CONVEX BODIES AND CONCAVE FUNCTIONS

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Abstract. We find properties that a class \( \mathcal{C} \) of closed bounded convex subsets of a Banach space \( E \) and a mapping \( p : \mathcal{C} \rightarrow \mathbb{R}^+ \) should satisfy in order to obtain the following result:

**Theorem.** Let \( \mathcal{C} \) and \( p : \mathcal{C} \rightarrow \mathbb{R}^+ \) satisfy these properties, and let \( K \) be a closed convex subset of \([0, 1] \times E\) such that for every \( t \in [0, 1] \) the set \( K(t) = \{ z \in E; (t, z) \in K \} \) is an element of \( \mathcal{C} \). Suppose that a concave continuous function \( f : [0, 1] \rightarrow \mathbb{R} \) is given such that

\[
0 < f(t) < p(K(t)), \quad \text{for every } t \in [0, 1].
\]

Then there exists a closed convex subset \( L \) of \([0, 1] \times E\) such that \( L \subset K \),

\[
L(t) = \{ z \in E; (t, z) \in L \} \in \mathcal{C} \quad \text{and} \quad f(t) = p(L(t)) \quad \text{for every } t \in [0, 1].
\]

Some examples and applications are given to the study of Steiner symmetrization and of the Riesz decomposition property for concave continuous functions.

**Introduction**

The aim of this paper was originally to study the following problem: Let \( A \) be a convex body in \( \mathbb{R}^d \) and \( \bar{A} \) be its Steiner symmetrical with respect to some fixed hyperplane \( H \). If \( C \) is a convex body such that \( C \subset \bar{A} \) and \( C \) is symmetric with respect to \( H \), does there exist a convex body \( B \subset A \) such that the Steiner symmetrical \( \bar{B} \) of \( B \) with respect to \( H \) is equal to \( C \)? Using the general result (Theorem 2) mentioned in the abstract, we prove that the answer is yes if \( d = 2 \) and generally no if \( d \geq 3 \). We study also the following problem (Corollary 3): given a direction \( u \in S_{d-1} \) and a convex body \( A \) in \( \mathbb{R}^d \) such that all its sections by hyperplanes orthogonal to \( u \) are homothetical to some fixed convex body \( D \), how does one characterize the convex bodies \( B \subset A \) with the same property? We show (Corollary 4) that, given any convex body \( A \) in \( \mathbb{R}^d \), there exists a convex body \( B \subset A \) whose sections by hyperplanes orthogonal to some fixed direction \( u \) have prescribed mean width.
Definition. Let \( E \) be a normed space, and let \( \mathcal{C} \) be a class of closed bounded convex subsets of \( E \) such that:

- \( \emptyset \notin \mathcal{C} \) and there exists \( C \in \mathcal{C} \) with nonempty interior.
- \( \lambda C + \mu D \in \mathcal{C} \) for every \( C, D \in \mathcal{C} \) and \( \lambda, \mu \geq 0 \).
- \( x + C \in \mathcal{C} \) for every \( x \in E \) and \( C \in \mathcal{C} \).

For every family \( \mathcal{F} \) of elements of \( \mathcal{C} \), totally ordered by inclusion,

\[
\bigcap_{C \in \mathcal{F}} C \in \mathcal{C}.
\]

Let \( p : \mathcal{C} \to \mathbb{R}_+ \) be such that:

- \( p \) is increasing: if \( C, D \in \mathcal{C} \), \( C \subset D \), then \( p(C) \leq p(D) \).

- \( p(\lambda C + \mu D) = \lambda p(C) + \mu p(D) \) for every \( C, D \in \mathcal{C} \) and \( \lambda, \mu \geq 0 \).

- \( p(x + C) = p(C) \) for every \( x \in E \) and \( C \in \mathcal{C} \).

For every family \( \mathcal{F} \) of elements of \( \mathcal{C} \), totally ordered by inclusion,

\[
p \left( \bigcap_{C \in \mathcal{F}} C \right) = \inf_{C \in \mathcal{F}} p(C).
\]

If \((E, \mathcal{C}, p)\) satisfies all these properties, we shall say that \((E, \mathcal{C}, p)\) is admissible.

Lemma 1. Let \( h \) and \( f \) be two concave continuous functions on \([0, 1]\) such that \( 0 \leq f \leq h \) and \( f \neq 0, h \). Then there exists a concave continuous function \( g \) on \([0, 1]\) satisfying \( f \leq g \leq h \), \( g \neq f, h \), and at least one of the following properties:

1. For some \( a, b \in [0, 1] \), \( a < b \), \( g \) is affine on \([a, b]\) and coincides with \( h \) on \([0, a) \cup (b, 1]\).
2. \( g \) is affine on \([0, 1]\), and \( g = \rho h \) for some \( \rho \in ]0, 1[ \).
3. \( g(0) = \rho h(0) \) for some \( \rho \in ]0, 1[ \), and for some \( a \in [0, 1[ \), \( g \) is affine on \([0, a]\) and coincides with \( h \) on \([a, 1]\).
4. \( g(1) = \rho h(1) \) for some \( \rho \in ]0, 1[ \), and for some \( b \in [0, 1[ \), \( g \) is affine on \([b, 1]\) and coincides with \( h \) on \([0, b]\).

Proof. Let \( \alpha_n, \beta_n \), \( n \in \mathbb{N} \), be the interiors in \( \mathbb{R} \) of the connected components of the set \( \{ f < h \} = \{ t \in [0, 1] ; f(t) < h(t) \} \); then we have \( \{ f < h \} \cap [0, 1] = \bigcup_{n \in \mathbb{N}} [\alpha_n, \beta_n[ \); the proof can be reduced to two cases:

I. For some \( n \in \mathbb{N} \), there exists \( t_0 \in ]\alpha_n, \beta_n[ \) such that, for every neighborhood \( V \) of \( t_0 \), \( h \) is not affine on \( V \). Then by the continuity of \( f \) and \( h \), for some \( \varepsilon > 0 \) such that \( [t_0 - \varepsilon, t_0 + \varepsilon] \subset ]\alpha_n, \beta_n[ \), we have

\[
\theta h(t_0 - \varepsilon) + (1 - \theta) h(t_0 + \varepsilon) > \max_{t \in [t_0 - \varepsilon, t_0 + \varepsilon]} f(t)
\]

for every \( \theta \in [0, 1] \). Then we define \([a, b] = [t_0 - \varepsilon, t_0 + \varepsilon]\) and we take the function \( g \) as defined in (1). Since \( h \) is concave but not affine on \([t_0 - \varepsilon, t_0 + \varepsilon]\), we have \( g < h \).

II. For every \( n \in \mathbb{N} \) and every \( t \in ]\alpha_n, \beta_n[ \), \( h \) is affine on some neighborhood of \( t \). Then by compactness and connectivity, using the continuity of \( h \), we get that \( h \) is affine on each of the intervals \([\alpha_n, \beta_n] \). If \( 0 < \alpha_n < \beta_n < 1 \), we have \( f(\alpha_n) = h(\alpha_n) \) and \( f(\beta_n) = h(\beta_n) \); it follows then from the concavity of \( f \) that \( f = h \) on \([\alpha_n, \beta_n[ \), which is of course absurd from the definition of \([\alpha_n, \beta_n[ \). Thus one of the following properties holds:

1. \( \{ f < h \} = [0, 1] \) and \( h \) is affine on \([0, 1]\).
(b) For some $0 < a < 1$, $h(a) = f(a)$, $[0, a] \subset \{f < h\}$, and $h$ is affine on $[0, a]$. 
(c) For some $0 < \beta < 1$, $h(\beta) = f(\beta)$, $[\beta, 1] \subset \{f < h\}$, and $h$ is affine on $[\beta, 1]$. 

By continuity, case (a) reduces to conclusion (2).

In case (b), by the continuity of $f$, for some $0 < \varepsilon < \alpha$ and $0 < \rho < 1$, we have

$$\max_{t \in [0, \varepsilon]} f(t) < \rho \min_{t \in [0, \varepsilon]} h(t).$$

Taking $a = \varepsilon$, we define $g$ satisfying property (3). Case (c) is completely analogous to case (b) and yields property (4).

**Theorem 2.** Let $(E, \mathcal{E}, p)$ be admissible, and let $K$ be a closed convex subset of $[0, 1] \times E$ such that for every $t \in [0, 1]$ the set $K(t) = \{z \in E; (t, z) \in K\}$ is an element of $\mathcal{E}$. Suppose that a concave continuous function $f: [0, 1] \to \mathbb{R}$ is given such that

$$0 < f(t) < p(K(t)), \quad \text{for every } t \in [0, 1].$$

Then there exists a closed convex subset $L$ of $[0, 1] \times E$ such that $L \subset K$, 

$L(t) = \{z \in E; (t, z) \in L\} \in \mathcal{E}$ and $f(t) = p(L(t))$ for every $t \in [0, 1]$.

**Proof.** Let $\mathcal{D}$ be the set of all closed convex subsets $M$ of $K$ such that for every $t \in [0, 1]$, $M(t) \in \mathcal{E}$ and $p(M(t)) \geq f(t)$. Then it is easy to see that, ordered by inclusion, $\mathcal{D}$ is inductive. By Zorn's lemma, $\mathcal{D}$ thus has a minimal element, say $N$; define $h: [0, 1] \to \mathbb{R}_+$ by $h(t) = p(N(t))$. It is clear that $f$ and $h$ are concave and continuous on $[0, 1]$ and satisfy $h \geq f$; we shall show that $h = f$. Suppose $f < h$; then given the function $g$ obtained by Lemma 1, we shall construct $N' \in \mathcal{D}$, $N' \subset N$, such that $g(t) = p(N'(t))$ for every $t \in [0, 1]$. But, since $f < g < h$, this contradicts the minimality of $N$. Let us consider the four cases of Lemma 1:

(1) Let $N'(t) = N(t)$ if $t \in [0, a] \cup [b, 1]$ and $N'(\lambda a + (1 - \lambda)b) = \lambda N(a) + (1 - \lambda)N(b)$, for $0 \leq \lambda \leq 1$. Then define $N' = \{(t, z); t \in [0, 1], z \in N'(t)\}$. It is clear that $N' \in \mathcal{D}$, $N' \subset N$, and $p(N'(t)) = g(t)$ for every $t \in [0, 1]$.

(2) Let $x_0 \in N(0)$ and $x_1 \in N(1)$, and define for $t \in [0, 1]$, $N'(t) = (1 - t)(x_0 + p(N(0) - x_0)) + t(x_1 + p(N(1) - x_1))$.

(3) Let $x \in N(0)$; define $N'(t) = (1 - \frac{t}{a})(x + p(N(0) - x)) + \frac{t}{a}N(a)$ for $t \in [0, a]$ and $N'(t) = N(t)$ for $t \in [a, 1]$.

(4) is analogous to (3).

**Remark.** If we had a concave continuous function $f$ defined on $[a, b]$, $0 \leq a < b \leq 1$, satisfying $0 \leq f(t) \leq p(K(t))$ for every $t \in [a, b]$, the conclusion of Theorem 2 would still hold, with $L(t) = \emptyset$ for $t \notin [a, b]$.

**Examples.** We give some examples of admissible $(E, \mathcal{E}, p)$. We refer to [B-Z] for all the undefined terminology about convex sets.

(1) Let $E$ be a Banach space, and let $D$ be a bounded closed convex subset of $E$ with 0 in its interior; let $\mathcal{E}_D$ be the family of all subsets of $E$ of the form $x + \mu D$, $x \in E$, $\mu \geq 0$; and let $p_D: \mathcal{E}_D \to \mathbb{R}_+$ be defined by $p_D(x + \mu D) = \mu$; observe that for $\lambda$, $\mu \geq 0$, one has $x + \mu D \subset y + \lambda D$ if and only if $\mu \leq \lambda$ and $x - y \in (\lambda - \mu)D$; it is easy to verify, using the completeness of $E$, that $(E, \mathcal{E}_D, p_D)$ is admissible.
(2) Let $\mathcal{K}^d$ be the class of all compact convex subsets of $E = \mathbb{R}^d$. If $p: \mathcal{K}^d \to \mathbb{R}^+$ satisfies the first three properties of admissible mappings, then it follows from Hahn-Banach theorem and the Riesz representation theorem that

$$p(K) = \int_{S^{d-1}} H_K(u) d\nu(u) \quad \text{for } K \in \mathcal{K}^d$$

for some positive measure $\nu$ on the sphere $S^{d-1}$ such that $\int_{S^{d-1}} u d\nu(u) = 0$ ($H_K$ denotes the support function of $K$: $H_K(u) = \max \{\langle x, u \rangle \colon x \in K\}$, where $\langle x, u \rangle$ is the scalar product of $x, u \in \mathbb{R}^d$). In particular such a $p$ satisfies automatically the fourth property of admissible mappings (see [F]). For instance, if $K_j, 2 \leq j \leq d$, are convex bodies in $\mathbb{R}^d$, one can consider as $p(K)$, the mixed volume $V(K, K_2, \ldots, K_d)$ of $K$ together with $K_2, \ldots, K_d$ or

$$V(K, B, \ldots, B) = h_m(K) = \frac{1}{\omega_{d-1}} \int_{S^{d-1}} H_K(u) d\sigma(u),$$

where $B$ is the Euclidean ball and $\sigma$ is the rotation invariant measure on $S^{d-1}$ with total mass $\omega_{d-1}$ (this is the mean width of $K$, and it coincides with $1/2\pi$ times the perimeter of $K$ if $d = 2$).

As applications of Theorem 2 to these examples, we get the following corollaries.

**Corollary 3.** Let $A$ be a body in $\mathbb{R}^d$, such that for some axis of direction $u \in S^{d-1}$, the sections $A(t) = \{z \in A; \langle z, u \rangle = t\}$ are all homothetic (whenever they are nonempty) to some fixed convex body $D \subset \{z; \langle z, u \rangle = 0\}$. Let $f$ be a nonnegative continuous function on $[a, b] \subset [0, 1]$ such that $f^{1/(d-1)}$ is concave and

$$f(t) \leq \text{vol}(A(t)) \quad \text{for every } t \in [a, b],$$

where $\text{vol}(\cdot)$ denotes the volume in hyperplanes orthogonal to $u$. Then there exists a body $B$ such that $B \subset A$, and for $t \in [a, b]$, the sections $B(t) = \{z \in B; \langle z, u \rangle = t\}$ are homothetic to $D$ and satisfy $f(t) = \text{vol}(B(t))$.

**Proof.** With the notation of Example (1), apply Theorem 2 to $E = \mathbb{R}^{d-1}$, $\mathcal{C} = \mathcal{C}_D$, and $p = p_D$. □

**Remark.** If the sections $A(t)$ of $A$ are not all homothetic, then Corollary 3 is no longer true, even if the problem is only to find a convex body $B \subset A$ such that $f(t) = \text{vol}(B(t))$ for every $t \in [a, b]$:

Embed $\mathbb{R}^{d-1}$ into $\mathbb{R}^d$ by $X = (X, 0)$, and suppose that $A = \text{conv}(A_0, A_1)$, where $A_0$ and $A_1$ are two nonhomothetic convex bodies of $\mathbb{R}^{d-1}$ and $\text{conv}(\cdot, \cdot)$ means convex hull. Then for $t \in [0, 1]$, $A(t), t = ((1-t)A_0 + tA_1, t)$, and if

$$f(t) = ((1-t)\text{vol}(A_0)^{1/(d-1)} + t\text{vol}(A_1)^{1/(d-1)})^{d-1},$$

then $f^{1/(d-1)}$ is concave (in fact, even affine) on $[0, 1]$, $f(0) = \text{vol}(A_0)$ and $f(1) = \text{vol}(A_1)$. By the Brunn-Minkowski theorem and its equality case, since $A_0$ and $A_1$ are not homothetic,

$$0 < f(t) < \text{vol}(A(t)) \quad \text{for every } t \in ]0, 1[, $$

But if a convex body $B$ in $\mathbb{R}^d$ satisfies $B \subset A$ and $\text{vol}(B(t)) \geq f(t)$ for every $t \in [0, 1]$, then clearly $B(0) = A_0$ and $B(1) = A_1$, so that $A = \text{conv}(A_0, A_1) \subset B \subset A$. It follows that $\text{vol}(B(t)) > f(t)$ for every $t \in ]0, 1[$.
Corollary 4. Let $A$ be a convex body in $\mathbb{R}^d$, and let $u \in S^{d-1}$. Let $f$ be a nonnegative continuous concave function on $[a, b]$ such that for every $t \in [a, b]$ we have

$$f(t) \leq h_m(\{z \in A; \langle z, u \rangle = t\}).$$

Then there exists a convex body $B \subset A$ such that $f(t) = h_m(\{z \in B; \langle z, u \rangle = t\})$ for every $t \in [a, b]$ and $\{z \in B; \langle z, u \rangle = t\} = \emptyset$ if $t \notin [a, b]$.

Proof. With the notation of Example (2), apply Theorem 2 to $E = \mathbb{R}^{d-1}$, $\mathcal{C} = \mathcal{H}^{d-1}$, and $p = h_m$ (mean width). □

Remark. The preceding corollary, applied to the case $d = 3$, allows us to find bodies inside $A$, with given (by $f$) perimeters of all the plane sections orthogonal to some fixed direction.

Let us apply Corollary 3 with $d = 2$: if $A$ is a convex body in $\mathbb{R}^2$, then for some interval $[a, b]$ of $\mathbb{R}$ and some concave continuous functions $g_1$ and $g_2$ on $[a, b]$ satisfying $g_1 + g_2 \geq 0$, we can write

$$A = \{(x, y); x \in [a, b], -g_2(x) \leq y \leq g_1(x)\}.$$

Now if $f$ is any concave continuous function on $[a, b]$ such that

$$0 \leq f(x) \leq g_1(x) + g_2(x) \quad \text{for every } x \in [a, b],$$

Corollary 3 says that there is a convex body $B \subset A$ such that for every $x \in [a, b]$,

$$f(x) = \max\{y; (x, y) \in B\} - \min\{y; (x, y) \in B\}.$$

This means also that there exists concave continuous functions $h_1$ and $h_2$ on $[a, b]$ such that $h_1 \leq g_1$, $h_2 \leq g_2$, and $f = h_1 + h_2$. This is the so-called Riesz decomposition property for concave continuous functions on $[a, b]$, with the pointwise order (clearly the hypothesis $g_1 + g_2 \geq 0$ is here irrelevant). The proof of Theorem 2 is inspired from this classical result in potential theory (see [M-S] and also [A]). Observe that the Riesz decomposition property can also be proved by using the unique decomposition of the extreme points of the order-segment $[0, g_1 + g_2] = \{f; f \text{ concave continuous}, 0 \leq f \leq g_1 + g_2\}$ and then the Krein-Milman theorem.

It should be noticed that this property is no longer true for functions of more than one variable:

Proposition 5. Let $K$ be a convex body in $\mathbb{R}^d$, $d \geq 1$. Suppose that the cone $V(K)$ of all the continuous concave functions on $K$ satisfies the Riesz decomposition property, that is: for any $f$, $g_1$, $g_2 \in V(K)$ such that $f \leq g_1 + g_2$, there exists $h_1$, $h_2 \in V(K)$ such that $h_1 \leq g_1$, $h_2 \leq g_2$, and $f = h_1 + h_2$. Then $d = 1$ (and $K$ is a segment).

Proof. We shall first prove that if $K$ is a convex body in $\mathbb{R}^2$, $V(K)$ does not satisfy this property; then we extend this result to any value of $d \geq 3$ (for $d = 1$, we refer to the above comments).

(1) Let $P$ be a convex body in $\mathbb{R}^2$, and select two points $X_1$ and $X_2$ in $P$ such that if $Y_1 Y_2$ is the chord of $P$ passing through $X_1$ and $X_2$, then $d(Y_1, X_1) = d(X_2, Y_2)$ (where $d(\ , \ )$ denotes the Euclidean distance); moreover, we can suppose that in the case when $P$ is a quadrangle $Y_1 Y_2$ is not a diagonal of $P$. 

Embed $\mathbb{R}^2$ into $\mathbb{R}^3$ by $X = (X, 0)$. We define a convex body $A$ of $\mathbb{R}^3$ by

$$A = \text{conv}(P, (X_1, 1), (X_2, -1))$$

and two functions $g_1, g_2 \in V(P)$ by the identity

$$A = \{ (X,t); X \in P, -g_2(X) < t < g_1(X) \}.$$

Let $a = g_1(X_2) = g_2(X_1) > 0$ and $C = \text{conv}(P, (X_1, 1 + a), (X_2, 1 + a))$, and for $X \in P$ define $f(X) = \max \{ t; (X, t) \in C \}$; it is clear that $f \in V(P)$, $f = 0$ on $\partial P$ (the boundary of $P$), $f(X_1) = f(X_2) = 1 + a$, and $0 \leq f \leq g_1 + g_2$.

Thus, if $f$ could be written as $f = h_1 + h_2$, with $h_i \in V(P)$ and $h_i \leq g_i$ for $i = 1, 2$, we would have $h_i = g_i$ on $\partial P$ and in $X_1, X_2$, for $i = 1, 2$, and thus $h_i = g_i$ on all $P$. Under our assumptions, this is impossible for the following reasons.

Since $X_1$ and $X_2$ are interior points of $P$, there exist two points $Z_1, Z_2 \in \partial P$, lying respectively in the two open half-planes separated by the line $X_1X_2$, such that the line through $Z_i$ parallel to $X_1X_2$ supports $P$ at $Z_i$, $i = 1, 2$. It follows that the hyperplane of $\mathbb{R}^3$ passing through $Z_i$, $(X_1, 1)$, and $(X_2, 1)$ supports $C$ in $Z_i$, $i = 1, 2$. Thus $f$ is affine on each of the triangles $Z_iX_1X_2$, $i = 1, 2$. The equality $f = h_1 + h_2 = g_1 + g_2$ implies then that $g_1$ and $g_2$ are affine on each of these triangles; but then, since the graphs of $g_1$ and $g_2$ are cones with respective vertices $(X_1, 1)$ and $(X_2, 1)$ and basis $P$, the segments $Y_iZ_j$ ($i, j = 1, 2$) would be in $\partial P$. This implies that $P$ is the quadrangle $Y_1Y_2Y_3Y_4$, which has $X_1X_2$ as a diagonal. We get a contradiction with the hypothesis.

(2) If $K$ is a convex body in $\mathbb{R}^d$, select a two-dimensional affine subspace $E$ of $\mathbb{R}^d$, passing through the interior of $K$, and define $P = E \cap K$. Let

$$A' = \text{conv}(K, (X_1, 1), (X_2, -1)),$$

and

$$C' = \text{conv}(K, (X_1, 1 + a), (X_2, 1 + a)),$$

where $X_1$ and $X_2$ are chosen in $P \subset E$, like they were in the preceding paragraph. Then it is clear that $A' \cap (E \times \mathbb{R}) = \text{conv}(P, (X_1, 1), (X_2, -1))$ and $C' \cap (E \times \mathbb{R}) = \text{conv}(P, (X_1, 1 + a), (X_2, 1 + a))$. If we define $f', g'_1, g'_2$ with respect to $C'$ and $A'$, we get $f'|_E = f$ and $g'|_E = g_i$, $i = 1, 2$, and we can apply the preceding result. □

Let $A$ be a convex body in $\mathbb{R}^d$, $H$ be some hyperplane of $\mathbb{R}^d$, and $\tilde{A}$ be the Steiner symmetral of $A$ with respect to $H$. If for some $u \in S^{d-1}$, $H = \{ x \in \mathbb{R}^d; \langle x, u \rangle = 0 \}$, let $P = \{ x \in H; x + tu \in A \text{ for some } t \in \mathbb{R} \}$ be the image of $A$ by the orthogonal projection onto $H$, and for $x \in P$, let $A(x) = \{ t \in \mathbb{R}; x + tu \in A \}$ ($A(x)$ is a segment); then we have

$$\tilde{A} = \left\{ x + \lambda u; x \in P, \lambda \in \frac{1}{2}(A(x) - A(x)) \right\}.$$

Given a convex body $C$ in $\mathbb{R}^d$, symmetric with respect to $H$ and such that $C \subset \tilde{A}$, does there exist a convex body $B \subset A$ such that its Steiner symmetral $B$ with respect to $H$ satisfies $\tilde{B} = C$? By Corollary 3 or 4, the answer is yes if $d = 2$; but Proposition 5 shows that it is generally no if $d \geq 3$. 

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One can ask, however, given the hyperplane $H$, what are the convex bodies $A$ in $\mathbb{R}^d$, $d \geq 3$, such that the following property holds:

\( (*) \) For every convex body $C \subset A$, symmetric with respect to $H$, there exists a convex body $B \subset A$ such that $B = C$.

If we suppose that $A$ is smooth and strictly convex, we have the following answer:

**Fact.** If $A = \{(X, t); X \in \mathcal{P}, t \in [-g_2(X), g_1(X)]\}$ as in the proof of Proposition 5 and if $A$ is smooth and strictly convex and satisfies $(*)$ with respect to the hyperplane $\mathbb{R}^{d-1} = \{t = 0\}$, then for some $\delta \geq 0$ and some affine function $u$ on $\mathbb{R}^{d-1}$ one has

$$g_2 + u = \delta(g_1 - u) \quad \text{on } \mathcal{P}.$$ 

**Proof.** By $(*)$, since $0 \leq g = g_1 + g_2$, there exists an affine function $u$ on $\mathbb{R}^{d-1}$ such that $-g_2 \leq u \leq g_1$ on $P$. Since $A$ is strictly convex, we have $g_1 = g_2 = u$ on $\partial P$. Changing $g_1$ into $g_1 - u$ and $g_2$ into $g_2 + u$, we can suppose that $u = 0$, so that $g_1 = g_2 = 0$ on $\partial P$. Now let $X_1 \neq X_2$ in the interior of $P$, and let $M_j = (X_j, g(X_j))$ for $j = 1, 2$. We define a concave function $f$ on $P$ by $\{(X, t); X \in P, 0 \leq t \leq f(X)\} = \text{conv}(M_1, M_2, \partial P)$. It follows from $(*)$ that $f = h_1 + h_2$ for some concave functions $h_i$ on $P$ such that $0 \leq h_1 \leq g_1$, $i = 1, 2$. Observe that $g_i(X_j) = h_i(X_j)$ for $i, j = 1, 2$. Let $E$ be some affine hyperplane through $M_1 M_2$ intersecting $\mathbb{R}^{d-1}$ on a tangent hyperplane to $P$ in some point $Y \in \partial P$. Then it is clear that $f$ and thus $h_1$ and $h_2$ must be affine on the triangle $X_1 X_2 Y$. Let $F$ be the two-dimensional affine space generated by these points, $G = P \cap F$, and $N_j = (X_j, h_1(X_j))$, $j = 1, 2$. Since $A$ is smooth, the convex body $G$ has a unique tangent line at $Y$, which is $T = E \cap F$. It follows that either the lines $M_1 M_2$, $N_1 N_2$, and $T$ are parallel or they intersect. Thus we get

$$\frac{(g_1 + g_2)(X_1)}{g_1(X_1)} = \frac{(g_1 + g_2)(X_2)}{g_1(X_2)}. \quad \Box$$

In the preceding proof, the hypotheses of smoothness and of strict convexity of $A$ can be replaced by the following assumptions: $\partial A$ does not contain any nontrivial segment orthogonal to $H$ and $\partial P$ is smooth. We conjecture that our characterization of property $(*)$ holds in the general case. It may be observed that if a convex body in $\mathbb{R}^d$, $d \geq 3$, satisfies $(*)$ with respect to any hyperplane $H$, then it follows from the Kakutani theorem that $A$ is an ellipsoid.

**Remark.** Let $A$ be given as in Corollary 3, and suppose 0 is in the interior of $D$. Then $A = \bigcup_{t \in [0, 1]}(h(t) + r(t)D, t)$ for some concave positive function $r: [0, 1] \rightarrow \mathbb{R}_+$ and some function $h: [0, 1] \rightarrow \mathbb{R}^{d-1}$. We proved in Corollary 3 that given a concave function $s: [0, 1] \rightarrow \mathbb{R}_+$ such that $0 \leq s \leq r$, there exists a function $k: [0, 1] \rightarrow \mathbb{R}_+$ such that $B = \bigcup_{t \in [0, 1]}(k(t) + s(t)D, t)$ is a convex body in $\mathbb{R}^d$ contained in $A$. One can ask also if, given a function $k: [0, 1] \rightarrow \mathbb{R}^{d-1}$, one can find some function $s: [0, 1] \rightarrow \mathbb{R}_+$ such that $B = \bigcup_{t \in [0, 1]}(k(t) + s(t)D, t)$ satisfies the same conclusion. It can be proved that the answer is positive if and only if

(a) If $k(t) = (k_1(t), \ldots, k_{d-1}(t))$, then the $k_i$, $i = 1, \ldots, d - 1$, are continuous and, as distributions, have second derivative $k_i''$ which are measures on $[0, 1]$. 

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(b) There exists $\alpha, \beta \in \mathbb{R}$ such that for every $x \in [0, 1],
\begin{align*}
N_D(k(x) - h(x)) + \alpha x + \beta + \int_0^1 G(x, y) d\mu(y) \leq r(x),
\end{align*}
where $N_D: \mathbb{R}^{d-1} \to \mathbb{R}^+$ is defined by $N_D(y) = \inf\{\lambda > 0; y \in \lambda D\}$, $\mu$ is the total variation measure on $[0, 1]$, with respect to $N_D$, of the vector measure $-k'' = (-k_1'', \ldots, -k_{d-1}'')$, and $G$ is the classical Green function of $[0, 1]: G(x, y) = x(1-y)$ if $0 < x \leq y \leq 1$ and $G(x, y) = y(1-x)$ if $0 \leq y < x \leq 1$. Observe that if the $k_i''$ have continuous density $f_i$ with respect to the Lebesgue measure $dy$ on $[0, 1]$, then
\begin{align*}
\int_0^1 G(x, y) d\mu(y) = \int_0^1 N_D((-f_i(x))_{i=1}^{d-1}) G(x, y) dy.
\end{align*}

REFERENCES


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