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NEW GENERALIZATIONS OF JENSEN'S FUNCTIONAL EQUATION

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ABSTRACT. Let f be an unknown entire function of a complex variable, and let s, t be real variables. We consider Jensen's functional equation

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2},$$

where x, y are complex variables. Replacing x and y by s and it in the above equation and taking the absolute values of the resulting equality one obtains the functional equation

$$\left|f\left(\frac{s+it}{2}\right)\right|=\left|\frac{f(s)+f(it)}{2}\right|.$$

The main purpose of this paper is to solve a new generalization of the above equation.

1. INTRODUCTION

We consider Jensen's functional equation

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2} \tag{1}$$

(cf. [1, pp. 43–49, 139, 145, 286, 302; 2, pp. 142–146, 242–249, 255, 356–357; 3; 12]) where f is an unknown entire function of a complex variable z and x, y are complex variables.

One can prove the following theorem.

Theorem A. *The only entire solution of* (1) *is given by* $f(z) = Az + B$ *where* A, B *are arbitrary complex constants.*

Proof. The proof is clear from operating $\partial^2/\partial y\partial x$ on both sides of (1) and setting $y = x$ in the resulting equality. □

In what follows f denotes an unknown entire function of a complex variable z.

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In this paper we shall give a new generalization of (1). To this end we start with a few preliminary remarks. We consider the following two Cauchy equations (cf. [1, pp. 31–42; 2, pp. 11–24]) and the quadratic equation:

$$f(x+y) = f(x) + f(y) \tag{2}$$

and

$$f(x+y) = f(x)f(y), \tag{3}$$

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \tag{4}$$

where x, y are complex variables.

If we replace x and y by s and it in (2) (3), (4), respectively, where s, t are real variables, and we take the absolute values in the resulting equations, then we obtain the following three functional equations:

$$|f(s+it)| = |f(s) + f(it)| \quad \text{(Robinson's functional equation; cf. [13])},$$

$$|f(s+it)| = |f(s)f(it)| \quad \text{(Hille's functional equation; cf. [10], [11])},$$

where s, t are real variables, and

$$|f(s+it) + f(s-it)| = 2|f(s) + f(it)| \quad \text{(cf. [8])}.$$

In [6], it is proved that Robinson's functional equation is equivalent to the following Hille's functional equation (cf. [10, 11]):

$$|f(s+it)|^2 = |f(s)|^2 + |f(it)|^2,$$

where s, t are real variables.

For some recent generalizations of Hille's functional equation one may refer to [5, 7, 14].

In a similar way, replacing x and y by s and it in (1), where s, t are real variables, and taking the absolute values of the resulting equality yields the functional equation

$$\left| f\left(\frac{s+it}{2}\right) \right| = \left| \frac{f(s) + f(it)}{2} \right|, \tag{5}$$

where s, t are real variables.

In this paper we shall solve the functional equation (5). In fact, we shall prove the following theorem.

Theorem 1. *The only entire solution of* (5) *is given by* $f(z) = Az + B$ *where* A, B *are arbitrary complex constants.*

In [9] the functional equation

$$\left| f\left(\frac{s+it}{2}\right) \right|^2 = \left| \frac{f(s) + f(it)}{2} \right|, \tag{6}$$

where s, t are real variables, was solved and the following theorem was proved.

Theorem B. *The only entire solutions of* (6) *are given by* $f(z) \equiv 0$ *and*

$$f(z) = e^{i\theta}\cos(az),$$

where θ *is an arbitrary real constant and* a *is an arbitrary real or purely imaginary constant.*

In the following we shall obtain a new generalization of (5) and (6), proving the following theorem:

Theorem 2. *The only entire solutions of the functional equations*

$$\left|f\left(\frac{s+it}{2}\right)\right|^m = \left|\frac{f(s)+f(it)}{2}\right|^n, \tag{7}$$

where s, t are real variables and m, n are arbitrarily fixed positive integers, are given as follows:

(i) *If $m = n$, i.e., $m = 1$ and $n = 1$, the only entire solution of* (7) *is given by that in Theorem* 1.

(ii) *If $m = 2n$, i.e., $m = 2$ and $n = 1$, the only entire solutions of* (7) *are given by those in Theorem* B.

(iii) *If $m \neq n$ and $m \neq 2n$, the only entire solutions of* (7) *are complex constants with modulus* 0 *or* 1.

The purpose of this paper is to prove Theorem 2 by using Theorem 1.

2. Proofs of main results

To prove Theorems 1 and 2 we shall apply the following lemmas.

Lemma 1. *If f is an entire function of a complex variable z, then the conjugate function of f denoted by g, i.e., $g(z) = \overline{f(\bar{z})}$, is also an entire function of z.*

Proof. Cf. [4, p. 28]. □

Lemma 2. *Let h, k be entire functions of a complex variable z, and let p, q be arbitrarily fixed positive integers.*

(i) *If $h(z)^p = k(z)^p$ with $h(0) = k(0) = 1$ for all complex z, then $h(z) = k(z)$ for all complex z.*

(ii) *If $p \neq q$ and if $h(z)^p = h(z)^q$ with $h(0) = 1$ for all complex z, then $h(z) = 1$ for all complex z.*

Proof. To prove (i) by the hypothesis that $h(z)^p = k(z)^p$ for all complex z we obtain

$$\prod_{j=0}^{p-1}(h(z) - w_j k(z)) = 0 \tag{8}$$

for all complex z, where $w_j = \exp(2j\pi i/p)$ $(j = 0, 1, 2, \dots, p-1)$.

Since the ring of all entire functions has no divisors of zero, then by (8) for some j_0 $(0 \le j_0 \le p-1)$, it follows that

$$h(z) - w_{j_0}k(z) = 0 \tag{9}$$

for all complex z.

Setting $z = 0$ in (9) and using the hypothesis that $h(0) = k(0) = 1$ yields

$$w_{j_0} = 1. \tag{10}$$

From (9), (10) we have $h(z) = k(z)$ for all complex z.

To prove (ii), by hypothesis we obtain

$$h(z)^p = h(z)^q \tag{11}$$

for all complex z.

Since, by hypothesis, $p \neq q$, we examine two cases.

Case 1: *Let* $p > q$. By (11) we obtain

$$(h(z)^{p-q} - 1)h(z)^q = 0 \tag{12}$$

for all complex z.

Since, by hypothesis, $h(0) = 1$, we have

$$h(z) \not\equiv 0. \tag{13}$$

Since the ring of all entire functions has no divisors of zero, by (13) we can divide both sides of (12) by $h(z)^q$. Hence we have $h(z)^{p-q} = 1$, and therefore

$$h(z)^{p-q} = 1^{p-q} \tag{14}$$

for all complex z.

By (14), $h(0) = 1$, and by Lemma 2(i) we have $h(z) = 1$ for all complex z.

Case 2: *Let* $p < q$. In this case the proof is similar to that of Case 1. □

Remark. The method of proving Theorems 1 and 2 is to introduce the conjugate function of f (cf. Lemma 1) denoted by

$$g(z) = \overline{f(\bar{z})} \tag{15}$$

for all complex z.

In what follows let g denote the conjugate function of f.

Proof of Theorem 1. We introduce the function g which is the conjugate function of f (cf. (15)). By Lemma 1, g is also an entire function of a complex variable z.

Our aim is to prove that $f'(z) = \text{const}$. Squaring both sides of (5) yields

$$4\left|f\left(\frac{s+it}{2}\right)\right|^2 = |f(s) + f(it)|^2. \tag{16}$$

By the formula $|\gamma|^2 = \gamma\bar{\gamma}$ for all complex γ and by (16) we obtain

$$4f\left(\frac{s+it}{2}\right)\overline{f\left(\frac{s+it}{2}\right)} = (f(s) + f(it))(\overline{f(s)} + \overline{f(it)}). \tag{17}$$

Replacing z by $\bar{z}$ in (15) yields

$$\overline{f(z)} = g(\bar{z}) \tag{18}$$

for all complex z.

By (18) we have

$$\begin{cases} \overline{f\left(\frac{s+it}{2}\right)} = g\left(\frac{s-it}{2}\right), \\ \overline{f(s)} = g(s), \\ \overline{f(it)} = g(\overline{it}) = g(-it). \end{cases} \tag{19}$$

Substituting (19) into (17) yields

$$4f\left(\frac{s+it}{2}\right)g\left(\frac{s-it}{2}\right) = (f(s) + f(it))(g(s) + g(-it)) \tag{20}$$

for all real s, t.

Here f, g are entire functions. Hence, by (20) and by the Identity Theorem we have

$$(21)\qquad 4f\left(\frac{x+y}{2}\right)g\left(\frac{x-y}{2}\right)=(f(x)+f(y))(g(x)+g(-y))$$

for all complex x, y.

Replacing x, y by $x+y$, $x-y$ in (21), respectively, yields

$$(22)\qquad 4f(x)g(y)=(f(x+y)+f(x-y))(g(y+x)+g(y-x))$$

for all complex x, y.

We may assume that

$$(23)\qquad f(z)\not\equiv 0.$$

Setting $x=0$ in (22) yields

$$(24)\qquad 2f(0)g(y)=(f(y)+f(-y))g(y)$$

for all complex y.

By (15) and (23) we obtain

$$(25)\qquad g(z)\not\equiv 0.$$

Since the ring of all entire functions has no divisors of zero, by (25) we can divide both sides of (24) by $g(y)$. Hence we have

$$(26)\qquad 2f(0)=f(y)+f(-y)$$

for all complex y.

Differentiating both sides of (26) yields

$$(27)\qquad f'(-y)=f'(y)$$

for all complex y.

Differentiating both sides of (22) with respect to x and setting $x=0$ in the resulting equality yields

$$(28)\qquad 2f'(0)g(y)=(f'(y)+f'(-y))g(y)$$

for all complex y.

Since the ring of all entire functions has no divisors of zero, by (25) we can divide both sides of (28) by $g(y)$. Hence we have

$$(29)\qquad 2f'(0)=f'(y)+f'(-y)$$

for all complex y.

Substituting (27) into (29) yields

$$(30)\qquad f'(y)=f'(0)$$

for all complex y.

By (30) we obtain

$$f(y)=Ay+B$$

for all complex y, where $A=f'(0)$ and B is a complex constant. □

Proof of Theorem 2. We examine three cases.

Case A. If $m=n$, i.e., $m=1$, $n=1$, the proof follows from Theorem 1.

Case B. If $m = 2n$, i.e., $m = 2$, $n = 1$, the proof follows from Theorem B.

Case C. Let $m \neq n$ and $m \neq 2n$.

In this case we introduce the conjugate function g (cf. (15)). By Lemma 1, g is also an entire function of a complex variable z.

Our aim is to prove that f is a complex constant with modulus 0 or 1.

Squaring both sides of (7), using the formula $|\gamma|^2 = \gamma\bar{\gamma}$ for all complex γ, and applying a similar method to that in the proof of Theorem 1 yields

$$4^n(f(x)g(y))^m = ((f(x+y) + f(x-y))(g(y+x) + g(y-x)))^n \tag{31}$$

for all complex x, y.

Setting $s = t = 0$ in (7) and using the hypothesis $m \neq n$ yields $f(0) = 0$ or $|f(0)| = 1$.

We discuss the two subcases of Case C.

Case C_1: *Let* $f(0) = 0$. Setting $z = 0$ in (15) and using $f(0) = 0$ yields

$$g(0) = 0. \tag{32}$$

Setting $y = x$ in (31) and using $f(0) = 0$ and (32) yields

$$4^n(f(x)g(x))^m = (f(2x)g(2x))^n \tag{33}$$

for all complex x.

f and g are entire functions. Therefore, if we set

$$l(x) = f(x)g(x) \tag{34}$$

for all complex x, then l is also an entire function.

Our aim is to prove that $l(x) \equiv 0$. Assume that this is not the case. Then $l(x) \not\equiv 0$. Furthermore, l is an entire function. Hence, l has the power series expansion of the form

$$l(x) = \sum_{m=r}^{+\infty} l_m x^m \tag{35}$$

for all complex x, where $l_r \neq 0$ and r is a positive integer by $l(0) = 0$ which follows from $f(0) = 0$ and (34).

By (33) and (34) we obtain

$$4^n l(x)^m = l(2x)^n \tag{36}$$

for all complex x.

By hypothesis we have $m \neq n$.

We consider two subcases of C_1.

Case C_{11}: *Let* $m < n$.

Substituting (35) into (36) and equating the coefficients of x^{mr} yields $4^n l_r^m = 0$, and so $l_r = 0$, which contradicts the fact that

$$l_r \neq 0. \tag{37}$$

Case C_{12}: *Let* $m > n$. Similarily, as above, we arrive at the contradiction. Hence

$$l(x) \equiv 0. \tag{38}$$

From (15), (34), and (37) we obtain $f(z) = 0$ for all complex z.

Case C_2: *Let* $|f(0)| = 1$. In this case we normalize f. We define θ to be

$$\theta \overset{\text{def}}{=} \arg(f(0)). \tag{39}$$

If we set

$$F(z) \overset{\text{def}}{=} e^{-i\theta} f(z) \tag{40}$$

for all complex z, it follows from (7) and (39) that F satisfies the functional equation

$$\left| F\left(\frac{s+it}{2}\right)\right|^m = \left|\frac{F(s)+F(it)}{2}\right|^n \tag{41}$$

for all real s, t and, by the hypothesis $|f(0)| = 1$ and (38) and (39), that

$$F(0) = 1. \tag{42}$$

Now we introduce the conjugate function of F denoted by G, i.e.,

$$G(z) = \overline{F(\bar{z})} \tag{43}$$

for all complex z.

By a similar method of getting (22) from (15) we get from (41)

$$4^n(F(x)G(y))^m = ((F(x+y)+F(x-y))(G(y+x)+G(y-x))^n \tag{44}$$

for all complex x, y.

Setting $z = 0$ in (43) and using (42) yields

$$G(0) = 1. \tag{45}$$

Our aim is to prove that $F(z) \equiv 1$.

Setting $x = 0$ in (44) and using (42) gives

$$2^n G(y)^m = (F(y)+F(-y))^n G(y)^n \tag{46}$$

for all complex y.

We discuss two subcases of C_2.

Case C_{21}: *Let* $m < n$. By (45) we have

$$G(y) \not\equiv 0. \tag{47}$$

Since the ring of all entire functions has no divisors of zero, by (47) we can divide both sides of (46) by $G(y)^m$. Hence we have

$$2^n = (F(y)+F(-y))^n G(y)^{n-m} \tag{48}$$

for all complex y.

Replacing y by $-y$ in (48) yields

$$2^n = (F(-y)+F(y))^n G(-y)^{n-m}$$

for all complex y.

Dividing (48) by the above equality yields

$$G(-y)^{n-m} = G(y)^{n-m} \tag{49}$$

for all complex y.

Since $G(-y)$, $G(y)$ are entire functions of y with $G(-0) = G(0) = 1$ which follows from (45), by (49), the fact that $n - m$ is a positive integer, and by Lemma 2(i), we obtain

$$G(-y) = G(y) \tag{50}$$

for all complex y.

By (43) and (50) we obtain

$$F(-y) = F(y) \tag{51}$$

for all complex y.

By (48) and (51) we have

$$F(y)^n G(y)^{n-m} = 1 \tag{52}$$

for all complex y.

By the symmetry property of F and G we can interchange F and G in (52). So we obtain

$$G(y)^n F(y)^{n-m} = 1 \tag{53}$$

for all complex y.

Dividing (52) by (53) side by side yields

$$F(y)^m = G(y)^m \tag{54}$$

for all complex y.

Since $F(y)$, $G(y)$ are entire functions of y with $F(0) = G(0) = 1$ which follows from (42), (45), by (54), and by Lemma 2(i), we obtain

$$F(y) = G(y) \tag{55}$$

for all complex y.

By (52) and (54) we have

$$F(y)^{2n-m} = 1^{2n-m} \tag{56}$$

for all complex y.

Since $F(y)$ is an entire function of y with $F(0) = 1$, by (56), the fact that $2n - m$ is a positive integer, and by Lemma 2(i), we have

$$F(y) = 1$$

for all complex y, i.e., $F(z) \equiv 1$.

Case C_{22}: *Let* $m > n$ with $m \neq 2n$. Dividing both sides of (46) by $G(y)^n$ and getting $F(-y) = F(y)$ for all complex y by using a similar method to that in Case C_{21} yields

$$G(y)^{m-n} = F(y)^n \tag{57}$$

for all complex y.

By the symmetry property of F and G we can interchange F and G. Hence, we obtain $F(y)^{m-n} = G(y)^n$, so

$$G(y)^n = F(y)^{m-n} \tag{58}$$

for all complex y.

Multiplying (57) and (58) side by side yields $F(y)^m = G(y)^m$ for all complex y.

Consequently, we obtain

$$F(y) = G(y) \tag{59}$$

for all complex y (cf. (54), (55)).

By (57) and (59) we obtain

$$F(y)^{m-n} = F(y)^n \tag{60}$$

for all complex y.

Since, by hypothesis, $m \neq 2n$, the two positive integers $m-n$, n are different. Furthermore, $F(0) = 1$. Hence, by (60) and Lemma 2(ii) we have $F(y) \equiv 1$, for all complex y, i.e., $F(z) \equiv 1$.

Thus we have established

$$F(z) \equiv 1. \tag{61}$$

By (40) and (61) we obtain $f(z) \equiv e^{i\theta}$, where θ is a real constant.

Consequently, f is a complex constant with modulus 0 or 1. □

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