AN ANSWER TO A CONJECTURE
ON THE COUNTABLE PRODUCTS OF \( k \)-SPACES

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(Communicated by James E. West)

Abstract. In this paper the author shows:

Theorem (CH). There is a \( k_\omega \)-space \( X \) which is not locally compact but for which \( X^\omega \) has a \( k \)-system.

This answers a conjecture of Y. Tanaka

1. Introduction

Let \( \mathcal{C} \) be a compact covering of a space \( X \). Then \( X \) is said to have the weak topology with respect to \( \mathcal{C} \) if \( U \subseteq X \) is open (closed) in \( X \) whenever \( U \cap C \) is open (closed) in \( C \) for each \( C \in \mathcal{C} \). Following Arhangel'skii [1] such a covering is called a \( k \)-system. We recall that a space \( X \) is a \( k \)-space if it has the weak topology with respect to the cover consisting of all compact subsets of \( X \). Then a space with a \( k \)-system is precisely a \( k \)-space. If \( \mathcal{C} \) is point-countable (countable), \( X \) is said to have a point-countable \( k \)-system (be a \( k_\omega \)-space).

Recall a space \( X \) has countable tightness \( t(X) \leq \omega \) if whenever \( x \in X \), then \( x \in \overline{C} \) for some countable \( C \subset X \). Arhangel'skii [2] has shown that \( X^\omega \) has a \( k \)-system if \( X \) is a locally compact space. Y. Tanaka has shown that if \( X \) has a point-countable \( k \)-system and \( t(X) \leq \omega \), then \( X^\omega \) has a \( k \)-system if and only if \( X \) is locally compact. In [3–5] Tanaka posed the following question: Let \( X \) have a (point-) countable \( k \)-system. Then is \( X \) locally compact if \( X^\omega \) has a \( k \)-system? In this paper we give a negative answer under (CH). Namely, under (CH), there is a \( k_\omega \)-space \( X \) which is not locally compact, but \( X^\omega \) has a \( k \)-system.

2. Results

For the ordinal \( \omega_2 + 1 \), let \( Z \) be the topological sum of countably many copies of \([0, \omega_2]\) with the order topology. Then let \( X \) be the quotient space obtained from \( Z \) by identifying all the \( \omega_2 \)'s. Let \( f: Z \to X \) be a canonical projection. Then \( f \) is a closed mapping and \( X \) is a \( k_\omega \)-space. Let \( \mathcal{C} = \{ C_n = \bigcup_{i \leq n} [0, \omega_2]; n < \omega \} \); then \( \mathcal{C} \) is a countable \( k \)-system of \( Z \). It

Received by the editors September 20, 1989 and, in revised form, March 21, 1990.

1991 Mathematics Subject Classification. Primary 54D50; Secondary 54B10.

Key words and phrases. \( k \)-space, \( k \)-system, locally compact, ordinal.

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0002-9939/94 $1.00 + $.25 per page

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will not cause confusion if we denote a $k$-system of $X$ by $\mathcal{C}$. Let $\mathcal{C}^\omega = \{ \prod_{i<\omega} C_n ; C_n \in \mathcal{C} \}$; then $\mathcal{C}^\omega$ is a $k$-system of the space $X^\omega$ and $\mathcal{C}^\omega$ is a compact covering of the space $X^\omega$. $f : Z \to X$ is a continuous mapping, and so is $f^\omega = \prod_{i<\omega} f_i : Z^\omega \to X^\omega$, where $f_i = f$.

**Lemma 1.** Suppose $\mathcal{C}$ is a compact subset in $X^\omega$ or $Z^\omega$. Then $X^\omega$ or $Z^\omega$ has a $\prod_{i<\omega} C_n \in \mathcal{C}^\omega$ such that $C \subseteq \prod_{i<\omega} C_n$.

**Proof.** By [4, Lemma 6].

**Lemma 2.** Let $f^\omega : Z^\omega \to X^\omega$ be a continuous mapping. For any $U \subseteq X$,

$$(f^\omega)^{-1}(U) \text{ is open in } Z^\omega \iff (\prod_{i<\omega} C_n) \cap U \text{ is open in } \prod_{i<\omega} C_n \text{ for each }\prod_{i<\omega} C_n \in \mathcal{C}^\omega.$$

**Proof.** Suppose $(\prod_{i<\omega} C_n) \cap U$ is open in $\prod_{i<\omega} C_n$ for each $\prod_{i<\omega} C_n \in \mathcal{C}^\omega$ and $f^\omega|\prod_{i<\omega} C_n : \prod_{i<\omega} C_n \to \prod_{i<\omega} C_n$ is continuous. Let

$$A = \left( f^\omega \prod_{i<\omega} C_n \right)^{-1} \left[ \left( \prod_{i<\omega} C_n \right) \cap U \right] = (f^\omega)^{-1}(U) \cap \left( \prod_{i<\omega} C_n \right);$$

then $A$ is open in $\prod_{i<\omega} C_n \subseteq Z^\omega$. Since $Z^\omega$ has a $k$-system $\mathcal{C}^\omega$ by Lemma 1, $(f^\omega)^{-1}(U)$ is open in $Z^\omega$.

Now assume $(f^\omega)^{-1}(U)$ is open in $Z^\omega$. Then $A$ is open in $\prod_{i<\omega} C_n$ for each $\prod_{i<\omega} C_n \in \mathcal{C}^\omega$. Therefore $\prod_{i<\omega} C_n - A$ is compact in $\prod_{i<\omega} C_n$, and so is $f^\omega(\prod_{i<\omega} C_n - A) = \prod_{i<\omega} C_n ((\prod_{i<\omega} C_n) \cap U)$. Then $U \cap (\prod_{i<\omega} C_n)$ is open in $\prod_{i<\omega} C_n$.

**Theorem (CH).** The space $X$ is a $k_\omega$-space which is not locally compact. But $X^\omega$ has a $k$-system.

**Proof.** By Lemma 1 it suffices to prove that given $U \subseteq X^\omega$, if $U \cap (\prod_{i<\omega} C_n)$ is open in $\prod_{i<\omega} C_n$ for each $\prod_{i<\omega} C_n \in \mathcal{C}^\omega$, then $U$ is open in $X^\omega$. Let $x \in U$; the general proof follows from the next three cases.

(A) $x = (\alpha_i)$, $\alpha_i \neq \omega_j$, $i < \omega$. $(f^\omega)^{-1}(\alpha_i) = (f^\omega)^{-1}(U)$.

(B) $x = (\omega_2, \omega_2, \ldots) = (\omega_2)$. First we prove that there is an $N < \omega$ such that for each $\prod_{i<\omega} C_n \in \mathcal{C}^\omega$, there is a $\delta \subseteq \{0, \omega_2\}$ with $(\bigcup_{i \leq \eta_i} [\delta, \omega_2] i) \times \cdot \cdot \cdot \times \bigcup_{i \leq \eta_n} [\delta, \omega_2] i) \times \prod_{i > N} C_n \subseteq U$.

Suppose that for each $n < \omega$ there is a $\prod_{i<\omega} C_n \in \mathcal{C}^\omega$ such that for each $\delta \subseteq \{0, \omega_2\}$, $((\bigcup_{i \leq \eta_i} [\delta, \omega_2] i) \times \cdot \cdot \cdot \times \bigcup_{i \leq \eta_n} [\delta, \omega_2] i) \times \prod_{i > N} C_n) \subseteq U \neq \emptyset$.

Then there is a $y_\delta = (\delta_1, \delta_2, \ldots, \delta_n, x_\delta_1, x_\delta_2, \ldots)$ such that

$$y_\delta \in \left( \bigcup_{i < \eta_i} [\delta, \omega_2] i \right) \times \cdot \cdot \cdot \times \left( \bigcup_{i < \eta_n} [\delta, \omega_2] i \right) \times \prod_{i > N} C_n \subseteq U.$$

Let $B_n = \{ (\delta_1, \delta_2, \ldots, \delta_n, x_\delta_1, x_\delta_2, \ldots) ; \delta < \omega_2 \}. Then B_n \cap U = \emptyset$ and $B_n \subseteq \prod_{i<\omega} C_n$. We show $B_n \cap (\{ \omega_2 \} \times \{ \omega_2 \} \times \ldots \times \{ \omega_2 \} \times \prod_{i > N} C_n)$ is nonempty. In fact, suppose $B_n \cap (\{ \omega_2 \} \times \{ \omega_2 \} \times \ldots \times \{ \omega_2 \} \times \prod_{i > N} C_n) = \emptyset$. Since $B_n$ and $\{ \omega_2 \} \times \{ \omega_2 \} \times \ldots \times \{ \omega_2 \} \times \prod_{i > N} C_n$ are compact in $\prod_{i<\omega} C_n$, $\{ \omega_2 \} \times \ldots \times \{ \omega_2 \} \times \prod_{i > N} C_n$
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\[ \prod_{i>n} C_{n_i} \] has a neighborhood \( W = (\bigcup_{i \leq n_1} [\delta_0, \omega_2]) \times \cdots \times (\bigcup_{i \leq n_n} [\delta_0, \omega_2]) \) \times \prod_{i>n} C_{n_i} \) in \( \prod_{i<\omega} C_{n_i} \) such that \( W \cap \overline{B}_n = \emptyset \). On the other hand, we let \[ A_n = \{ (\delta_1', \delta_2', \ldots, \delta_n', x_{\delta_1}, x_{\delta_2}, \ldots) \in B_n; \delta_0 \leq \delta < \omega_2 \}; \]
then \( A_n \neq \emptyset \), \( A_n \subset B_n \), and \( A_n \subset W \). Therefore, \( A_n = A_n \cap B_n \subset W \cap \overline{B}_n = \emptyset \), a contradiction.

Hence there is a \( (\omega_2, \omega_2, \ldots, \omega_2, x_{\omega_1}, x_{\omega_2}, \ldots) \in \overline{B}_n \). Since \( (\prod_{i<\omega} C_{n_i}) \cap U \) is open in \( \prod_{i<\omega} C_{n_i} \), \( (\omega_2, \omega_2, \ldots, \omega_2, x_{\omega_1}, x_{\omega_2}, \ldots) \in \overline{B}_n \) and \( B_n \subset \prod_{i<\omega} C_{n_i} - [(\prod_{i<\omega} C_{n_i}) \cap U] \); hence, \( (\omega_2, \omega_2, \ldots, \omega_2, x_{\omega_1}, x_{\omega_2}, \ldots) \notin U \). Let \[ B = \{ (\omega_2, x_{\omega_1}, x_{\omega_2}, \ldots), (\omega_2, \omega_2, x_{\omega_1}, x_{\omega_2}, \ldots), \ldots \}; \]
then \( B \cap U = \emptyset \). \( \prod_1(B) = \{ \omega_2 \}, \prod_{i+1}(B) = \{ \omega_2, x_{\omega_1}, x_{\omega_2-1}, \ldots, x_{\omega_1} \}, i = 1, 2, \ldots \). For each \( i < \omega \), \( \prod_i(B) \) is finite in \( X \), so there is a \( C_{n_i} \in \mathcal{E} \) such that \( B \subset \prod_{i<\omega} C_{n_i} \). Since \( U \cap (\prod_{i<\omega} C_{n_i}) \) is open in \( \prod_{i<\omega} C_{n_i} \), we have \( \overline{B} \subset \prod_{i<\omega} C_{n_i} - [(\prod_{i<\omega} C_{n_i}) \cap U] \). This \( B \) is a sequence converging to \( (\omega_2, \omega_2, \ldots) \) in \( \prod_{i<\omega} C_{n_i} \). Indeed, take a neighborhood \( W = (\bigcup_{i \leq n_1} [\delta_0, \omega_2]) \times \cdots \times (\bigcup_{i \leq n_n} [\delta_0, \omega_2]) \times \prod_{i>n} C_{n_i} \) of \( (\omega_2, \omega_2, \ldots) \) in \( \prod_{i<\omega} C_{n_i} \). Then \( \prod_{i<n}(\omega_2, \omega_2, x_{\omega_1+1}, x_{\omega_2+1}, \ldots, \omega_2, \omega_2, x_{\omega_1+2}, x_{\omega_2+2}, \ldots, \ldots) \subset W \).
Therefore, \( (\omega_2, \omega_2, \ldots) \in B \subset \prod_{i<\omega} C_{n_i} - [(\prod_{i<\omega} C_{n_i}) \cap U], (\omega_2) \notin U \).
It is a contradiction with \( (\omega_2) \in U \). Second we look for a neighborhood \( W \) of \( (\omega_2) \) in \( X \) such that \( (\omega_2) \in W \subset U \). Let \( \mathcal{E} = \{ \prod_{i<\omega} C_{n_i} ; C_{n_i} \in \mathcal{E} \} = \{ K_\alpha ; \alpha < \omega_1 \}. \) By the previous arguments, there is an \( N < \omega \) such that for each \( K_\alpha = \prod_{i<\omega} C_{n_i} \) there is a \( \delta_\alpha < \omega_2 \) such that \( (\prod_{i \leq n_1} [\delta_0, \omega_2]) \times \cdots \times (\bigcup_{i \leq n_n} [\delta_0, \omega_2]) \times \prod_{i \geq N} C_{n_i} \subset U \). Let \( \delta = \sup_{\alpha < \omega_1} \delta_\alpha \); then \( \delta < \omega_2 \) and \( (\prod_{i \leq n_1} [\delta, \omega_2]) \times \cdots \times (\bigcup_{i \leq n_n} [\delta, \omega_2]) \times \prod_{i \geq N} C_{n_i} \subset U \) for any \( \prod_{i<\omega} C_{n_i} \in \mathcal{E} \).
Thus let \( V = \bigcup_{i \leq \omega} \prod_{i \leq \omega} [\delta, \omega_2] \), be a neighborhood of the point \( (\omega_2) \) in \( X \); then \( \bigcup_{\alpha < \omega_1} \prod_{i \leq \omega} [\delta, \omega_2] \times \prod_{i \geq N} C_{n_i} \subset W \subset U \).

(C) \( x = (\alpha_1, \omega_2, \alpha_2, \omega_2, \ldots, \alpha_n, \omega_2, \alpha_{n+1}, \omega_2, \ldots) \), \( \alpha_i \neq \omega_2 \). First we show that there is an \( N < \omega \) and a \( \delta_0 \in (0, \alpha_1) \), \( i = 1, 2, \ldots, N \), such that for each \( \delta_0' \in (\delta_i, \alpha_i) \), \( i = 1, 2, \ldots, N \), there is a \( V \in \mathcal{Z} \) with \( \{ \delta_0' \} \times V \times \{ \delta_0' \} \times \cdots \times \{ \delta_0' \} \times V \times X \times X \times \cdots \subset U \), where \( \mathcal{Z} = \{ \prod_{i \leq \omega} [\alpha, \omega_2] ; \alpha < \omega_2 \}. \) Suppose for each \( n < \omega \) and each \( \delta_0 \in (0, \alpha_1) \), \( i = 1, 2, \ldots, n \), there is a \( \delta_0^i \in (\delta_i, \alpha_i) \), \( i = 1, 2, \ldots, n \), such that for each \( V \in \mathcal{Z} \), \( \{ \delta_0^i \} \times V \times \{ \delta_0^i \} \times \cdots \times \{ \delta_0^i \} \times V \times X \times X \times \cdots - U \neq \emptyset \). Take any \( \delta_0 \in (0, \alpha_1) \). For each \( n < \omega \) and each \( \delta_0 \in (0, \alpha_1) \), \( i = 1, 2, \ldots, n \), there is a \( \delta_0^i \in (\delta_i, \alpha_i) \), \( i = 1, 2, \ldots, n \), such that for each \( V \in \mathcal{Z} \) there is an \( x_v = (\delta_1, \alpha_1, \delta_2, \alpha_2, \ldots, \delta_n, \alpha_n, x_{\omega_1}, x_{\omega_2}, \ldots) \) such that \( x_v \in \{ \delta_0^i \} \times V \times \{ \delta_0^i \} \times \cdots \times \{ \delta_0^i \} \times V \times X \times X \times \cdots - U \). Let \( B_n = \{ x_v = (\delta_1, \alpha_1, \delta_2, \alpha_2, \ldots, \delta_n, \alpha_n, x_{\omega_1}, x_{\omega_2}, \ldots) ; V \in \mathcal{Z} \}. \) There are two possible cases.

Case 1. \( |B_n| < \aleph_2 \). By \( |\mathcal{Z}| = \aleph_2 \) there is an \( x_v' \) which repeats at least \( \aleph_2 \) times in \( B_n \). Then the point \( x_v' = (\beta_1, \omega_2, \beta_2, \omega_2, \ldots, \beta_n, \omega_2, x_{\omega_1}, x_{\omega_2}, \ldots) \),

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If some $2^k$th ($k \leq n$) coordinate of $x_{\omega'}$ is not $\omega_2$, then let this $2^k$th ($k \leq n$) coordinate be $\alpha$ and $\mathcal{Z}_1 = \{V \in \mathcal{Z}; V \supseteq V_{\omega+1}\}$. Since $\mathcal{Z}$ is linearly ordered by inclusion, $|\mathcal{Z}_1| \leq \aleph_1$; therefore, $x_{\omega'}$ repeats at most $\aleph_1$ times in \(\{x_\nu \in B_n; V \in \mathcal{Z}_1\}\). Hence there is a $\delta \geq \alpha + 1$ such that $x_\delta = x_{\omega'}$. On the other hand, if $\alpha \in V_\delta = \bigcup_{i<\omega}[\delta, \omega_2]_i$, then $\alpha \geq \delta \geq \alpha + 1$, which is a contradiction. By the $x_{\omega'} \in B_n$ and $B_n \cap U = \emptyset$, it follows that $x_{\omega'} \notin U$.

**Case 2.** $|B_n| = \aleph_2$. Since $|\{\Pi_{i<\omega}C_n; C_n \in \mathcal{C}\}| = \aleph_1$, there is a $\Pi_{i<\omega}C_n \in \mathcal{C}$ such that $|\{(\Pi_{i<\omega}C_n) \cap B_n\}| = \aleph_2$. Let $\mathcal{Z}_1 = \{V \in \mathcal{Z}; x_\nu \in B_n \cap (\Pi_{i<\omega}C_n)\}$; then $\mathcal{Z}_1$ is cofinal in $\mathcal{Z}$. Let $A_n = \{x_\nu \in B_n \cap (\Pi_{i<\omega}C_n); V \in \mathcal{Z}_1\}$ (hence $A_n \subset \Pi_{i<\omega}C_n$) and $K = \{\alpha_1\} \times \{\omega_2\} \times \{\alpha_2\} \times \{\omega_2\} \times \cdots \times \{\alpha_n\} \times \{\omega_2\} \times \Pi_{i>n}C_n$; then we will show that $A_n \cap K \neq \emptyset$. Indeed, suppose $A_n \cap K = \emptyset$. Now, for any $x \in A_n \cap K$, there is a neighborhood $W = [\beta_1', \alpha_1'] \times (\bigcup_{i \leq n_1}[\delta, \omega_2]) \times \cdots \times [\beta_n', \alpha_n'] \times (\bigcup_{i \leq n_2}[\delta, \omega_2]) \times \Pi_{i \geq n_2}C_n$ of $K$ in $\Pi_{i<\omega}C_n$ such that $W \cap A_n = \emptyset$. Let $A_n^* = \{x_\nu \in A_n; V \subseteq \bigcup_{i<\omega}[\delta, \omega_2]\}$; then $A_n^* \subset W$ and $|A_n^*| = \aleph_2$. On the other hand, $A_n^* \subset A_n$, so $A_n^* = A_n^* \cap A_n \subset W \cap A_n = \emptyset$, a contradiction. Thus there is an $x_{\omega} \in C_{n+\omega}$ ($i < \omega$) such that

$$
(\alpha_1, \omega_2, \alpha_2, \omega_2, \ldots, \alpha_n, \omega_2, x_{\omega_1}, x_{\omega_2}, \ldots) \\
\in A_n \subset \prod_{i<\omega}C_n - \left[\left(\prod_{i<\omega}C_n\right) \cap U\right].
$$

Hence $(\alpha_1, \omega_2, \alpha_2, \omega_2, \ldots, \alpha_n, \omega_2, x_{\omega_1}, x_{\omega_2}, \ldots) \notin U$. Therefore, for either Case 1 or 2, there is a $\delta_i \in (\delta, \alpha_i)$, $i = 1, 2, \ldots, n$, such that $y_n = (\delta_n, \omega_2, \delta_n, \omega_2, \ldots, \delta_n, \omega_2, x_{\omega_1}, x_{\omega_2}, \ldots) \notin U$.

Let $B = \{y_n; n = 1, 2, \ldots\}$. Then $\Pi_1(B) = \{\delta_{i+1}, i < \omega\}$, $\Pi_2(B) = \{\omega_2\}$, $\Pi_{2n+1}(B) = \{x_1, 2n-1, x_2, 2n-3, \ldots, x_n, 1\} \cup \{\delta_{n+1}, n+1; i < \omega\}$, $n = 1, 2, \ldots$, $\Pi_{2n}(B) = \{x_1, 2n-2, x_2, 2n-4, \ldots, x_{n-1}, 2\} \cup \{\omega_2\}$, $n = 2, 3, \ldots$, and $B \cap U = \emptyset$. Obviously, for each $i < \omega$ there is a $C_m \in \mathcal{C}$ such that $\Pi_i(B) \subset C_m$; therefore, $B \subset \Pi_{i<\omega}C_m$. Let $K = (\delta_1, \alpha_1) \times (\omega_2) \times \cdots \times (\delta_n, \alpha_n) \times (\omega_2) \times (\delta_{n+1}, \alpha_{n+1}) \times (\omega_2) \times \cdots$; then $K \cap \bigcap_{i < \omega}C_m = \emptyset$. In fact, suppose $\bigcap_{i < \omega}C_m \subset K$, where we assume $K \subset \Pi_{i<\omega}C_m$. Then there is a neighborhood $W = [B_1, \alpha_1] \times (\bigcup_{i \leq m_1}[\delta, \omega_2]) \times \cdots \times [\beta_n, \alpha_n] \times (\bigcup_{i \leq m_2}[\delta, \omega_2]) \times \Pi_{i \geq n}C_m$ of $K$ in $\Pi_{i<\omega}C_m$ such that $W \cap A_n = \emptyset$. On the other hand, $\emptyset \neq \{y_n, \ldots, \} \subset B \cap W = \emptyset$, a contradiction. Take any $x \in K \cap \bigcap_{i < \omega}C_m \subset \emptyset$, $x = (\delta_1', \omega_2, \delta_2', \omega_2, \ldots, \delta_n', \omega_2, \ldots)$. By $U \cap \Pi_{i<\omega}C_m$ being open in $\Pi_{i<\omega}C_m$ and $B \subset \Pi_{i<\omega}C_m - (\{\Pi_{i<\omega}C_m\} \cap U)$, it follows that $(\delta_1', \omega_2, \ldots, \delta_n', \omega_2, \delta_{n+1}', \omega_2, \ldots) \in \bigcup_{i \leq \omega}C_m - \{\Pi_{i<\omega}C_m\} \cap U$. Therefore, $(\delta_1', \omega_2, \ldots, \delta_n', \omega_2, \delta_{n+1}', \omega_2, \ldots) \notin U$. Hence it is proven that for each $\delta_i \in (0, \alpha_i)$, $i < \omega$, there is a $\delta_i' \in (\delta_i, \alpha_i)$, $i < \omega$, such that $(\delta_i', \omega_2, \ldots, \delta_n', \omega_2, \omega_2, \ldots) \notin U$. Let $A = \{(\delta_1', \omega_2, \ldots, \delta_n', \omega_2, \ldots); \delta_i' \in (0, \alpha_i), i < \omega\}$; then $A \cap U = \emptyset$ and $A \subset K_1 = [0, \alpha_1] \times (\omega_2) \times \cdots \times [0, \alpha_n] \times (\omega_2) \times \cdots$. By $U \cap K_1$ being open in $K_1$, it follows that $(\alpha_1, \omega_2, \ldots, \alpha_n, \omega_2, \ldots) \in A \subset K_1 - (K_1 \cap U)$. This contradicts the fact
(\alpha_1, \omega_2, \ldots, \omega_n, \omega_2, \alpha_{n+1}, \omega_2, \ldots) \in U$. Second, we look for a neighborhood \(W\) of \((\alpha_1, \omega_2, \ldots, \alpha_n, \omega_2, \ldots)\) in \(X^\omega\) such that \(W \subset U\). We already showed that there is an \(N < \omega\) and a \(\delta_i \in (0, \alpha_i), \ i = 1, 2, \ldots, N\), such that for each \(\delta_i' \in (\delta_i, \alpha_i), \ i = 1, 2, \ldots, N\), there exists a \(V \in \mathcal{V}\) with \(\{\delta_i'\} \times V \times \{\delta_2'\} \times V \times \cdots \times \{\delta_N'\} \times V \times X \times X \times \cdots \subset U\). On the other hand,

\(|[0, \alpha_1] \times [0, \alpha_2] \times \cdots \times [0, \alpha_N]| \leq \aleph_1 < \aleph_2\).

Then for each \((\delta_1', \delta_2', \ldots, \delta_N') \in (\delta_1, \alpha_1) \times \cdots \times (\delta_N, \alpha_N)\), there is an \(\alpha < \omega_2\) such that \(\{\delta_1'\} \times V_\alpha \times \cdots \times \{\delta_N'\} \times V_\alpha \times X \times X \times \cdots \subset U\). Let \(\delta = \sup_{\alpha < \omega_1} \alpha\). Then \(\delta < \omega_2\), \(V_\delta \in \mathcal{V}\), and for each \((\delta_1', \delta_2', \ldots, \delta_N') \in (\delta_1, \alpha_1) \times \cdots \times (\delta_N, \alpha_N)\), \(\{\delta_1'\} \times V_\delta \times \cdots \times \{\delta_N'\} \times V_\delta \times X \times X \times \cdots \subset U\). Therefore, \((\delta_1, \alpha_1) \times V_\delta \times \cdots \times (\delta_N, \alpha_N) \times V_\delta \times X \times X \times \cdots \subset U\).

(D) In general, if \(x \in U\), we can show that \(U\) is a neighborhood of \(x\) in \(X^\omega\) by using the technique of (B) or (C).

Remark. In the previous theorem, we can replace "not locally compact" by "not even a \(k'\)-space". Indeed, let \(Z_0 = [1, \omega_0], Z_n = [0, \omega_2]^n\) for \(n = 1, 2, \ldots,\) with order topology. Let \(Z\) be the topological sum of \(Z_n, \ i < \omega\); then \(Z\) is a locally compact Hausdorff space. Let \(Y\) be the quotient space obtained from \(Z\) by identifying each "\(n\)" in \(Z_0\) and "\(\omega_2\)" in \(Z_n\) for \(n = 1, 2, \ldots\). Then \(Y\) is a \(k_\omega\)-space but not a \(k'\)-space. In fact, \(\omega_0 \notin K \cap (\bigcup_{n<\omega} [0, \omega_2]^n)\) for each compact subset \(K \subset Y\). Let \(g: Y \to Y/[1, \omega_0] = X\) be a canonical projection. Then \(g\) is a prefect mapping, as is \(g^\omega\). Since \(X^\omega\) has a \(k\)-system, so does \(Y^\omega\). Then \(Y\) has the desired property.

ACKNOWLEDGMENT

The author wishes to thank Y. Tanaka for his suggestions.

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