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FREE PRODUCT VON NEUMANN ALGEBRAS OF TYPE III

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ABSTRACT. In this paper we will show that most free products of von Neumann algebras with respect to nontracial states produce type III_λ factors $(\lambda \neq 0)$. In addition, for all such λ, examples can be obtained with the component algebras being finite dimensional. Finally, conditions will be given to ensure that these free products will be full factors.

1. INTRODUCTION

The study of free products of operator algebras is a new one that has seen rapid and impressive progress in recent years. This growth was touched off by Voiculescu, who showed that free products are crucial in extending probability theory to the noncommutative setting of operator algebra theory.

Unlike the case for tensor products, free products of von Neumann algebras depend upon a choice of state on each component algebra. Much attention in the literature has been given to free products of von Neumann algebras with respect to traces. Such free products are always of type II_1. We show that outside this setting free product algebras are often of type III. In fact, when we consider nontracial states whose centralizers contain several orthogonal unitaries, the free product will be a type III_λ factor $(\lambda \neq 0)$. If, in addition, these unitaries have a group structure, the free product will be full. This gives a rather natural extension of the well-known result that von Neumann algebras generated by free groups are full.

We also show that free products can be of type III even when the component centralizers are not large. Indeed, it is conjectured that nontrivial free products with respect to nontracial states are always of type III.

Two preprints have recently come to our attention that relate to the subject of this paper. Dykema has found a condition independent of ours that guarantees type III factors. Also Radulescu has shown $M_2 * L^\infty[0, 1]$ is a type III_λ factor and has identified its core using random matrices.

2. PRELIMINARIES

In this paper, we consider the (von Neumann algebraic) free product of two von Neumann algebras with respect to faithful, normal states. The construction

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and notation are taken from Voiculescu [10]. In addition, his paper is a good source for those facts about free product algebras that are stated without proof in this section.

To review, suppose $(\mathcal{M}_1, \phi_1)$ and $(\mathcal{M}_2, \phi_2)$ are von Neumann algebras equipped with faithful, normal states. Then we can use the GNS construction to get $(\mathcal{H}_1, \pi_1, \xi_1)$ and $(\mathcal{H}_2, \pi_2, \xi_2)$ respectively. Notice that because of the faithfulness and normality of the states, we can, and often will, identify $\pi_i(\mathcal{M}_i) \cong \mathcal{M}_i$, $i = 1, 2$.

Define the pointed free product Hilbert space, $(\mathcal{H}, \xi_0) = (\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$, as follows: identify ξ_0 with ξ_1 and ξ_2 (which we assume to be unit vectors), and set

$$\mathcal{H} = \mathbb{C}\xi_0 \oplus \sum_{r \geq 1} \sum_{i_1 \neq i_2 \neq \cdots \neq i_r}^{\oplus} \mathcal{H}_{i_1}^0 \otimes \mathcal{H}_{i_2}^0 \otimes \cdots \otimes \mathcal{H}_{i_r}^0 \tag{1}$$

where $\mathcal{H}_{i_j}^0 = \mathcal{H}_{i_j} \ominus \mathbb{C}\xi_{i_j}$ and each $i_j \in \{1, 2\}$. Notice that $\mathcal{L}(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$ act naturally on $\mathcal{H}$ from the left when we identify $\mathcal{H} \cong \mathcal{H}_i \otimes L_i$ where L_i are those tensors in $\mathcal{H}$ not beginning in $\mathcal{H}_i^0$, $i = 1, 2$. In particular, we get the inclusions $\lambda_i : \mathcal{M}_i \hookrightarrow \mathcal{L}(\mathcal{H})$, $i = 1, 2$.

Now we can define the free product $(\mathcal{M}, \phi) = (\mathcal{M}_1, \phi_1) * (\mathcal{M}_2, \phi_2)$ as follows: $\mathcal{M} = [\lambda_1(\mathcal{M}_1) \cup \lambda_2(\mathcal{M}_2)]''$ and $\phi = \phi_1 * \phi_2 = (\cdot\, \xi_0 | \xi_0)$.

The free product state, ϕ, is normal (being a vector state), and because our construction yields separating and cyclic vectors, ξ_1 and ξ_2, ϕ is faithful.

One very important property that ϕ enjoys is that $\mathcal{M}_1$ and $\mathcal{M}_2$ are free with respect to it in the following sense: $\phi(a_1 a_2 \cdots a_n) = 0$ whenever for $1 \leq i \leq n$, $a_i \in \mathcal{M}_{k_i}^0$, $k_i \in \{1, 2\}$, $k_i \neq k_{i+1}$ $(1 \leq i \leq n-1)$. Here $\mathcal{M}_i^0 = \{x \in \mathcal{M}_i \mid \phi_i(x) = 0\}$, $i = 1, 2$. By linearity of ϕ, we get the following recursive formula for the situation where each $a_i \in \mathcal{M}_{k_i}$ rather than in $\mathcal{M}_{k_i}^0$:

$$\phi(a_1 \cdots a_n) = \sum_{r=1}^{n} \sum_{1 \leq i_1 < \cdots < i_r \leq n} (-1)^{r+1} \phi_{k_{i_1}}(a_{i_1}) \cdots \phi_{k_{i_r}}(a_{i_r}) \phi(a_1 \cdots \check{a}_{i_1} \cdots \check{a}_{i_r} \cdots a_n) \tag{2}$$

where "ˇ" indicates terms that are omitted.

From (2) we can see the well-known fact that ϕ extends ϕ_1 and ϕ_2.

We also observe that the algebraic free product decomposes into

$$\mathcal{M}_0 = \mathbb{C} \oplus \sum_{r \geq 1} \sum_{i_1 \neq i_2 \neq \cdots \neq i_r}^{\oplus} \mathcal{M}_{i_1}^0 \otimes \mathcal{M}_{i_2}^0 \oplus \cdots \oplus \mathcal{M}_{i_r}^0$$

where each tensor product is algebraic. We refer to elements of each summand as words and each summand as a type of word. This is an orthogonal decomposition with respect to ϕ. To see this, apply equation (2) to $\phi(y^* x)$ where x and y are words. Computation shows that the right-hand side of the equation will be zero unless y and x are the same type of word. Explicitly, if $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$ where, for $1 \leq i \leq n$, x_i, $y_i \in \mathcal{M}_{k_i}^0$, $k_i \in \{1, 2\}$, $k_i \neq k_{i+1}$ $(1 \leq i \leq n-1)$, then

$$\phi(y^* x) = \prod_{i=1}^{n} \phi_{k_i}(y_i^* x_i).$$

This can be seen by induction on n when one observes that (2) implies

$$\phi(y_n^* \cdots y_2^* y_1^* x_1 x_2 \cdots x_n) = \phi_{k_1}(y_1^* x_1)\phi(y_n^* \cdots y_2^* x_2 \cdots x_n)$$

for $n \geq 2$.

With this decomposition we can see that whenever $\alpha_i \in \operatorname{Aut}(\mathcal{M}_i)$ leaves ϕ_i invariant $(i = 1, 2)$, there is a unique $\alpha_1 * \alpha_2 \in \operatorname{Aut}(\mathcal{M})$ that extends each a_i. Namely, $\alpha_1 * \alpha_2(x) = \prod_{i=1}^n \alpha_{k_i}(a_i)$ where x is the word $a_1 a_2 \cdots a_n$ with each $a_i \in \mathcal{M}_{k_i}^0$. Notice that $\alpha_1 * \alpha_2(x)$ is a word of the same type as x.

Another well-known property of free products in which we are particularly interested is that ϕ is a trace iff ϕ_1 and ϕ_2 both are. Thus free products with respect to traces give finite algebras. Our goal will be to show that under most circumstances if ϕ is not a trace, $\mathcal{M}$ will not support a trace (even a semifinite one).

Because we are concerned with the type of free product algebras, we identify the modular automorphism group:

Lemma 1. *For* $(\mathcal{M}, \phi) = (\mathcal{M}_1, \phi_1) * (\mathcal{M}_2, \phi_2)$ *we have* $\sigma_t^\phi = \sigma_t^{\phi_1} * \sigma_t^{\phi_2}$.

Proof. For any faithful, normal state, ϕ, on a von Neumann algebra, $\mathcal{M}$, the modular automorphism group, σ_t^ϕ, is uniquely characterized by the KMS conditions:

(a) $\phi \circ \sigma_t^\phi = \phi$.

(b) For all x, $y \in \mathcal{M}$, there exists $F_{x,y}^\phi$—a function holomorphic on the strip,

$$\{z \in \mathbb{C} \mid 0 < \operatorname{Im} z < 1\},$$

and continuous and bounded on the boundary such that for all $t \in \mathbb{R}$:

$$F_{x,y}^\phi(t) = \phi(\sigma_t^\phi(x)y) \quad \text{and} \quad F_{x,y}^\phi(t+i) = \phi(y\sigma_t^\phi(x)).$$

Since condition (a) holds for both ϕ_1 and ϕ_2, each ϕ_i leaves $\mathcal{M}_i^0$ invariant. So $\sigma_t^{\phi_1} * \sigma_t^{\phi_2}$ is well defined. It is easy to see that $\sigma_t^{\phi_1} * \sigma_t^{\phi_2}$ satisfies condition (a).

To show that condition (b) is satisfied as well, first consider x and y, words as above. As was mentioned before, $\sigma_t^{\phi_1} * \sigma_t^{\phi_2}$ preserves the type of a word, so

$$\phi(\sigma_t^{\phi_1} * \sigma_t^{\phi_2}(x)y) = \phi(y\sigma_t^{\phi_1} * \sigma_t^{\phi_2}(x)) = 0,$$

unless x and y^* are the same type of word. Otherwise, say $x = x_1 x_2 \cdots x_n$ and $y = y_n y_{n-1} \cdots y_1$ where, for $1 \leq i \leq n$, x_i, $y_i \in \mathcal{M}_{k_i}^0$, $k_i \in \{1, 2\}$, $k_i \neq k_{i+1}$ $(1 \leq i \leq n-1)$. Then we have already shown that

$$\phi(\sigma_t^{\phi_1} * \sigma_t^{\phi_2}(x)y) = \prod_{i=1}^n \phi_{k_i}(\sigma_t^{\phi_{k_i}}(x_i)y_i).$$

Likewise,

$$\phi(y\sigma_t^{\phi_1} * \sigma_t^{\phi_2}(x)) = \prod_{i=1}^n \phi_{k_i}(y_i\sigma_t^{\phi_{k_i}}(x_i)).$$

Hence the function $F_{x,y}^\phi$ in condition (b) is $\prod_{i=1}^n F_{x_i,y_i}^{\phi_{k_i}}$. This same procedure works if x and y are sums of words as well and thus for any x, $y \in \mathcal{M}_0$. By normality, we can find an $F_{x,y}^\phi$ for general x, $y \in \mathcal{M}$. $\square$

3. Determining the type of a free product

We now give conditions that will lead to free product factors of type III. The main result of this section will be:

Theorem 2. *Suppose* $(\mathcal{M}_1, \phi_1)$ *and* $(\mathcal{M}_2, \phi_2)$ *are two von Neumann algebras equipped with faithful, normal states such that* $\mathcal{M}_{1,\phi_1}$ *contains two mutually orthogonal unitaries (with respect to* ϕ*) and* $\mathcal{M}_{2,\phi_2}$ *contains three mutually orthogonal unitaries; then the free product algebra,* $\mathcal{M}$, *is a factor. If, in addition, either* ϕ_1 *or* ϕ_2 *is nontracial,* $\mathcal{M}$ *is a type* III_λ *factor* $(\lambda \neq 0)$.

In order to prove this result we need the following lemma from Avitzour [1]:

Lemma 3. *Let* $(\mathcal{M}_1, \phi_1)$ *and* $(\mathcal{M}_2, \phi_2)$ *be two von Neumann algebras such that there exist* $a \in \mathcal{U}(\mathcal{M}_1)$ *and* $b, c \in \mathcal{U}(\mathcal{M}_2)$ *satisfying* $\phi_1(a) = \phi_2(b) = \phi_2(c) = \phi_2(b^*c) = 0$, $a \in \mathcal{M}_{1,\phi_1}$, *and* $b \in \mathcal{M}_{2,\phi_2}$; *then for all* $x \in \mathcal{M}_0 = \mathrm{Alg}(\lambda_1(\mathcal{M}_1) \cup \lambda_2(\mathcal{M}_2))$:

$$\phi(x) \in \overline{\mathrm{co}}^{\mathrm{n}}\{uxu^* : u \in G(a, b, c)\}$$

where $G(a, b, c)$ *is the group in* $\mathcal{M}_0$ *generated by* a, b, *and* c *and by* $\overline{\mathrm{co}}^{\mathrm{n}}$ *we mean the norm closure of the convex hull.*

We first note that if u and v are unitaries orthogonal to each other relative to ϕ_1, then u^*v is orthogonal to $u^*u = 1$ and thus $a = u^*v$ satisfies the conditions of Lemma 3. Likewise, three orthogonal unitaries in $\mathcal{M}_{2,\phi_2}$ provide us with b and c. Thus the conditions of Theorem 2 are equivalent to those of the lemma plus the assumption that $c \in \mathcal{M}_{2,\phi_2}$.

Under these assumptions, since $\mathcal{M}_\phi \supset \mathcal{M}_{\phi_1} \vee \mathcal{M}_{\phi_2}$ (which can be seen, for instance, from our characteristic of σ_t^ϕ), $\mathcal{M}_\phi \supset G(a, b, c)$; hence $\mathcal{M}_\phi' \subset G(a, b, c)'$. So by the lemma, for all $x \in \mathcal{M}_\phi' \cap \mathcal{M}_0$, $x = \phi(x) \in \mathbb{C}$. By normality we have the following very important corollary to the lemma:

Corollary 4. *Under the conditions of Theorem 2,* $\mathcal{M}_\phi' \cap \mathcal{M} = \mathbb{C}$.

Corollary 4 will immediately prove our theorem once we review Connes's classification of type III factors [2]. Connes defined the following invariant:

$$S(\mathcal{M}) = \bigcap\{\mathrm{Sp}(\Delta_\phi) \mid \phi \text{ is a faithful, semifinite, normal weight}\}$$

where Δ_ϕ is the modular operator associated with ϕ. He then classified type III factors as follows:

Definition 1. A factor $\mathcal{M}$ is said to be of type III_λ, $0 < \lambda < 1$, if

$$S(\mathcal{M}) = \{\lambda^n | n \in \mathbb{Z}\} \cup \{0\},$$

of type III_0 if

$$S(\mathcal{M}) = \{0, 1\},$$

and of type III_1 if

$$S(\mathcal{M}) = \mathbb{R}_+.$$

Although determining this invariant may seem daunting, Connes also showed that if there is a faithful, semifinite, normal weight, ϕ, such that $\mathcal{M}_\phi$ is a factor, then $S(\mathcal{M}) = \mathrm{Sp}(\Delta_\phi)$.

Proof of Theorem 2. Notice that Corollary 4 immediately implies that $\mathscr{M}$ is a factor. We can indeed say much more: If either ϕ_1 or ϕ_2 is nontracial, then $\mathscr{M}$ is a factor of type III. For suppose that $\mathscr{M}$ were semifinite. Then the modular automorphism group, σ_t^ϕ, would be inner. But it is a general fact that if $\sigma_t^\phi = \operatorname{Ad} u_t$ then for all $t \in \mathbb{R}$, $u_t \in \mathscr{M}_\phi' \cap \mathscr{M}_\phi$. (This can easily be seen by applying the facts: for all $x \in \mathscr{M}_\phi$, $x = \sigma_t^\phi(x) = u_t x u_t^*$, and for all $x \in \mathscr{M}$, $\phi(x) = \phi \circ \sigma_t^\phi(x) = \phi(u_t x u_t^*)$). Hence $\sigma_t^\phi = \mathrm{id}$, i.e., ϕ would have to be a trace. This shows $\mathscr{M}$ is a type III factor when ϕ is not a trace.

Staying with this type III case, we note that by Corollary 4, $\mathscr{M}_\phi$ is a factor; hence $S(\mathscr{M}) = \operatorname{Sp}(\Delta_\phi)$. Since a type III factor Δ_ϕ is an unbounded operator, $\operatorname{Sp}(\Delta_\phi) \neq \{0, 1\}$. So our free product factor, $\mathscr{M}$, cannot be of type III_0. With this we have proven Theorem 2. □

Given a factor of type III_λ, $\lambda \neq 0$, the value of λ can be found by determining another invariant of Connes, the modular period, $T(\mathscr{M})$:

Definition 2. For a von Neumann algebra, $\mathscr{M}$, $T(\mathscr{M}) = \{t \in \mathbb{R} \mid \sigma_t^\phi \in \operatorname{Int}(\mathscr{M})\}$ where ϕ is a faithful, normal, semifinite weight.

We remark that $T(\mathscr{M})$ does not depend on the choice of ϕ by the Connes-Radon-Nikodym Theorem.

Connes also showed that for a factor not of type III_0, $T(\mathscr{M})$ determines the type (assuming the factor has separable predual). In particular,

$$T(\mathscr{M}) = \left\{ \frac{2\pi k}{\ln \lambda} \mid k \in \mathbb{Z} \right\}$$

for type III_λ factors and

$$T(\mathscr{M}) = \{0\}$$

for type III_1 factors.

It turns out that $T(\mathscr{M})$ is a particularly useful invariant for us to work with. For if $t_0 \in T(\mathscr{M})$, then $\sigma_{t_0}^\phi = \operatorname{Ad} u$ with $u \in \mathscr{M}_\phi' \cap \mathscr{M}_\phi$. This provides us with the following corollary to Lemma 3:

Corollary 5. *Under the conditions and notation of Theorem* 2,

$$T(\mathscr{M}) = \{t \in \mathbb{R} \mid \sigma_t^\phi = \mathrm{id}\}$$

and therefore

$$T(\mathscr{M}) = \{t \in \mathbb{R} \mid \sigma_t^{\phi_1} = \mathrm{id}\} \cap \{t \in \mathbb{R} \mid \sigma_t^{\phi_2} = \mathrm{id}\}.$$

Notice that the last assertion in the corollary follows easily from the characterization: $\sigma_t^\phi = \sigma_t^{\phi_1} * \sigma_t^{\phi_2}$ since $\sigma_t^\phi = \mathrm{id}$ iff $\sigma_t^{\phi_1} = \mathrm{id}$ and $\sigma_t^{\phi_2} = \mathrm{id}$.

Proposition 6. *There exist free product factors of type* III_λ *for all* $\lambda \neq 0$.

Proof. Corollary 5 will allow us to construct such factors explicitly. Assuming the conditions of Theorem 2, whenever $\sigma_t^{\phi_1}$ is T_1-periodic and $\sigma_t^{\phi_2}$ is T_2-periodic with $T_1/T_2 \in \mathbb{Q}$, $T(\mathscr{M}) = \{n\alpha \mid n \in \mathbb{Z}\}$ where α is the least common integer multiple of T_1 and T_2. Thus $\mathscr{M}$ would be a type III_λ factor with $\lambda = e^{-2\pi/a}$. If $T_1/T_2 \notin \mathbb{Q}$, then $T(\mathscr{M}) = \{0\}$, so $\mathscr{M}$ would be of type III_1 with ϕ almost-periodic.

Such periodicities are easily manipulated even for the finite matrix algebras, M_n. Consider M_n with the state: $\phi = \tau_n(h\cdot)$ where τ_n is the normalized trace and

$$h = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

is a positive matrix with $\tau_n(h) = 1$. Then $\sigma_t^\phi = \operatorname{Ad} h^{it}$. Thus varying $\lambda_1, \ldots, \lambda_n$ can adjust the periodicity of σ_t^ϕ to any positive real number.

Clearly not all such choices of eigenvalues will produce states that conform to the restrictions of Theorem 2. For instance, take M_2 with any ϕ nontracial. Then there do not exist any unitaries in its centralizer that have value zero. In contrast, (M_2, τ) has three orthogonal unitaries:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By inflating by M_n, we see that $(M_n \otimes M_2, \psi \otimes \tau_2)$ with ψ an arbitrary state, is a $2n \times 2n$ matrix algebra with three orthogonal unitaries in its centralizer. Thus this offers a simple example of matrix algebras with states having arbitrary periodicity and that have sufficient "bulk" to their centralizers. □

We now turn out attention to the centralizer, $\mathscr{M}_\phi$. This will given us important insight into $\mathscr{M}$ even when $\mathscr{M}_1$ and $\mathscr{M}_2$ do not satisfy the conditions of Theorem 2. Unfortunately, we do not have $\mathscr{M}_\phi = (\mathscr{M}_{1,\phi_1}, \phi_1) * (\mathscr{M}_{2,\phi_2}, \phi_2)$ in general because many more elements might be in $\mathscr{M}_\phi$. To see this suppose $(\mathscr{M}_1, \phi_1)$ is such that ϕ_1 has a point spectrum. Then there exists $x \in \mathscr{M}_1$ with $\sigma_t^{\phi_1}(x) = \lambda^{it}x$ for some λ a positive real number. If $\mathscr{M}_{2,\phi_2}$ is nontrivial, there exists $y \in \mathscr{M}_{2,\phi_2}$ with $\phi_2(y) = 0$ and so $xyx^* \in \mathscr{M}_\phi$ (since $\sigma_t^{\phi_1}(x^*) = \lambda^{-it}x^*$) but is not in the free product of the component centralizers. This sort of complications cannot happen when ϕ_1 and ϕ_2 have continuous spectra:

Lemma 7. *Suppose $(\mathscr{M}_1, \phi_1)$ and $(\mathscr{M}_2, \phi_2)$ are von Neumann algebras with faithful, normal states such that $\sigma_t^{\phi_1}$ and $\sigma_t^{\phi_1}$ have continuous spectra; then*

$$\mathscr{M}_\phi = (\mathscr{M}_{1,\phi_1}, \phi_1) * (\mathscr{M}_{2,\phi_2}, \phi_2).$$

Proof. We need only show $\mathscr{M}_\phi \subset (\mathscr{M}_{1,\phi_1}, \phi_1) * (\mathscr{M}_{2,\phi_2}, \phi_2)$. Considering $\mathscr{M}$ represented on the free product space, $\mathscr{H}$, we let $U(t) = \Delta_\phi^{it}$ be the modular operator of ϕ. As a consequence of Lemma 1, $U(t) = U_1(t) * U_2(t)$ where $U_1(t) = \Delta_{\phi_1}^{it}$ and $U_2(t) = \Delta_{\phi_2}^{it}$. This is well defined because each $U_i(t)$ leaves ϕ_i invariant (see [10] for the explicit definition of a free product of operators). As a result of [4],

$$[\mathscr{M}_\phi \xi_0] = \{\xi \in \mathscr{H} \mid U(t)\xi = \xi, \ \forall t \in \mathbb{R}\}.$$

Similarly, $\mathscr{H}_{1,0} = [\mathscr{M}_1\xi_1]$ and $\mathscr{H}_{2,0} = [\mathscr{M}_2\xi_2]$ are the fixed vectors in $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. So our goal is to show that $[\mathscr{M}_\phi\xi_0] \subset (\mathscr{H}_{1,0}, \xi_1) * (\mathscr{H}_{2,0}, \xi_2)$ within $\mathscr{H}$.

We will write for $i = 1, 2$, $\mathscr{H}_i = \mathscr{H}_{i,0} \oplus \mathscr{H}_{i,1}$ and denote $\mathscr{H}_{i,0}^0 = \xi_i^\perp \cap \mathscr{H}_{i,0}$. Recall that $\mathscr{H}$ decomposes as in (1) and $U(t)$ preserves this decomposition.

Thus it suffices to consider ϕ in an alternating tensor product of $\mathcal{H}_1^0$ and $\mathcal{H}_2^0$ and to show $U(t)\xi = \xi$ implies that ξ is in an alternating tensor product of $\mathcal{H}_{1,0}^0$ and $\mathcal{H}_{2,0}^0$. For simplicity we will consider the case $\xi \in \mathcal{H}_1^0 \otimes \mathcal{H}_2^0$ and observe that the same argument will work for any alternating tensor product.

We decompose

$$\mathcal{H}_1^0 \otimes \mathcal{H}_2^0 = (\mathcal{H}_{1,0}^0 \otimes \mathcal{H}_{2,0}^0) \oplus (\mathcal{H}_{1,0}^0 \otimes \mathcal{H}_{2,1}) \oplus (\mathcal{H}_{1,1} \otimes \mathcal{H}_{2,0}^0) \oplus (\mathcal{H}_{1,1} \otimes \mathcal{H}_{2,1}).$$

Consider $f_1(t) = (U_1(t)\eta_1|\eta_1)$ for some fixed $\eta_1 \in \mathcal{H}_1$. Since f_1 is a continuous positive-definite function, by Borcher's theorem $f_1 = \int_{\mathbb{R}} e^{2\pi i p t}\, d\mu_1(p) = \hat{\mu}_1(t)$. Also fixing $\eta_2 \in \mathcal{H}_2$ we get $f_2 = \hat{\mu}_2$. Observe that

$$((U_1(t) \otimes U_2(t))(\eta_1 \otimes \eta_2)|\eta_1 \otimes \eta_2) = f_1(t) = \widehat{\mu_1 * \mu_2}(t).$$

By assumption, $U_i(t)$ has continuous spectrum on $\mathcal{H}_{i,1}$ for $i = 1, 2$. Thus for $\eta_1 \in \mathcal{H}_{1,1}$, μ_1 has no point masses. In other words η_1 is fixed under $U_1(t)$ iff its corresponding measure has a point mass. But a convolution of two measures has a point mass iff both measures do. This shows that for $\xi = \eta_1 \otimes \eta_2$ to be fixed under $U_1(t) \otimes U_2(t)$, $\xi \in \mathcal{H}_{1,0}^0 \otimes \mathcal{H}_{2,0}^0$.

A finite sum, $\sum_{i=1}^n \eta_{1,i} \otimes \eta_{2,i}$ that is fixed under $U_1(t) \otimes U_2(t)$ must also be in $\mathcal{H}_{1,0}^0 \otimes \mathcal{H}_{2,0}^0$ by the same reasoning as above since finite sums of continuous measures are continuous. Finally, for general $\xi \in \mathcal{H}_1^0 \otimes \mathcal{H}_2^0$ approximation by finite sums guarantees our result.

It only remains to note that for tensors of length n, the convolution of n measures has a point mass iff all n measures do. □

As a simple consequence of Lemma 7, consider $(\mathcal{M}_1, \phi_1)$ and $(\mathcal{M}_2, \phi_2)$ where each state is ergodic (i.e., the centralizers are trivial) and has continuous spectrum (and so therefore each algebra is a type III_1 factor). Here $(\mathcal{M}, \phi)$ will have ϕ ergodic; hence, $\mathcal{M}$ is a type III_1 factor (see [4] for examples of algebras with ergodic states). This example demonstrates a case of a type III free product factor which does not satisfy the conditions of Theorem 2 (since the centralizers are trivial).

We summarize this last result along with another that shows instances in which type III free product algebras occur:

Proposition 8. *Assume $(\mathcal{M}_1, \phi_1)$ and $(\mathcal{M}_2, \phi_2)$ are von Neumann algebras with faithful, normal states and $(\mathcal{M}, \phi)$ is their free product.*

(a) *If $\mathcal{M}_1$ is a type* III *algebra, then so is $\mathcal{M}$.*

(b) *If ϕ_1 and ϕ_2 are both ergodic with continuous spectra, then $\mathcal{M}$ is a type* III_1 *factor.*

Proof. For (a), observe that $\sigma_t^\phi(\mathcal{M}_1) = \mathcal{M}_1$. So by Takesaki [7] there exists a normal conditional expectation, $E: \mathcal{M} \to \mathcal{M}_1$. Now since the range of E is of type III, so is $\mathcal{M}$ by Tomiyama [9]. We have already proven statement (b). □

It seems clear that the conditions of Theorem 2 are far from necessary in order to guarantee a type III free product. Our feeling is that it is only a lack of machinery that denies us a more general statement. With this sentiment in mind, we end this section with a conjecture:

Conjecture 9. *Whenever $(\mathcal{M}_1, \phi_1)$ is a von Neumann algebra with a nontracial state, the free product $(\mathcal{M}, \phi) = (\mathcal{M}_1, \phi_1) * (\mathcal{M}_2, \phi_2)$, with $\mathcal{M}_2 \neq \mathbb{C}$ and ϕ_2 arbitrary, is a type* III *algebra.*

It is also possible that in this general setting type III_0 free products might not occur just as in the result of Theorem 2.

4. Fullness of the free product

In this section we look at circumstances in which a free product algebra will be full.

Definition 3. A von Neumann algebra $\mathscr{M}$ is said to be full if $\mathrm{Int}(\mathscr{M}) \subset \mathrm{Aut}(\mathscr{M})$ is closed (in the usual u-topology).

Connes [3] showed that for $\mathscr{M}$ separable, fullness is equivalent to a lack of nontrivial sequences that asymptotically commute with elements of $\mathscr{M}_*$. To make this more precise we will need some definitions. For all that follows we will denote for $x \in \mathscr{M}$ and ϕ, a normal state:

$$\|x\|_\phi = \phi(x^*x)^{1/2},$$
$$\|x\|_\phi^\# = [\tfrac{1}{2}(\phi(x^*) + \phi(xx^*))]^{1/2}.$$

We note that when ϕ is faithful, these norms give the σ-strong and σ^*-strong topologies on bounded subsets of $\mathscr{M}$ respectively.

Definition 4. For a von Neumann algebra, $\mathscr{M}$, with a faithful, normal state, ϕ, a sequence $\{x_n\} \in l^\infty(\mathbb{N}, \mathscr{M})$ is said to be:

(a) *central* if for all $y \in \mathscr{M}$, $[x_n, y]$ converges to zero σ^*-strongly (equivalently $\lim_{n\to\infty} \|[x_n, y]\|_\phi^\# = 0$);
(b) *strongly central* if $\lim_{n\to\infty} \|\omega(x_n \cdot) - \omega(\cdot x_n)\| = 0$ for all $\omega \in \mathscr{M}_*$;
(c) *trivial* if there exists $\{a_n\} \in l^\infty(\mathbb{N}, Z(\mathscr{M}))$ such that $x_n - a_n$ converges to zero σ^*-strongly (equivalently $\lim_{n\to\infty} \|x_n - a_n\|_\phi^\# = 0$).

Now we state Connes's result as:

Proposition 10. *A separable von Neumann algebra is full iff every strongly central sequence is trivial.*

Every strongly central sequence is central, and for $\mathscr{M}$ of type II_1, the central sequences and strongly central sequences coincide [3]. In fact one reason Connes introduced the concept of strongly central sequences was to generalize results of McDuff on central sequences in II_1 algebras [5].

For type II_1 factors, fullness is equivalent to not having property Γ; so $\mathscr{R}$, the AFD II_1 factor, in particular is not full. The simplest examples of full II_1 factors are the group von Neumann algebras, $\mathscr{R}(\mathbb{F}_n)$, i.e., the von Neumann algebra generated by the left regular representation of the free group on n elements $(n \geq 2)$. This result follows from Pukanszky [6, Lemma 10]. He showed that if s_1 and s_2 are two generators in $\mathbb{F}_n$, then

$$\|x - \tau(x)\|_\tau \leq 14 \max\{\|[x, u(s_1)]\|_\tau, \|[x, u(s_2)]\|_\tau\}$$

for all $x \in \mathscr{R}(\mathbb{F}_n)$. Thus all central sequences are trivial. A slight variation on his argument shows $\mathscr{R}(G)$ is full when $G = G_1 * G_2$ with $|G_1| \geq 2$ and $|G_2| \geq 3$. Since by Voiculescu [10] $\mathscr{R}(G) = (\mathscr{R}(G_1), \tau_1) * (\mathscr{R}(G_2), \tau_2)$ where each τ_i is normalized trace, we see that Pukanszky's lemma gives conditions for fullness of free products with respect to trace. We now generalize this result for the nontracial case.

Theorem 11. *Suppose* $(\mathcal{M}_1, \phi_1)$ *and* $(\mathcal{M}_2, \phi_2)$ *have the property that the centralizer of each component algebra,* $\mathcal{M}_{i,\phi_i}$, *contains a discrete group,* G_i, *of orthogonal unitaries,* $i = 1, 2$, *where* $|G_1| \geq 2$, $|G_2| \geq 3$; *then the free product algebra,* $\mathcal{M}$, *is full.*

Proof. Fix $a \in G_1 \backslash \{e_1\}$ and $b, c \in G_2 \backslash \{e_2\}$, $b \neq c$. As in the result of Pukanszky discussed earlier, we will want to show that for all $x \in \mathcal{M}$:

$$\|x - \phi(x)\|_\phi \leq 14 \max\{\|[x, a]\|_\phi, \|[x, b]\|_\phi, \|[x, c]\|_\phi\}. \tag{3}$$

Since $a, b, c \in \mathcal{M}_\phi$, it is easy to check that (3) implies the same inequality with the $\|\cdot\|_\phi$ norm replaced by $\|\cdot\|_\phi^\#$. Thus by proving (3) we will show that all central (and therefore strongly central) sequences are trivial. On the Hilbert space level this inequality can be expressed as

$$\|x\xi_0 - (x\xi_0|\xi_0)\xi_0\| \leq 14 \max\{\|[x, a]\xi_0\|, \|[x, b]\xi_0\|, \|[x, c]\xi_0\|\}.$$

Notice that $xa\xi_0 = Sa^*x^*\xi_0 = Sa^*Sx\xi_0 = J\Delta^{1/2}a^*\Delta^{-1/2}Jx\xi_0$. Since $a \in \mathcal{M}_\phi$, we have $xa\xi_0 = Ja^*Jx\xi_0$. The same applies to b and c. Thus we need to show

$$\|\xi - (\xi|\xi_0)\xi_0\| \leq 14 \max\{\|[\xi, a]\|, \|[\xi, b]\|, \|[\xi, c]\|\} \tag{4}$$

for all $\xi \in \mathcal{H}$ where ξa is the usual right multiplication: $Ja^*J\xi$.

For $i = 1, 2$ define $\mathcal{K}(g) = \mathbb{C}g\xi_i \subset \mathcal{H}_i$ for each $g \in G_i$. Also define $\mathcal{K}(0_i) = (l^2(G_i)\xi_i)^\perp \subset \mathcal{H}_i$. For all $g, g' \in G_i$, $g\mathcal{K}(g') = \mathcal{K}(gg') = \mathcal{K}(g)g'$. Since left and right multiplication by $g \in G_i$ leaves $l^2(G_i)$ invariant, we also have $g\mathcal{K}(0_i) = \mathcal{K}(0_i) = \mathcal{K}(0_i)g$. Thus we define a left and right action of G_i on $\widetilde{G}_i = G_i \cup \{0_i\}$ by multiplication in G_i if $g \in \widetilde{G}_i \backslash \{0_i\}$ and $g'0_i = 0_i g' = 0_i$ for all $g' \in G_i$. Then we have shown for $i = 1, 2$:

$$\mathcal{H}_i = \sum_{g \in \widetilde{G}_i}^{\oplus} \mathcal{K}(g) \tag{5}$$

and $g'\mathcal{K}(g) = \mathcal{K}(g'g)$, $\mathcal{K}(g)g' = \mathcal{K}(gg')$ for all $g' \in G_i$ and $g \in \widetilde{G}_i$.

Now consider $\Phi = \widetilde{G}_1 * \widetilde{G}_2$. By this notation we mean the set-theoretic free product of $\widetilde{G}_1$ and $\widetilde{G}_2$ with the identification $e = e_1 = e_2$:

$$\Phi = \{e\} \cup \{g_1 g_2 \cdots g_n | g_i \in \widetilde{G}_{j_i} \backslash \{e\}, j_i \neq j_{i+1}\}.$$

For such a word, $g_1 g_2 \cdots g_n$, define $\mathcal{K}(g_1 g_2 \cdots g_n) = \mathcal{K}(g_1) \otimes \mathcal{K}(g_2) \otimes \cdots \otimes \mathcal{K}(g_n)$.

Since our free product Hilbert space, $\mathcal{H}$, decomposes as in equation (1) and each $\mathcal{H}_i$ decomposes as in (5), we get

$$\mathcal{H} = \sum_{g \in \Phi}^{\oplus} \mathcal{K}(g)$$

and for $i = 1, 2$ whenever $g' \in G_i$

$$g'\mathcal{K}(g) = \mathcal{K}(g'g) \quad \text{and} \quad \mathcal{K}(g)g' = \mathcal{K}(gg').$$

We are now in a position to mimic Pukanszky's proof of the 14ε lemma. His proof relies on decomposing the Hilbert space, $l^2(G)$, with respect to the elements of G where G is the free product of two groups. Here we decompose with respect to Φ which is a free product of sorts but which does not have a group multiplication.

We have already shown that $\mathrm{Ad}(a)$, $\mathrm{Ad}(b)$, and $\mathrm{Ad}(c)$ take elements of Φ to elements of Φ. Because inner automorphisms are invertible, we have: $a\Phi a^* = \Phi$, $b\Phi b^* = \Phi$, and $c\Phi c^* = \Phi$. Let S be the subset of Φ consisting of those words ending in $\widetilde{G}_1\backslash\{e\}$.

Claim: (a) $S \cup aSa^* = \Phi\backslash\{e\}$.

(b) S, bSb^* and cSc^* are pairwise disjoint.

Proof of Claim: (a) For any $g \in \Phi\backslash\{e\}$ whenever g is a word ending in $\widetilde{G}_2\backslash\{e\}$, $a^*ga \in S$.

(b) Each of these three sets consists of words ending $\widetilde{G}_1\backslash\{e\}$, b^*, and c^* respectively.

With this claim proven we may now proceed to prove inequality (4). From here on, the proof is almost identical to Pukanszky's. We write out the details here for the reader's convenience. We write for any $\xi \in \mathscr{H}$, $\xi = \sum_{g\in\Phi}\xi(g)$ where $\xi(g) \in \mathscr{H}(g)$, and we will denote $c(g) = \|\xi(g)\|$.

We assume for some $\varepsilon > 0$

$$\|a^*\xi a - \xi\| = \|[\xi, a]\| < \varepsilon$$

and

$$\|b^*\xi b - \xi\| = \|[\xi, b]\| < \varepsilon, \qquad \|c^*\xi c - \xi\| = \|[\xi, c]\| < \varepsilon.$$

We have $a^*\xi a = \sum_{s\in\Phi} a^*\xi(s)a$, but $\xi'(a^*sa) = a^*\xi(s)a \in \mathscr{H}(a^*sa)$ and $\|\xi'(a^*sa)\| = \|\xi(s)\|$, i.e., $\|\xi'(g)\| = c(aga^*)$ for all $g \in \Phi$. Hence we have

$$\begin{aligned}\varepsilon^2 > \|a^*\xi a - \xi\|^2 &= \left\|\sum_{g\in\Phi}\xi'(g) - \xi(g)\right\|^2 = \sum_{g\in\Phi}\|\xi'(g) - \xi(g)\|^2 \\ &\geq \sum_{g\in\Phi}|c(aga^*) - c(g)|^2.\end{aligned}$$

So

$$\left(\sum_{g\in\Phi}|c(aga^*) - c(g)|^2\right)^{1/2} < \varepsilon.$$

The same inequality holds for b and c.

For $A \subset \Phi$ define $\mu(A) = \sum_{g\in A} c(g)^2$. Now

$$|\mu(aSa^*)^{1/2} - \mu(S)^{1/2}| \leq \left(\sum_{g\in\Phi}|c(aga^*) - c(g)|^2\right)^{1/2} < \varepsilon.$$

Let $K = \mu(\Phi - \{e\})$. Then

$$|\mu(aSa^*) - \mu(S)| \leq |\mu(aSa^*)^{1/2} + \mu(S)^{1/2}||\mu(aSa^*)^{1/2} - \mu(S)^{1/2}| \leq 2K^{1/2}\varepsilon.$$

So $\mu(aSa^*) \leq \mu(S) + 2K^{1/2}\varepsilon$. But $K \leq \mu(S) + \mu(aSa^*) \leq 2\mu(S) + 2K^{1/2}\varepsilon$. Therefore, $\mu(S) \geq K/2 - K^{1/2}\varepsilon$.

But we also have $|\mu(bSb^*) - \mu(S)| \leq 2K^{1/2}\varepsilon$. Thus $\mu(bSb^*) \geq \mu(S) - 2K^{1/2}\varepsilon \geq K/2 - 3K^{1/2}\varepsilon$. The same holds for c: $\mu(cSc^*) \geq K/2 - 3K^{1/2}\varepsilon$.

Now $K \geq \mu(S) + \mu(bSb^*) + \mu(cSc^*) \geq 3K/2 - 7K^{1/2}\varepsilon$. Solving, we find $14\varepsilon \geq K^{1/2}$. But

$$K = \sum_{g \in \Phi \setminus \{e\}} c(g)^2 = \sum_{g \in \Phi} c(g)^2 - (\xi|\xi_0)^2 = \|\xi - (\xi|\xi_0)\xi_0\|^2 .$$

Thus $\mathscr{M}$ is full. □

As an example of a full free product algebra we recall the example from the proof of Proposition 6:

$$(M_n \otimes M_2,\ \psi_1 \otimes \tau_2) * (M_m \otimes M_2,\ \psi_2 \otimes \tau_2).$$

Since I, u, and v form a group of orthogonal unitaries in M_2 each component algebra above will satisfy the conditions of Theorem 11.

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