notation, we will denote the closure and boundary of $D$ by $\overline{D}$ and $\partial D$ respectively, and for $u, v \in X$ we use $\text{seg}[u, v]$ to denote the segment $\{tu + (1 - t)v : t \in [0, 1]\}$. We will also use $B(x; r)$ and $\overline{B}(x; r)$ to stand for the open ball $\{z \in X : \|x - z\| < r\}$ and the closed ball $\{z \in X : \|x - z\| \leq r\}$ respectively. We denote the distance between the sets $A$ and $B$ by $\text{dist}(A, B)$, i.e.,

$$\text{dist}(A, B) = \inf\{\|a - b\| : a \in A, b \in B\}.$$  

II. Preliminaries

The main objective of this paper is to extend Theorem 1 of Morales [11]. We begin by stating this result under the following proposition.

**Proposition 1** ([11]). Let $X$ and $X^*$ be uniformly convex Banach spaces, let $D$ be a bounded open subset of $X$, and let $T : \overline{D} \to X$ be a uniformly continuous mapping which is locally pseudo-contractive on $D$. Suppose there exists $z \in D$ such that

(3) $\|z - Tz\| < \|x - Tx\| \quad \text{for all } x \in \partial D.$

Then there exists a unique path $t \to z_t \in D$, $t \in [0, 1)$, satisfying

(4) $z_t = tT(z_t) + (1 - t)z$,

where the strong $\lim_{t \to 1^-} z_t$ exists, and this limit is a fixed point for $T$.

As a consequence of the proof of this previous result, the following can easily be derived.

**Corollary 1.** Let $X$ and $X^*$ be uniformly convex Banach spaces, let $D$ be a bounded open subset of $X$, and let $T : \overline{D} \to X$ be a continuous mapping which is locally pseudo-contractive on $D$. Suppose there exists $z \in D$ such that (3) holds. Then there exists a unique path $t \to z_t \in D$, $t \in [0, 1)$, satisfying (4). If, in addition, this path $\{z_t : 0 \leq t < 1\}$ satisfies

(*) $\text{dist}(\{z_t\}, \partial D) > 0$,

then the strong $\lim_{t \to 1^-} z_t$ exists, and this limit is a fixed point of $T$.

In view of Corollary 1, we should observe that uniform continuity is essential to prove condition (*). On the other hand, due to the well-known fact that
every locally nonexpansive mapping is globally nonexpansive on convex sets, condition (*) can also be shown (see [7]). This fact allows us to derive the following special case, which was obtained earlier by Brück et al. [4].

**Corollary 2.** Let $X$ and $X^*$ be uniformly convex Banach spaces, let $D$ be an open subset of $X$, and let $T: D \rightarrow X$ be a continuous mapping which is locally nonexpansive on $D$. Suppose (3) holds for some $z \in D$. Then there exists a unique path $t \rightarrow z_t \in D$, $t \in [0, 1)$, satisfying (4). If, in addition, this path \{z_t : 0 \leq t < 1\} is bounded, then the strong limit $\lim_{t \rightarrow 1^-} z_t$ exists, and this limit is a fixed point of $T$.

### III. Main result

Now we are ready to extend Proposition 1 by replacing the uniform continuity by mere continuity. In fact, this process will take place by reformulating the original problem into a problem involving locally nonexpansive mappings. However, using this argument, we will lose some properties which will not allow us to use the result of [11].

**Theorem 1.** Let $X$ and $X^*$ be uniformly convex Banach spaces, let $D$ be a bounded open set of $X$, and let $T: D \rightarrow X$ be a continuous mapping which is locally pseudo-contractive on $D$. Suppose there exists $z \in D$ such that

$$\|z - Tz\| < \|x - Tx\| \quad \text{for all } x \in \partial D.$$  

Then there exists a unique path $t \rightarrow z_t \in D$, $t \in [0, 1)$, satisfying

$$z_t = tTz_t + (1 - t)z,$$

where the strong $\lim_{t \rightarrow 1^-} z_t$ exists, and this limit is a fixed point of $T$.

Before proving this result, we need to introduce some basic facts that will be used in the proof of the main theorem. We begin with a lemma, whose proof can be found in Kirk [7]. It might be worthwhile to mention that the existence of the path $t \rightarrow z_t$ was previously established by Kirk and Morales [8] for general Banach spaces. Therefore, it is the strong convergence of this path which is actually at stake.

**Lemma 1** (cf. Fact II of [7]). Let $D$ be an open subset of a Banach space $X$, and suppose $F: D \rightarrow X$ is a continuous mapping which is locally strongly accretive on $D$. Let $u = F(x)$, $x \in D$, and let $S = \text{seg}[u, v]$ such that $\text{seg}[u, v] \subset F(D)$ for some $v \in X$. Then $v \in F(D)$ and there exists a unique path (up to parametrization) whose image $Y$ begins at $x$, ends at a point $w \in F^{-1}(v)$, and for which $F(Y) = S$. Moreover, the inverse of the restriction of $F$ to $\Gamma$ is a Lipschitz mapping of $S$ onto $Y$.

Our next lemma can also be found in Kirk [7]. For complete details of its proof, see pages 94 and 97 of [7] respectively.

**Lemma 2.** Let $X$ be a Banach space, and let $T: D \rightarrow X$ be as in Theorem 1, satisfying (3) for some $z \in D$. Suppose $Fx = 2x - Tx$. Then:

(i) $\text{seg}[z, Fz] \subset F(D)$.

(ii) Let $x \in D$ such that $\|x - Tx\| < \rho = \|z - Tz\|/3$.

Then $B(Fx; \rho) \subset F(D)$.
Proposition 2. Let $X$ be a Banach space, let $D$ be a connected open subset of $X$, and let $T : D \to X$ be a continuous mapping which is locally pseudo-contractive on $D$. Then the mapping $Fx = 2x - Tx$ is globally one-to-one on $D$.

Proof. We first observe that $F$ is continuous on $\overline{D}$ and locally strongly accretive on $D$ and, thus, locally expansive on $D$. This means, for each $x \in D$, there exists a neighborhood $U$ such that for $u, v \in U$

$$
\|u - v\| \leq \|Fu - Fv\|.
$$

Also, as a consequence of Deimling's domain invariance theorem [5, Theorem 3], $F$ maps open subsets of $D$ onto open sets of $X$. Now we are ready to show that $F$ is a one-to-one mapping on $D$. To see this, let $y \in F(D)$. Since $F(D)$ is open, there exists $v \in F(D)$ so that $\text{seg}[v, y] \subset F(D)$. Choose $u \in D$ such that $v = Fu$. Then, by Lemma 1, there exists a unique path $\gamma : [0, 1] \to D$ so that $\gamma(0) = u$, $\gamma(1) = x$ for some $x \in F^{-1}(y)$, and for which $F(\gamma([0, 1])) = \text{seg}[v, y]$. Suppose there is $x_1 \in D$ such that $x_1 \neq x$ and $x_1 \in F^{-1}(y)$. Let $B(x_1; \eta) \subset D$ for some $\eta > 0$. Then there exists $\varepsilon > 0$ for which $B(y; \varepsilon) \subset F(B(x_1; \eta))$. On the other hand, due to the continuity of $F$ at $x$, there exists $\delta > 0$ such that $F(B(x; \delta)) \subset B(y; \varepsilon)$. Now, if we consider the restriction of $F$ to $\tilde{D} = D\setminus B(x; \delta/2)$, it follows that $\text{seg}[v, y] \subset F(\tilde{D})$.

Once again, there exists a (unique) path $\gamma_1 : [0, 1] \to \tilde{D}$ such that $\gamma_1(0) = u$, $\gamma_1(1) = x_2$ for some $x_2 \neq x$, and for which $F(\gamma_1([0, 1])) = \text{seg}[v, y]$. This contradicts the uniqueness of the path $\gamma$. This implies $F^{-1}(y) = \{x\}$, and hence $F$ is a homeomorphism from $D$ onto $F(D)$.

Proof of Theorem 1. In view of Proposition 2, the mapping $F$ is not necessarily one-to-one on $\overline{D}$. Therefore, we will redefine the domain of $T$ to assure that $F$ is also invertible on the boundary of its domain. Due to Theorem 2 of [10], we may select $w \in D$ such that

$$
\|w - Tw\| < \|z - Tz\|.
$$

We now replace $D$ by the open set $D_0$ defined by

$$
D_0 = \{x \in D : \|x - Tx\| < \|z - Tz\|\}.
$$

Then $\partial D_0 \subset D$ and

$$
\|w - Tw\| < \|x - Tx\| \quad \text{for} \ x \in \partial D_0.
$$

This means the path $t \mapsto w_t$ for which $w$ satisfies (5) exists and is uniquely defined on $[0, 1)$ (see Lemma 3 of [11]). By Lemma 2(i), we know that $\text{seg}[w, Fw] \subset F(D_0)$, and since by Proposition 2 $F^{-1}$ exists on $F(D_0)$, we derive that $F^{-1}$ is nonexpansive on $\text{seg}[w, Fw]$ and

$$
\|w - F^{-1}(w)\| \leq \|w - F(w)\| < \|x - Tx\|
$$

for all $x \in \partial D_0$. Due to the fact $\partial F(D_0) = F(\partial D_0)$, we may say that for each $y \in \partial F(D_0)$, there exists $x \in \partial D_0$ such that $y = Fx$ and

$$
\|w - F^{-1}(w)\| < \|y - F^{-1}(y)\|.
$$
Consequently, by Corollary 2, there exists a unique path \( t \to u_t \in F(D_0) \), \( t \in [0, 1) \), satisfying the equation

\[
u_t = tF^{-1}(u_t) + (1-t)w
\]

where the \( \lim_{t \to 1^-} u_t \) exists and is a fixed point of \( F^{-1} \). Due to uniqueness of the path \( t \to w_t \), \( F^{-1}(u_t) = w_t \) where \( s = 1/(2-t) \), and hence the strong \( \lim_{t \to 1^-} w_t \) exists. Since this limit exists for every \( w \in D_0 \) that satisfies (5), we choose a sequence \( \{z^n\} \) in \( D_0 \) such that \( z^n \to z \). For each \( z^n \), the corresponding path can be written as

\[
z^n_t = tT(z^n_t) + (1-t)z^n, \quad t \in [0, 1].
\]

Let \( \eta > 0 \) such that \( B(z; \eta) \subset D \), and let \( k \in \mathbb{N} \) such that \( z^n \in B(z; \eta/4) \) for all \( n \geq k \). From (6) and the fact that each \( z^n \in D_0 \), we have

\[
\|z^n_t - z^n\| = \frac{t}{1-t}\|z^n - T(z^n_t)\| \leq \frac{t}{1-t}\|z - Tz\|.
\]

Then there exists \( t_0 \in (0, 1) \) for which \( \|z^n_t - z^n\| < \eta/4 \) for \( t \in [0, t_0] \) and \( n \in \mathbb{N} \). This implies that \( z^n_t \in B(z; \eta/2) \) for all \( t \in [0, t_0] \) and for all \( n \geq k \). Since \( T \) is globally pseudo-contractive on \( B(z; \eta) \), there exists \( j \in J(z^n_t - z^n) \) so that

\[
\langle z^n_t - z^n_m, j \rangle = t\langle Tz^n_t -TZ^n_m, j \rangle + (1-t)\langle z^n - z^n_m, j \rangle
\]

\[
\leq t\|z^n_t - z^n_m\|^2 + (1-t)\|z^n - z^n_m\|, \quad n, m \geq k \text{ and } t \in [0, t_0].
\]

This means the sequence \( \{z^n_t\}_{n=1}^{\infty} \) is a convergent sequence for each \( t \in [0, t_0] \), say, \( \lim_{n \to \infty} z^n_t = \hat{z}_t \). Then we may obtain that

\[
\|z^n_t - z^n\| \leq \|z^n - z^n_m\| \quad \text{for all } m, n \geq k \text{ and } t \in [0, t_0].
\]

Once again due to uniqueness of the path, \( \hat{z}_t = z_t \) for all \( t \) for which \( \{z^n_t\} \) is convergent. We now define the set

\[
E = \{s \in [0, 1]: \|z^n_t - z^n_m\| \leq \|z^n - z^n_m\| \text{ for all } t \in [0, s], n, m \geq n_s, \text{ for some } n_s \in \mathbb{N} \}.
\]

Since \( t_0 \in E \) and \( z_{t_0} \in D \), there exist \( \delta > 0 \) and \( l \in \mathbb{N} \) such that \( B(z_{t_0}; \delta) \subset D \) and

\[
\|z^n_{t_0} - z_{t_0}\| < \delta/5 \quad \text{for all } n \geq l.
\]

Hence there exists \( \alpha > 0 \) for which

\[
z^n_t \in B(z_{t_0}; \delta/4) \quad \text{for all } t \in (t_0 - \alpha, t_0 + \alpha).
\]

This implies that \( \|z^n_t - z_{t_0}\| < \delta/2 \) for all \( n \geq l \) and \( t \in (t_0 - \alpha, t_0 + \alpha) \). Otherwise, we may choose \( j \geq l \) and \( t_1 \in (t_0 - \alpha, t_0 + \alpha) \) so that \( \|z^n_t - z_{t_0}\| = \delta/2 \). This implies \( z^n_{t_1}, z^n_{t_1} \in B(t_{t_0}; \delta) \), and since \( T \) is pseudo-contractive on \( B(z_{t_0}; \delta) \), we obtain

\[
\|z^n_{t_1} - z^n_{t_1}\| \leq \|z^n - z^n\| < \delta/5.
\]

This is a contradiction, since \( \|z^n_{t_1} - z^n\| \geq \delta/4 \). Therefore, \( (t_0 - \alpha, t_0 + \alpha) \subset E \).

Due to the continuity of the path \( t \to z^n_t \), we deduce that \( t_0 + \alpha \in E \).
means \([0, 1) \subseteq E\). It remains to show that \(1 \in E\). To see this, we first mention that

\[
\|z^n_t - T(z^n_t)\| = \left(\frac{1}{t} - 1\right) \|z^n - z^n_t\|.
\]

Since \(D\) is bounded, there exists \(s \in (0, 1)\) such that

\[
\|z^n_t - T(z^n_t)\| \leq \rho \quad \text{for all} \quad n \geq 1 \quad \text{and} \quad t \in [s, 1].
\]

Therefore, by Lemma 2(ii), \(B(F(z^n_t); \rho) \subseteq F(D)\). We choose \(k \in \mathbb{N}\) such that \(k > n_s\) and \(\|z^n - z^m\| < \rho\) for all \(n, m \geq k\). Then

\[
\|F(z^n_m) - F(z^n_t)\| \leq \|z^n - z^m\| \quad \text{for} \quad n, m \geq k \quad \text{and} \quad t \in [s, 1].
\]

Otherwise, we may find \(i, j \geq k\) and \(r \in (s, 1)\) such that \(\|F(z^n_i) - F(z^n_j)\| > \|z^n_i - z^n_j\|\), while \(\|F(z^n_i) - F(z^n_j)\| \leq \|z^n_i - z^n_j\|\). Then there exists \(t \in [s, r]\) such that

\[
\|z^n_i - z^n_j\| < \|F(z^n_i) - F(z^n_j)\| < \rho.
\]

Hence \(\text{seg}[F(z^n_i), F(z^n_j)] \subseteq F(D)\), and thus \(\|z^n_i - z^n_j\| \leq \|F(z^n_i) - F(z^n_j)\|\). Since

\[
F(z^n_i) - F(z^n_j) = (2 - \frac{1}{i})(z^n_i - z^n_j) + (\frac{1}{i} - 1)(z^n_i - z^n_j),
\]

we derive that \(\|F(z^n_i) - F(z^n_j)\| \leq \|z^n_i - z^n_j\|\). This contradicts (7). Therefore,

\[
\|z^n_i - z^n_j\| \leq \|z^n - z^m\| \quad \text{for} \quad n, m \geq k \quad \text{and} \quad t \in [s, 1].
\]

This implies \(1 \in E\), and hence \(E = [0, 1]\). This means there exists \(n_0 \in \mathbb{N}\) so that

\[
\|z^n_t - z^n_m\| \leq \|z^n - z^m\| \quad \text{for all} \quad n, m \geq n_0 \quad \text{and} \quad t \in [0, 1].
\]

In particular, \(\{z^n_t\}\) is a convergent sequence, say, to \(z_1\). Then for \(\varepsilon > 0\), (8) implies there exists \(n \in \mathbb{N}\) such that \(\|z^n_t - z_1\| < \varepsilon/3\) for all \(t \in [0, 1]\). Also, we may choose \(\delta > 0\) satisfying

\[
\|z^n_t - z_1\| < \varepsilon/3 \quad \text{for all} \quad t \in (1 - \delta, 1].
\]

Therefore,

\[
\|z_t - z_1\| \leq \|z_t - z^n_t\| + \|z^n_t - z^n_m\| + \|z^n_m - z_1\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]

for \(t \in (1 - \delta, 1]\). This completes the proof.

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