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INDEFINITE ELLIPTIC BOUNDARY VALUE PROBLEMS ON IRREGULAR DOMAINS

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ABSTRACT. We establish estimates for the remainder term of the asymptotics of the Dirichlet or Neumann eigenvalue problem

$$-\Delta u(x) = \lambda r(x)\, u(x), \quad x \in \Omega \subset \mathbb{R}^n,$$

defined on the bounded open set $\Omega \subset \mathbb{R}^n$; here, the "weight" r is a real-valued function on Ω which is allowed to change sign in Ω and the boundary $\partial\Omega$ is irregular. We even obtain error estimates when the boundary is "fractal".

These results—which extend earlier work of the authors [particularly, J. Fleckinger & M. L. Lapidus, Arch. Rational Mech. Anal. **98** (1987), 329-356; M. L. Lapidus, Trans. Amer. Math. Soc. **325** (1991), 465-529]—are already of interest in the special case of positive weights.

1. INTRODUCTION

In this paper, we study the influence of the irregularity of the weight r and of the boundary $\partial\Omega$ on the asymptotics of the eigenvalues for the following boundary value problem (in its variational formulation):

$$\text{(P)} \qquad -\Delta u = \lambda r u \qquad \text{in } \Omega \subset \mathbb{R}^n,$$

with Dirichlet or Neumann homogeneous boundary conditions.

Here, Ω is a bounded open set and r is a real-valued function on Ω that is allowed to change sign in Ω (in which case it is called an "*indefinite weight function*") and may be discontinuous. Under suitable assumptions on r and Ω ([Pl], [BS], [FF], [L1], [FL1], ...), there exists a countable sequence of positive [resp., negative] eigenvalues tending to $+\infty$ [resp., $-\infty$]; furthermore,

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the number of positive eigenvalues less than λ for the above problem, denoted by $N_i^+(\lambda; r, \Omega)$ (where $i = 0$ [resp., $i = 1$] corresponds to the Dirichlet [resp., Neumann] problem) satisfies

$$(1.1) \quad N_i^+(\lambda; r, \Omega) \sim W(\lambda; r_+, \Omega) := (2\pi)^{-n}\mathscr{B}_n\lambda^{n/2}\int_\Omega r_+^{n/2}, \quad \text{as } \lambda \to +\infty,$$

where $r_+ = \max(r, 0)$ and $\mathscr{B}_n$ is the volume of the unit ball in $\mathbb{R}^n$.

Of course, we also introduce $N_i^-(\lambda; r, \Omega)$, the number of negative eigenvalues larger than $\lambda < 0$.

We study here the asymptotics of the "remainder term":

$$(1.2) \qquad R_i^\pm(\lambda; r, \Omega) := N_i^\pm(\lambda; r, \Omega) - W(\lambda; r_\pm, \Omega)$$

and extend (in the special case of the Dirichlet and Neumann Laplacians) earlier results of the authors. In particular, we obtain estimates for the "remainder term" of (P) valid when $\partial\Omega$ is very irregular and even "fractal". We thereby obtain—as was suggested in [L2, Remark 3.5(c), p. 490]—the counterpart of [FL2, Theorem 2, p. 339] and [L2, Theorems 2.3 and 4.1, pp. 482–483 and pp. 510–511] for the Dirichlet and Neumann Laplacians *with an "indefinite weight function" and on an open set with "rough boundary"*. In the former case [FL2], the weight function r is allowed to be indefinite and discontinuous but the boundary $\partial\Omega$ is assumed to be (relatively) regular, whereas in the latter case [L2], the boundary is allowed to be "fractal" but $r \equiv 1$.

Our results show, in particular, that there is an interesting interplay between the singularities (or the oscillations) of the weight function r and the irregularities of the boundary $\partial\Omega$. (*See* esp. Theorem 2 and Remarks 3 and 4 below.) They are new and of (mathematical or physical) interest even when the weight function r is *positive*. They enable one, for example, to study the vibrations of "drums with fractal boundary" ([L2-4], [LF1,2]) and *variable* mass density. Other significant physical applications include the study of flows through porous media and of the vibrations of cracked bodies.

"Indefinite elliptic problems" (i.e., involving an indefinite weight function) occur naturally by linearization of many semilinear elliptic equations and are of broad interest in applied mathematics, engineering, physics, and biology; for example, transport theory, hydrodynamics, crystal coloration, laser theory, reaction diffusion equations, ... (See, e.g., [FL1,2] and the relevant references therein.) Recent mathematical works on these problems include [Be], [BS], [Fa1-3], [FF], [FL1-3], [He1-2], [HeKa], [KKZ], [KZ], [L1], [We].

The rest of this paper is organized as follows:

In Section 2, we consider the Dirichlet problem studied in [FL2, Section 3]. There we assume only that $\Omega_\pm^o$, the interior of $\Omega_\pm := \{x \in \Omega / \ r(x) \gtrless 0\}$, is "Jordan contented" (without the hypothesis of "segment property") and recover and strengthen the asymptotic estimate of [FL2, Theorem 2, p. 339].

In Section 3, by use of "Whitney-type coverings" (with dyadic cubes which become smaller near the boundary), we obtain a result for the Dirichlet problem without any condition on Ω. For the Neumann problem, we recover Theorem 2 of [FL2] (where an hypothesis was not made precise enough, as mentioned in [L2, Remark 3.5(d), p. 490] and [FL3], under some of the same conditions on Ω as in [FM, Mt1,2] where $r \equiv 1$.

The case when $\partial\Omega$ is "fractal" is also studied in Section 3 and hence, in the special case of the Dirichlet and Neumann Laplacians, this paper extends to indefinite weights some of the main results of [L2,3], esp. [L2, Theorems 2.1 and 4.1, pp. 479 and 510–511], where $r \equiv 1$. (See also [LF1,2] where a special case of this result is announced for the Dirichlet Laplacian.) The proof of our main result (Theorem 2 from Section 3) combines techniques from both [FL2] and [L2] (as well as [Mt2]) in order to deal with the oscillations and/or discontinuities of the (possibly) indefinite weight function, as well as with the roughness of the boundary.

2. The Dirichlet problem on a "Jordan contented set"

2.A. **Hypotheses and results.** (H_1) Ω is a nonempty bounded open set in $\mathbb{R}^n$ which is "Jordan contented"; the weight function r belongs to $L^\infty(\Omega)$ and is allowed to change sign; furthermore $|\Omega_+^o| > 0$, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$ and Ω_+^o is the interior of $\Omega_+ = \{x \in \mathbb{R}^n / \ r(x) > 0\}$.

Remark 1. Recall that $\omega \subset \mathbb{R}^n$ is said to be "*Jordan contented*" (in French, "*quarrable au sens de Jordan*") if it can be well approximated from within and without by a finite union of dyadic cubes (*see, e.g.*, [LoSt, Chapter 6, §§6-7] or [RS, p. 271]). In that case, its boundary $\partial\omega$ must have zero n-dimensional Lebesgue measure.

We now consider a covering of $\mathbb{R}^n$ by disjoint open cubes $(Q_\zeta)_{\zeta\in\mathbb{Z}^n}$ with side η (so that $\mathbb{R}^n = \bigcup_{\zeta\in\mathbb{Z}^n} \overline{Q_\zeta}$) and we set

$$I(\Omega_+^o) := \{\zeta \in \mathbb{Z}^n / \ Q_\zeta \subset \Omega_+^o\}, \quad \Omega'_+ = \Big(\bigcup_{\zeta\in I(\Omega_+^o)} \overline{Q_\zeta}\Big)^o, \tag{2.1a}$$

$$J(\Omega_+^o) := \{\zeta \in \mathbb{Z}^n / \ Q_\zeta \cap \overline{\Omega} \neq \varnothing\}, \tag{2.1b}$$

and

$$\omega = \Big(\bigcup_{\zeta\in J(\Omega_+^o)\setminus I(\Omega_+^o)} \overline{Q_\zeta}\Big)^o. \tag{2.1c}$$

We next introduce two hypotheses (which were made in [FL2]):

(H_2) Ω_+^o is "Jordan contented" and satisfies the "β-condition": given $\beta > 0$, there exist two positive constants c_0 and η_0 such that, for all $\eta \in (0, \eta_0]$,

$$\frac{\#(J(\Omega_+^o)\setminus I(\Omega_+^o))}{\#(I(\Omega_+^o))} \le c_0\eta^\beta,$$

where $\#A$ denotes the number of elements in the finite set A.

(H_3) r_+ satisfies the "γ-condition on Ω_+^o": given $\gamma \ge n$, there exists a positive constant c_1, which does not depend on ζ, such that, for all η small enough and for all $\zeta \in I(\Omega_+^o)$,

$$\rho_\zeta(r) := \|r_+ - r_\zeta\|^{n/2}_{L^{n/2}(Q_\zeta)} \le c_1\eta^\gamma, \tag{2.2a}$$

with $r_\zeta \ge 0$ defined by

$$r_\zeta^{n/2} := \eta^{-n}\int_{Q_\zeta} r^{n/2} = \eta^{-n}\|r\|^{n/2}_{L^{n/2}(Q_\zeta)}. \tag{2.2b}$$

Moreover, r_+ can be extended to a neighborhood of $\overline{\Omega_+}$ to a positive bounded function, still denoted by r_+. Then

Theorem 1. *Under hypotheses* (H_1) *to* (H_3), *for all* $\delta \in [\frac{1}{2(\nu+1)}, \frac{1}{2}]$, *where* $\nu := \min(\beta, \gamma - n)$, *we have*

$$R_0^+(\lambda; r, \Omega) = O(\lambda^{\frac{n-1}{2}+\delta}), \quad \text{as } \lambda \to +\infty.$$

This result is nothing else than Theorem 2, p. 339, in [FL2] for the Dirichlet problem, but with weaker hypotheses. An analogous result could of course be obtained for negative eigenvalues under analogous hypotheses on Ω_- and r_-. The exponent $\frac{n-1}{2} + \delta$ depends on the regularity of the weight function r (the larger γ, the "smoother" r) and of the boundary $\partial\Omega$ (the larger β, the "smoother" $\partial\Omega$) so that for smooth data, this exponent is as close as we want from $\frac{n-1}{2}$, the best possible exponent.

2.B. **Proof of Theorem 1.** We first note that for all $\rho > 0$,

$$N_0^+(\lambda; r, \Omega_+^o) \leq N_0^+(\lambda; r, \Omega) \leq N_0^+(\lambda; r_+ + \rho, \Omega).$$

Hence, by letting ρ tend to zero, exactly as in [L1], [FL1], ... , Theorem 1 when r changes sign can be derived from the case when r is positive.

Thus from now on in this section, we assume that

(H_4) r is positive, so that $\Omega = \Omega_+^0$ and $N_0^-(\lambda; r, \Omega) = 0$;

in that case, we write in short

$$N_i(\lambda; r, \Omega) := N_i^+(\lambda; r, \Omega) \tag{2.3}$$

and

$$I := I(\Omega_+^o), \quad J := J(\Omega_+^o). \tag{2.4}$$

The monotonicity of $N_0(\lambda; r, \Omega)$ with respect to Ω allows us to consider the problem on $(\overline{\Omega'} \cup \overline{\omega})^o$ which is larger than Ω; moreover, by use of the method of the Dirichlet-Neumann bracketing [CH, RS, Mt1, ... , FL1-2, L1-2, ...], we have

$$N_0(\lambda; r, \Omega') \leq N_0(\lambda; r, \Omega) \leq N_0(\lambda; r, (\overline{\Omega'} \cup \overline{\omega})^o) \leq N_1(\lambda; r, \Omega') + N_1(\lambda; r, \omega).$$

Hence, by subtracting the "Weyl term" $W(\lambda; r, \Omega)$, we get

$$A_0 \leq R_0(\lambda; r, \Omega) \leq A_1 + A_2, \tag{2.5}$$

where for $i = 0$ or 1,

$$A_i := \sum_{\zeta \in I} R_i(\lambda; r_\zeta, Q_\zeta) \tag{2.6}$$

and

$$A_2 := N_1(\lambda; r, \omega) - W(\lambda; r, \Omega \backslash \overline{\Omega'}) \leq N_1(\lambda; r, \omega). \tag{2.7}$$

The "interior terms" A_0 and A_1 can be handled exactly as the term A defined by Eq. (19.0) in [FL2], p. 342. (Note that in [FL2], the dimension is denoted by k instead of n.) Hence, for a given $\lambda > 0$ large enough, we choose (as in [FL2, Eq. (24), p. 343]):

$$\eta = \lambda^{-a} \quad \text{with } a \in (0, \delta] \tag{2.8}$$

and we deduce that there exists $c > 0$ such that the counterpart of Eq. (36) in [FL2, p. 345] holds:

$$R_0(\lambda; r, \Omega) \geq -c\lambda^{\frac{n-1}{2}+\delta}. \tag{2.9}$$

In the same manner, we obtain an upper bound for A_1, which follows from Eqs. (38) to (40) in [FL2, p. 346]:

$$|A_1| \le c\lambda^{\frac{n-1}{2}+\delta}. \tag{2.10}$$

For the "boundary term" $N_1(\lambda; r, \omega)$ which appears in (2.7), we again use the Dirichlet-Neumann bracketing and the monotonicity of $N_1(\lambda; r, \omega)$ with respect to r:

$$\begin{aligned} A_2 &\le N_1(\lambda; r, \omega) \le N_1(\lambda; M, \omega) = N_1(\lambda M; 1, \omega) \\ &\le \sum_{\zeta \in J\setminus I} N_1(\lambda M; 1, Q_\zeta) \le c(\#(J\setminus I))\eta^n \lambda^{n/2}, \end{aligned} \tag{2.11}$$

where $M < \infty$ is an upper bound for r_+; note that M exists by hypothesis (H_3). The last inequality is well known since the problem corresponding to $N_1(\lambda; 1, Q_\zeta)$ is the Laplacian on a cube "without weight", *i.e.*, with $r \equiv 1$.

By use of (H_1), (H_2), (2.4), and (2.8), there exists $c > 0$ such that

$$\#(J\setminus I) \le c\lambda^{a(n-\beta)} \text{ with } a \in (0, \delta]. \tag{2.12}$$

Hence, combining (2.11) with (2.12), we obtain as in [FL2, Eq. (28), p. 344]:

$$A_2 \le c\lambda^{\frac{n}{2}-a\beta} \le c\lambda^{\frac{n-1}{2}+\delta}, \tag{2.13}$$

provided that we choose a positive number a such that

$$\frac{1}{\nu}\left(\frac{1}{2} - \delta\right) \le a \le \delta. \tag{2.14}$$

Note that this choice of a is possible since by hypothesis of Theorem 1,

$$\frac{1}{2(\nu+1)} \le \delta \le \frac{1}{2}.$$

Theorem 1 for $r > 0$ follows from (2.5), (2.9), (2.10), and (2.13).

3. Irregular boundaries

3.A. **Introduction.** In the previous section, the upper bound for $N_0(\lambda; r, \Omega)$ was very simple to establish since by use of the monotonicity of $N_0(\lambda; r, \Omega)$ with respect to Ω, we could include Ω in a (finite) union of cubes. But this is not possible for the Neumann problem. Hence for obtaining more precise estimates at the boundary, we consider a covering of $\mathbb{R}^n$ by dyadic cubes. This partition is used in particular in [CH] and by many authors (*e.g.*, [Mt2], [L2], ...); it is sometimes referred to as a "Whitney-type covering".

Let O be an open set in $\mathbb{R}^n$. For a given $\eta_0 > 0$, we consider a covering of $\mathbb{R}^n$ by disjoint open cubes $(Q_{\zeta_q})_{\zeta_q \in \mathbb{Z}^n}$ with side

$$\eta_q = \eta_0 2^{-q}, \; q \in \mathbb{N}. \tag{3.1}$$

Set

$$I_0(O) = \{\zeta_0 \in \mathbb{Z}^n / \; Q_{\zeta_0} \subset O\}; \; O_0' = \Big(\bigcup_{\zeta_0 \in I_0(O)} \overline{Q_{\zeta_0}}\Big)^o \tag{3.2}$$

and

$$O_0'' = O \backslash \overline{O_0'} ;$$

$$(3.3) \qquad I_q(O) = \{\zeta_q \in \mathbb{Z}^n / \; Q_{\zeta_q} \subset O_{q-1}''\} ; \; O_q' = (\overline{O_{q-1}'} \cup (\bigcup_{\zeta_q \in Iq(O)} \overline{Q_{\zeta_q}}))^o$$

and

$$O_q'' = O \backslash \overline{O}_q' \;, \; \text{ for } q \geq 1 .$$

We now weaken the "β-condition" which implies that Ω_+^o is a Jordan contented set; this enables us to deal with more irregular boundaries and even fractal boundaries, as in [L2-3].

3.B. **Hypotheses and results.** (H_1') Ω is a (nonempty) bounded open set in $\mathbb{R}^n$ with $n \geq 1$; the weight function r belongs to $L^\infty(\Omega)$ and is allowed to change sign; furthermore, $|\Omega_+^o| > 0$.

(H_2') Ω_+^o is such that there exists $d \in [n-1 ; n)$ satisfying

$$\mathscr{M}^*(d ; \partial\Omega_+^o) := \limsup_{\epsilon \to 0+} \epsilon^{-(n-d)} |\tilde{\Gamma}_\epsilon| < \infty ,$$

with

$$\tilde{\Gamma}_\epsilon := \Gamma_\epsilon \cap \Omega_+^o \text{ and } \Gamma_\epsilon := \{x \in \mathbb{R}^n / \operatorname{dist}(x , \partial(\Omega_+^o)) < \epsilon\} ;$$

here, dist (, .) denotes the Euclidean distance in $\mathbb{R}^n$.

(H_3') r_+ satisfies the "γ-condition" on Ω_+^o (adapted to the covering Q_{ζ_q}) with $\gamma > d$; *i.e.*, there exists $c_2 > 0$ which does not depend on $\zeta_q \in I_q(\Omega_+^o)$ and there exists $\eta_1 > 0$ such that for all $\eta \in (0, \eta_1]$, for all $q \in \mathbb{N}$,

$$\rho_\zeta(r) \leq c_2 \eta_q^\gamma \;,$$

where ρ_ζ is defined as in (2.2a).

(H_4') For the Neumann problem, we assume that (H_2') holds with $\tilde{\Gamma}_\epsilon$ replaced by Γ_ϵ; we assume in addition that Ω_+^o satisfies *either* the "extension property" (there exists a continuous linear extension of $H^1(\Omega)$ onto $H^1(\mathbb{R}^n)$, *e.g.*, [L2, Definition 4.3, p. 510]) *or* the "(C') condition" ([Mt2, pp. 154-156]: there exists a finite covering of $\partial\Omega$ by open sets $(O_j)_{j=1,\dots,J_0'}$, with $O_j \subset \mathbb{R}^n$; there are open sets U_j' in $\mathbb{R}^{n-1}$, positive numbers h_j, upper semi-continuous functions $\varphi_j : (-h_j , 2h_j) \to U_j'$, and C^1 diffeomorphisms $\theta_j : O_j \cap \Omega \to V_j$ with

$$V_j = \{\xi = (\xi_1 , \xi') \in \mathbb{R}^n / \; \xi' \in U_j' ; \; \varphi_j(\xi') < \xi_1 < 2h_j\} .$$

Furthermore, for all x and y in $O_j \cap \Omega$, there exists $s > 0$ such that the path $\mathscr{L}_s(x , y)$ associated to the reunion of segments

$$[\theta_j(x) , \theta_j(x) + se_1] \cup [\theta_j(x) + se_1 , \theta_j(y) + se_1] \cup [\theta_j(y) + se_1 , \theta_j(y)]$$

lies within $O_j \cap \Omega$ (here, e_1 denotes the first vector of the standard basis in $\mathbb{R}^n$). Moreover, if we denote by $\rho_j(x , y)$ the infimum of the lengths of all such $\mathscr{L}_s(x , y)$ in $O_j \cap \Omega$, there exists k_0 such that for all $j \in \{1 , 2 , \dots , J_0\}$,

$$\rho_j(x , y) \leq k_0 \operatorname{dist}(x , y).$$

Remark 2. (a) If the open set ω satisfies the "segment property" [Ag, p. 11], then it satisfies the "(C') condition"; this is the case, for example, if $\partial\omega$ is

Lipschitz. The "(C') condition" was first introduced in [FM, p. 914, §3], and then used in [Mt1-2], [FL1-2], [L1-4]. Roughly, it means that the boundary $\partial\Omega$ is not "too long": the Euclidean distance $\operatorname{dist}(x, y)$ is equivalent to $d_\Omega(x, y)$, the minimal length of continuous paths from x to y *within* Ω. Note that if the "(C') condition" is not satisfied, then the usual Weyl asymptotic formula (1.1) may fail for the Neumann problem. (See, *e.g.*, the counterexamples when $i = 1$ and $r \equiv 1$ in [FM] and [Mt2, §VII.1, pp. 200–204].)

(b) When $n = 2$, the simply connected domain ω satisfies the "extension property" if and only if it is a *quasidisk, i.e.*, the image of the unit disk under a quasiconformal mapping. (*See, e.g.*, [L2, Example 4.2, p. 510] and the references therein.) The boundary $\partial\omega$ is then a *quasicircle* (roughly, a distorted circle) and may be extremely irregular [L2, p. 510].

(c) Intuitively, the more "regular" the boundary $\partial\Omega_+^\circ$, the *smaller* we may choose d in hypothesis (H_2'). Similarly, the more "regular" the weight function r, the *larger* we may choose γ in hypothesis (H_3'). For example, if $r \equiv 1$, as in [L2], then clearly, any $\gamma > 0$ is suitable. Further, it follows from (the argument provided in) [FL2, Example 2, p. 333] that if r is Hölder continuous of order $\theta \in (0, 1)$ and bounded away from zero on Ω_+°, then it satisfies the "γ-condition" on Ω_+° with $\gamma = n + (n\theta/2)$; in particular, $\gamma > n$ in that case.

Theorem 2. *Under the above hypotheses, we have for* $i = 0$ *or* 1:

(i) *If* $d \in (n-1, n)$ (*the "fractal case"*)

$$R_i^+(\lambda; r, \Omega) := N_i^+(\lambda; r, \Omega) - W(\lambda; r_+, \Omega) = O(\lambda^{t/2}), \text{ as } \lambda \to +\infty,$$

where $t := \max(d, d+n-\gamma)$. [*Hence* $t = d$ *if* $\gamma \geq n$ *and* $t = d+n-\gamma$ *if* $\gamma < n$.]

(ii) *If* $d = n-1$ (*the "nonfractal case"*),

$$R_i^+(\lambda; r, \Omega) = \begin{cases} O(\lambda^{\frac{n-1}{2}}\log\lambda) & \text{when } \gamma > n, \\ O(\lambda^{\frac{n-1}{2}+\frac{n-\gamma}{2}}) & \text{when } \gamma \leq n. \end{cases}$$

Theorem 2 extends to problems with indefinite weights earlier results obtained for $r \equiv 1$ ([Mt1-2], [L2-3]). It also makes more precise and extends to irregular boundaries results of [FL2] where an hypothesis needed to be completed (*see* [L2, Remark 3.5(d), p. 490]).

Remark 3. (a) Hypothesis (H_2') which replaces in this section the "β-condition" says that the (interior Bouligand-) Minkowski dimension D of $\partial\Omega_+^0$ is $\leq d$. (*See, e.g.*, [L2, Definition 2.1(b), pp. 474-475 and §3]. For relationships between the "β-condition" and the Minkowski dimension, *see* [L2, Corollary 3.3 and Remark 3.5, pp. 489-490].) When the weight r is "smooth enough" (compared with $\partial\Omega$), *viz* $\gamma \geq n$, then $t = d$ and we recover for $d > n-1$ results of [L2-3] in the special case of the (Dirichlet or Neumann) Laplacian. (*See* Remark 2(c) above and [L2, Theorem 2.3, pp. 482–483].) Indeed, the irregularity (fractality) of the boundary of Ω_+^o can be due to that of Ω $(\partial\Omega \cap \partial(\Omega_+^o))$ or that of r $(\partial(\Omega_+^o)\backslash\partial\Omega) : d < \gamma < n$.

(b) The use of the Minkowski dimension D for rough boundaries (in a related context) was first made by Brossard and Carmona in [BrCa]. In [BrCa], the authors also obtained (pre-Tauberian) error estimates (expressed in terms of D) for the short time asymptotics of the "partition function" $Z(t) = \operatorname{Trace}(e^{t\Delta})$, when Δ is the Dirichlet Laplacian and $r \equiv 1$.

(c) Strictly speaking and to be in keeping with the terminology used in [L2-4], the "fractal" (resp., "nonfractal") case is that when $D \neq n-1$ (resp., $D = n-1$). Recall from [L2] that we always have $D \in [n-1, n]$ and that if $0 < \mathscr{M}^*(d; \partial\Omega_+^o) < \infty$, then $d = D$, the (interior) Minkowski dimension of $\partial\Omega_+^o$.

Of course, an analogous theorem holds for $N_i^-(\lambda; r, \Omega)$ under analogous hypotheses.

As in the previous section, we will prove Theorem 2 for $r > 0$; the case where r changes sign can be derived as indicated above. Therefore, from now on, we assume that

(H_5') r is positive, hence $\Omega = \Omega_+^o$, and as above we set

$$N_i(\lambda; r, \Omega) = N_i^+(\lambda; r, \Omega). \tag{3.4}$$

We also let, in view of (3.3),

$$I_q := I_q(\Omega) \tag{3.5a}$$

and

$$\omega_q := \Omega_q''. \tag{3.5b}$$

3.C. **A lower bound.** We establish here a lower bound for $R_i(\lambda; r, \Omega) = N_i(\lambda; r, \Omega) - W(\lambda; r, \Omega)$, $i = 0$ or 1, when r is positive. The numbers γ and d are given, with $\gamma > d$. Choose $\lambda > 1$ and then $\eta_0 = \lambda^{-a}$ where $a > \frac{n-d}{2(\gamma-d)}$ with η_0 small enough so that (H_2') or (H_4') for the Dirichlet or Neumann problem, respectively, implies that there exists $c_3 > 0$ such that for all $q \in \mathbb{N}$,

$$(\#(I_q))\eta_q^n \leq c_3\, \eta_q^{n-d}, \tag{3.6}$$

where η_q is defined by (3.1) and I_q by (3.3) and (3.5a).

Since r is bounded, there exists $M > 0$ such that for almost all $x \in \Omega$, $|r(x)| \leq M$, and hence, λ being given, there exists $P \in \mathbb{N}$ such that, for all $q > P$,

$$N_0(\lambda; r, Q_{\zeta_q}) = 0. \tag{3.7}$$

We define

$$P := \max\{q \in \mathbb{N} /\ N_0(\lambda; M, Q_{\zeta_q}) \neq 0\}. \tag{3.8}$$

Note that the integer P depends on λ. By means of (3.3) and the usual inequalities on $N_i(\lambda; r, \Omega)$, we can write:

$$\begin{aligned} N_1(\lambda; r, \Omega) &\geq N_0(\lambda; r, \Omega) \geq N_0(\lambda; r, \Omega_P') \\ &\geq \sum_{q=0}^{P} \sum_{\zeta_q \in I_q} N_0(\lambda; r, Q_{\zeta_q}). \end{aligned} \tag{3.9}$$

Then, by subtracting the "Weyl term", we obtain

$$R_i(\lambda; r, \Omega) = N_i(\lambda; r, \Omega) - W(\lambda; r, \Omega) \geq A_1 + A_2 + A_3 + A_4, \tag{3.10}$$

with

$$A_1 := \sum_{q=0}^{P} \sum_{\zeta_q \in Iq} (N_0(\lambda; r, Q_{\zeta_q}) - N_0(\lambda; r_{\zeta_q}, Q_{\zeta_q})), \tag{3.10a}$$

$$A_2 := \sum_{q=0}^{P} \sum_{\zeta_q \in Iq} R(\lambda; r_{\zeta_q}, Q_{\zeta_q}), \tag{3.10b}$$

$$A_3 := \sum_{q=0}^{P} \sum_{\zeta_q \in Iq} (W(\lambda; r_{\zeta_q}, Q_{\zeta_q}) - W(\lambda; r, Q_{\zeta_q})), \tag{3.10c}$$

and

$$A_4 := -W(\lambda; r, \omega_P). \tag{3.10d}$$

We first note that, with ρ_ζ defined as in (2.2a),

$$|N_i(\lambda; r, Q_\zeta) - N_i(\lambda; r_\zeta, Q_\zeta)| \leq N_i(\lambda; |r - r_\zeta|, Q_\zeta) \leq \rho_\zeta(r)\lambda^{n/2}. \tag{3.11}$$

Therefore, by (H_3'), (3.1), (3.6), and (3.7), we have

$$\begin{aligned} -A_1 &\leq c \sum_{q=0}^{P} (\#(I_q))\, \eta_q^{\gamma} \lambda^{n/2} \leq c\lambda^{n/2} \sum_{q=0}^{P} \eta_q^{\gamma-d} \\ &\leq c\lambda^{n/2} \eta_0^{\gamma-d} \sum_{q=0}^{P} 2^{-q} \leq c' \lambda^{(n/2)-a(\gamma-d)} . \end{aligned} \tag{3.12}$$

We note that, when $d < \gamma \leq n$, we have $\frac{n-d}{\gamma-d} > 1$ and thus our choice of a preceding Eq. (3.6) implies $a > \frac{1}{2}$. Thus, for $d \in [n-1, n)$,

$$-A_1 \leq c\lambda^{t/2}, \quad \text{with } t = \max(d,\ d + n - \gamma). \tag{3.13}$$

We now consider A_2 (which is negative); it can be handled as in [FL2, Eq. (21), p. 342]: we know from [CH, Section 6.4] (or [RS, Proposition 2, pp. 266–267]) that there exists $c_0' > 0$ such that, for $i = 0$ or 1, for all $\wedge \geq 1$ and all $q \in \mathbb{N}$,

$$|R_i(\wedge; 1, Q_{\zeta_q})| \leq c_0'[1 + (\wedge\eta_q^2)^{(n-1)/2}] . \tag{3.14}$$

Consequently,

$$|A_2| \leq c_0' \sum_{q=0}^{P} (\#(I_q))[1 + (\lambda r_\zeta \eta_q^2)^{(n-1)/2}] . \tag{3.15}$$

We deduce from (3.8) that there exists $c_4 > 0$ satisfying

$$\frac{1}{2} c_4 \sqrt{\lambda} < 2^P < c_4 \sqrt{\lambda} . \tag{3.16}$$

Hence, by combining (3.1), (3.6), (3.15), and (3.16), and since r is bounded by M, we obtain

$$|A_2| \leq c' \sum_{q=0}^{P} \eta_q^{-d} + c'' \lambda^{(n-1)/2} \sum_{q=0}^{P} \eta_q^{n-1-d}.$$

Therefore,

$$|A_2| \le c\lambda^{d/2} \quad \text{for } d > n-1, \tag{3.17i}$$

$$|A_2| \le c\lambda^{(n-1)/2} \ln\lambda \quad \text{for } d = n-1. \tag{3.17ii}$$

By definition of r_{ζ_q} in (2.2), it is obvious in view of (3.10c) that

$$A_3 = 0. \tag{3.18}$$

Finally we deduce from (3.16), (H_1'), and (H_2') or (H_4') that

$$|A_4| = W(\lambda; r, \omega_P) = c\int_{\omega_P} (\lambda r)^{n/2} \le c\lambda^{n/2}|\omega_P| \le c\lambda^{n/2}\,\eta_P^{n-d} \le c\lambda^{d/2}. \tag{3.19}$$

Hence it follows from (3.10), (3.13), and (3.17) to (3.19) that, for r regular enough, *i.e.*, for $\gamma > d$, we have:

(i) When $d = n-1$,

$$R_i(\lambda; r, \Omega) \ge -c\lambda^{\frac{n-1}{2}} \ln\lambda \quad \text{if } \gamma > n \tag{3.20a}$$

$$\ge -c\lambda^{\frac{n-1}{2}+\frac{n-\gamma}{2}} \quad \text{if } d < \gamma \le n. \tag{3.20b}$$

(ii) When $d \in (n-1, n)$,

$$R_i(\lambda; r, \Omega) \ge -c\lambda^{t/2}, \tag{3.21a}$$

where

$$t = \max(d,\ d+n-\gamma). \tag{3.21b}$$

Remark 4. (a) In particular, when $r \equiv 1$, we have $\gamma \ge n$ [*see* Remark 2(c)]; if $d \in (n-1, n)$, we then recover, in the special case of the Laplacian, the estimate $R_i(\lambda; r, \Omega) \ge -c\lambda^{d/2}$ which has been established in [L2, Theorem 2.3, p. 482] and [L3]. As was shown in [L2], [L4], this estimate is *sharp* for every $d \in (n-1, n)$ and for $i = 0$ or 1.

(b) Note that when r is Hölder continuous on Ω and bounded away from zero, then $\gamma > n$ (by Remark 2(c) above) and hence $\gamma > d$ is automatically fulfilled.

(c) When $n = 1$ and $0 < d < 1$, the lower bound is of the form $-c\lambda^{d/2}\ln\lambda$ [resp. $-c\lambda^{d/2}$] when $\gamma > 1$ [resp. ≤ 1].

3.D. **Upper bound.** For the Dirichlet problem, we can simply include Ω in a finite union of cubes; up to a set with Lebesgue measure zero, we have

$$\Omega \subset O_P := \bigcup_{q=0}^{P} \bigcup_{\zeta_q \in Iq} Q_{\zeta_q} \cup \Big(\bigcup_{\zeta_P \in I_P'} Q_{\zeta_P}\Big),$$

where

$$I_P' = \{\zeta_P \in \mathbb{Z}^n /\ Q_{\zeta_P} \cap \partial\Omega \ne \varnothing\}.$$

We note that as for I_q in (3.6), we have

$$\#(I_P') \le c_3\eta_P^{-d}. \tag{3.22}$$

Therefore, we can write

(3.23)

$$N_0(\lambda; r, \Omega) \le N_0(\lambda; r, O_P) \le \sum_{q=0}^{P} \sum_{\zeta_q \in Iq} N_1(\lambda; r, Q_{\zeta_q}) + \sum_{\zeta_P \in I_P'} N_1(\lambda; r, Q_{\zeta_P}).$$

In (3.23), in order to extend r, we must assume λ (and hence P) large enough.

By subtracting the Weyl term, we obtain

(3.24)

$$R_0(\lambda; r, \Omega) \leq \sum_{q=0}^{P} \sum_{\zeta_q \in Iq} R_1(\lambda; r, Q_{\zeta_q}) + \sum_{\zeta_P \in I'_P} N_1(\lambda; r, Q_{\zeta_P}) \leq B_1 + B_2 + B_3 + B_4,$$

where

$$B_1 := \sum_{q=0}^{P} \sum_{\zeta_q \in Iq} (N_1(\lambda; r, Q_{\zeta_q}) - N_1(\lambda; r_{\zeta_q}, Q_{\zeta_q})), \tag{3.25a}$$

$$B_2 := \sum_{q=0}^{P} \sum_{\zeta_q \in Iq} (N_1(\lambda; r_{\zeta_q}, Q_{\zeta_q}) - W(\lambda; r_{\zeta_q}, Q_{\zeta_q})), \tag{3.25b}$$

$$B_3 := \sum_{q=0}^{P} \sum_{\zeta_q \in Iq} (W(\lambda; r_{\zeta_q}, Q_{\zeta_q}) - W(\lambda; r, Q_{\zeta_q})), \tag{3.25c}$$

and

$$B_4 := \sum_{\zeta_P \in I'_P} N_1(\lambda; r, Q_{\zeta_P}). \tag{3.25d}$$

The first three terms B_1, B_2, B_3 can be handled exactly as A_1, A_2, A_3 in the previous section (§3.C), and we obtain:

$$B_1 \leq c\lambda^{t/2} \quad \text{with } t = \max(d,\ d + n - \gamma), \tag{3.26}$$

$$B_2 \leq c\lambda^{d/2} \quad \text{for } d > n - 1, \tag{3.27a}$$

$$B_2 \leq c\lambda^{(n-1)/2} \ln \lambda \quad \text{when } d = n - 1, \tag{3.27b}$$

$$B_3 = 0. \tag{3.28}$$

Since r is bounded, we have

$$B_4 \leq \sum_{\zeta_P \in I'_P} N_1(\lambda M; 1, Q_{\zeta_P}) \leq c\lambda^{n/2} \eta_P^n(\#(I'_P))$$

so that, by (3.1), (3.16), and (3.22),

$$B_4 \leq c\lambda^{d/2}. \tag{3.29}$$

Hence, by combining (3.26) to (3.29) with (3.24), we have for $\gamma > d$,

(3.30)

$$\begin{aligned} R_0(\lambda; r, \Omega) &\leq c\lambda^{\frac{n-1}{2}} \ln \lambda \quad \text{when } d = n - 1 \text{ and } \gamma > n, \\ &\leq c\lambda^{\frac{n-1}{2} + \frac{n-\gamma}{2}} \quad \text{when } d = n - 1 \quad \text{and } d < \gamma \leq n, \\ &\leq c\lambda^{t/2} \quad \text{with } t = \max(d,\ d + n - \gamma), \quad \text{when } d \in (n - 1, n). \end{aligned}$$

For the Neumann problem, we cannot write (3.23); we simply have by means of definition (3.5b):

$$R_1(\lambda; r, \Omega) \leq B_1 + B_2 + B_3 + B'_4, \tag{3.31}$$

where B_1, B_2, B_3 are given by (3.25a-c) and

$$B'_4 := N_1(\lambda; r, \omega_P). \tag{3.32}$$

Since r is bounded,

$$B_4' \leq N_1(\lambda M\,;\, 1\,,\, \omega_P)\,. \tag{3.33}$$

We now have to study a Neumann problem (without weight function) on the boundary strip ω_P. We deal as in [FM], [Mt1-2] or [L2, pp. 496–497]. We denote by $\tilde{u}$ the extension of u to $\mathbb{R}^n$ with $\tilde{u} \in H^1(\mathbb{R}^n)$. On each cube Q_{ζ_q} we approximate $\tilde{u}$ by its mean value $\overline{u_{\zeta_q}}$. Set

$$v := \sum_{\zeta_P \in I_P'} \overline{u_{\zeta_P}}\, \mathbb{1}_{Q_{\zeta_P}}\,, \tag{3.34}$$

where

$$\mathbb{1}_{Q_{\zeta_P}}(x) := \begin{cases} 1 & \text{when } x \in Q_{\zeta_P}\,, \\ 0 & \text{when } x \notin Q_{\zeta_P}. \end{cases}$$

By use of inequality (1) in [FM, p. 915] or inequality 5.2 in [Mt1], we obtain

$$\|u - v\|^2_{L^2(\omega_P)} \;\leq\; c\eta_P^2\, \|u\|^2_{H^1(\Omega)}. \tag{3.35}$$

(*See* also [L2].) We note that v defined by (3.34) lies in a $(\#(I_P'))$-dimensional subspace of $H^1(\mathbb{R}^n)$. Hence in view of (3.22) and (3.35):

$$B_4' \leq c\lambda^{d/2}\,. \tag{3.36}$$

It follows from (3.26) to (3.28) combined with (3.31) and (3.36) that the counterpart of (3.30) also holds for $R_1(\lambda\,;\, r\,,\, \Omega)$ [instead of $R_0(\lambda\,;\, r\,,\, \Omega)$], and hence Theorem 2 is proved.

Remark 5. (a) Our results (Theorems 1 and 2) could be extended to more general elliptic operators, much as in [Mt1-2], [FL2], and [L2-3], although we chose not do so in order to keep our arguments reasonably short and simple.

(b) It would be interesting to investigate whether, under suitable hypotheses, the results of [LP1,2]—that establish when $n = 1$ (and $r \equiv 1$) the "modified Weyl-Berry conjecture" of [L2] concerning the existence of a (monotonic) asymptotic second term for $N(\lambda)$—can be extended to "fractal strings" ([L2], [L4], [LP1-2]) with *variable* mass density.

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