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DESCRIPTIONS OF CONDITIONAL EXPECTATIONS INDUCED BY NON-MEASURE-PRESERVING TRANSFORMATIONS

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ABSTRACT. Given a measure-preserving transformation T acting on a σ-finite measure space $(X, \mathscr{A}, m)$ and a σ-finite sigma algebra $\mathscr{B} \subset \mathscr{A}$, the conditional expectations $E(\cdot|\mathscr{B})$ acting on $L^\infty(\mathscr{A})$ and $E(\cdot|T^{-1}\mathscr{B})$ acting on $L^\infty(T^{-1}\mathscr{A})$ are known to be related by the formula $[E(f|\mathscr{B})] \circ T = E(f \circ T|T^{-1}\mathscr{B})$. In this note the conditional expectation $E(\cdot|T^{-1}\mathscr{B})$ is investigated in the non-measure-preserving case, and those transformations for which the above equation holds are characterized in terms of measurability conditions for $d(m \circ T^{-1})/dm$. It is precisely in the non-measure-preserving case that the measurability of $d(m \circ T^{-1})/dm$ plays an important role. Relatedly, it is shown that if composition by T intertwines $E(\cdot|\mathscr{B})$ and any mapping Λ, then Λ is a conditional expectation induced by a measure equivalent to m. These results were motivated by a result concerning induced conditional expectation operators on C^*-algebras, and the paper concludes with a brief description of this C^*-algebra setting.

PRELIMINARY REMARKS

Let $(X, \mathscr{A}, m)$ be a σ-finite space and $T: X \to X$ a measurable transformation. Throughout this paper we assume that $T^{-1}\mathscr{A} \subseteq \mathscr{A}$ and that $m \circ T \ll m$. We denote by h the Radon-Nikodym derivative $d(m \circ T^{-1})/dm$, and we will assume throughout that $h \in L^\infty$. We denote by $E^{\mathscr{B}}$ the conditional expectation $E(\cdot|\mathscr{B})$ considered as a bounded linear transformation from $L^\infty(\mathscr{A})$ onto $L^\infty(\mathscr{B})$. If $\mathscr{A} \supset \mathscr{B} \supset \mathscr{C}$, then $E^{\mathscr{C}}_{\mathscr{B}}$ denotes the appropriate conditional expectation from $L^\infty(\mathscr{B})$ onto $L^\infty(\mathscr{C})$. Note that $E^{\mathscr{C}}_{\mathscr{B}} E^{\mathscr{B}} = E^{\mathscr{C}}$. Finally, given a sigma algebra $\mathscr{B}$, we denote by $\mathscr{B}_n$ the sigma algebra $T^{-n}\mathscr{B}$.

Unless expressly stated otherwise, all functions and sets are assumed to be, or constructed so as to be, $\mathscr{A}$-measurable. All sigma algebras encountered are assumed to be sigma finite with respect to m. All set and function statements are to be interpreted as holding up to m-null sets. This includes statements regarding the disjointness of sets. For a given measurable function f, we let S_f be the support of f so that $S_f = \{f > 0\}$. The reader will note that at no point in the following discussion are an uncountable number of supports considered, and consequently none of the attendant "measurability versus

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selection" problems are encountered. Since S_h will be encountered often, we give it special notational status by calling it H.

The following facts will be applied often in this article:

- $T^{-1}H = X$ [4].
- $f \circ T = g \circ T$ if and only if $\chi_H f = \chi_H g$ [1]. (We shall apply this fact most often in the form: $f \circ T = 0$ if and only if $hf = 0$.)
- $E^{\mathscr{B}}(g) = g$ if and only if g is $\mathscr{B}$-measurable.
- If f is $\mathscr{B}$-measurable, then $E^{\mathscr{B}}(fg) = fE^{\mathscr{B}}(g)$.
- $|E^{\mathscr{B}}(fg)|^2 \le E^{\mathscr{B}}(|f|^2)\cdot E^{\mathscr{B}}(|g|^2)$ (conditional Cauchy-Schwarz inequality).
- If $f \ge 0$, then $E^{\mathscr{B}}(f) \ge 0$. If $f > 0$, then $E^{\mathscr{B}}(f) > 0$.
- For each nonnegative $\mathscr{A}$-measurable function f, $S_f \subset S_{E^{\mathscr{B}} f}$ [5, §1].
- For each $\mathscr{A}$-measurable set A, $S_{E^{\mathscr{B}} \chi_A}$ is the smallest $\mathscr{B}$-measurable set containing A [5, §1].

Main results

It is shown in [6] that if T is measure preserving (i.e., $h = 1$ a.e. dm), then for any $\mathscr{B}$,

$$\forall f \in L^\infty(\mathscr{B}), \quad (E^{\mathscr{B}} f) \circ T = E^{\mathscr{B}_1}_{\mathscr{A}_1}(f \circ T). \tag{1}$$

In particular, if T is measure preserving, then the mapping $\Lambda\colon L^\infty(\mathscr{A}_1) \to L^\infty(\mathscr{B}_1)$ given by $\Lambda(f \circ T) = (E^{\mathscr{B}} f) \circ T$ is well defined. Before considering the validity of (1) in situations where T is not measure preserving, we address the question of the existence of the map Λ. We shall see later that this question is related to the study of C^*-algebra conditional expectations.

Lemma 1. *Λ is well defined if and only if $H \in \mathscr{B}$.*

Proof. If Λ is well defined, then it is linear. Thus Λ is well defined if and only if whenever $f \in L^\infty(\mathscr{A})$ and $f \circ T = 0$, $(E^{\mathscr{B}} f) \circ T = 0$; equivalently, if $hf = 0$, then $hE^{\mathscr{B}} f = 0$, i.e., $\chi_H f = 0 \Rightarrow \chi_H E^{\mathscr{B}} f = 0$. Suppose first that H is in $\mathscr{B}$ and $\chi_H f = 0$. Then $E^{\mathscr{B}}(\chi_H f) = 0$. Since $H \in \mathscr{B}$, the required implication is established.

Now suppose that Λ is well defined. If $H = X$, then H is certainly $\mathscr{B}$-measurable. Assuming $m(X - H) > 0$, then $\chi_H(1 - \chi_H) = 0$, so $\chi_H E^{\mathscr{B}}(1 - \chi_H) = 0$. Thus $\chi_H = \chi_H E^{\mathscr{B}} \chi_H$. We may then apply $E^{\mathscr{B}}$ to both sides of the preceding equality and deduce that

$$E^{\mathscr{B}} \chi_H = E^{\mathscr{B}}(\chi_H E^{\mathscr{B}} \chi_H) = (E^{\mathscr{B}} \chi_H)^2.$$

Thus there is a $\mathscr{B}$-set H_1 such that $E^{\mathscr{B}} \chi_H = \chi_{H_1}$. It then follows (see the preliminary remarks) that $H_1 \supset H$. But this shows that $\chi_{H_1} - \chi_H \ge 0$, while

$$E^{\mathscr{B}}(\chi_{H_1} - \chi_H) = \chi_{H_1} - E^{\mathscr{B}} \chi_H = \chi_{H_1} - \chi_{H_1} = 0.$$

Thus, $H = H_1$, so in particular, $H \in \mathscr{B}$. □

Remark. The preceding result is summarized by the statement that the following diagram, wherein C_T is the operator of composition by T, is commutative exactly when $H \in \mathscr{B}$:

$$\begin{array}{ccc} L^\infty(\mathscr{A}) & \xrightarrow{C_T} & L^\infty(\mathscr{A}_1) \\ {\scriptstyle E^{\mathscr{B}}}\downarrow & & \downarrow{\scriptstyle \Lambda} \\ L^\infty(\mathscr{B}) & \xrightarrow[C_T]{} & L^\infty(\mathscr{B}_1) \end{array}$$

Later in this article we will briefly discuss C^*-algebra conditional expectations. This class includes all conditional expectations of the form $E^{\mathscr{C}}_{\mathscr{B}}$ (referred to henceforth as classical conditional expectations) and (when $H \in \mathscr{B}$) the map $\Lambda\colon L^\infty(\mathscr{A}_1) \to L^\infty(\mathscr{B}_1)$ defined above. Since $E^{\mathscr{B}_1}_{\mathscr{A}_1}$ is known to agree with Λ when T is measure preserving, it seems reasonable to examine the relationship between these mappings in general (with a proviso of course that H be in $\mathscr{B}$). To this end we now characterize $E^{\mathscr{B}_1}_{\mathscr{A}_1}$ in terms of h and $E^{\mathscr{B}}$. This characterization, presented in Proposition 3, does not assume the $\mathscr{B}$-measurability of H. We continue to use H_1 to denote the support of $E^{\mathscr{B}}\chi_H$ and note that since $T^{-1}H = X$, $T^{-1}H_1 = X$ as well. The following observation and its corollary will be called into use several times in this paper.

Lemma 2. *Let* $f \geq 0$. *Then* $S_{(E^{\mathscr{B}}f)\circ T} \supset S_{f\circ T}$.

Proof. $S_{E^{\mathscr{B}}_f} \supset S_f$ and consequently $T^{-1}S_{E^{\mathscr{B}}_f} \supset T^{-1}S_f$. But for any g, $T^{-1}S_g = S_{g\circ T}$. □

Corollary. $(E^{\mathscr{B}}h)\circ T > 0$ *a.e.*

Proposition 3. *For every* $f \in L^\infty(\mathscr{A})$,

$$E^{\mathscr{B}_1}_{\mathscr{A}_1}(f\circ T) = \frac{(E^{\mathscr{B}}(hf))\circ T}{(E^{\mathscr{B}}h)\circ T}.$$

Proof. Let $f \in L^\infty(\mathscr{A})$, and let $B \in \mathscr{B}$. Since $|E^{\mathscr{B}}(hf)|^2 \leq (E^{\mathscr{B}}h^2)(E^{\mathscr{B}}|f|^2)$, $S_{E^{\mathscr{B}}hf} \subset H_1$. Thus

$$\begin{aligned}\int_{T^{-1}B} E^{\mathscr{B}_1}(f\circ T)\,dm &= \int_{T^{-1}B} f\circ T\,dm = \int_B hf\,dm = \int_B E^{\mathscr{B}}(hf)\,dm \\ &= \int_B \chi_{H_1}E^{\mathscr{B}}(hf)\,dm \\ &= \int_{B\cap H_1} E^{\mathscr{B}}(hf)\,dm = \int_{B\cap H_1} E^{\mathscr{B}}h\frac{E^{\mathscr{B}}(hf)}{E^{\mathscr{B}}h}\,dm \\ &= \int_{B\cap H_1} h\frac{E^{\mathscr{B}}(hf)}{E^{\mathscr{B}}h}\,dm = \int_{T^{-1}(B\cap H_1)} \frac{(E^{\mathscr{B}}(hf))\circ T}{(E^{\mathscr{B}}h)\circ T}\,dm \\ &= \int_{T^{-1}B} \frac{(E^{\mathscr{B}}(hf))\circ T}{(E^{\mathscr{B}}h)\circ T}\,dm \quad \text{(noting that } T^{-1}H_1 = X\text{)}.\end{aligned}$$

Since $T^{-1}B$ is a generic member of $\mathscr{B}_1$, the proof is complete. □

The following corollary gives necessary and sufficient conditions for the validity of (1) in the non-measure-preserving case:

Corollary. *$E^{\mathscr{B}_1}_{\mathscr{A}_1}(f \circ T) = (E^{\mathscr{B}} f) \circ T$ for all $f \in L^\infty(\mathscr{A})$ if and only if h is $\mathscr{B}$-measurable.*

Proof. If h is $\mathscr{B}$-measurable, then by Proposition 3,

$$E^{\mathscr{B}_1}_{\mathscr{A}_1}(f \circ T) = \frac{(E^{\mathscr{B}}(hf)) \circ T}{(E^{\mathscr{B}} h) \circ T} = \frac{(h \circ T)(E^{\mathscr{B}} f) \circ T}{(h \circ T)} = (E^{\mathscr{B}} f) \circ T .$$

Conversely, suppose that the second and fourth expressions in the above equation are known to be equal for every bounded f in $L^\infty(\mathscr{A})$. Then, in particular (taking $f = h$), $(E^{\mathscr{B}} h^2) \circ T = (E^{\mathscr{B}} h)^2 \circ T$. This may be restated as $h(E^{\mathscr{B}} h^2) = h(E^{\mathscr{B}} h)^2$. We then apply $E^{\mathscr{B}}$ to deduce that $(E^{\mathscr{B}} h)(E^{\mathscr{B}} h^2) = (E^{\mathscr{B}} h)^3$. But since h is nonnegative, $E^{\mathscr{B}} h$ and $E^{\mathscr{B}}(h^2)$ have the same support (namely, H_1), so that $E^{\mathscr{B}}(h^2) = (E^{\mathscr{B}} h)^2$. We are now able to employ the variance type calculation:

$$\begin{aligned} 0 &\le E^{\mathscr{B}}((h - E^{\mathscr{B}} h)^2) = E^{\mathscr{B}}(h^2 - 2hE^{\mathscr{B}} h + (E^{\mathscr{B}} h)^2) \\ &= E^{\mathscr{B}}(h^2) - 2(E^{\mathscr{B}} h)^2 + (E^{\mathscr{B}} h)^2 = E^{\mathscr{B}}(h^2) - (E^{\mathscr{B}} h)^2 = 0 . \end{aligned}$$

But this is possible if and only if $h = E^{\mathscr{B}} h$ a.e., that is, h is $\mathscr{B}$-measurable. □

Our next result shows that (when defined) Λ *is* a classical conditional expectation, with respect to a measure equivalent to m. We will make use of the following test proved in [3, §3.4] for invariant measures for Markov operators (i.e., positive contractions) on an L^1 space. (In addition to the proof of this and related results, Krengel presents useful and interesting historical notes on these topics.)

Proposition 4. *Let M be a Markov operator on $L^1(X, \mathscr{C}, \nu)$, and let $C \in \mathscr{C}$. Then there exists a nonnegative L^1 function g with $C \subset S_g$ and $Mg = g$ if and only if for every nonnegative L^∞ function f with $\varnothing \neq S_f \subset C$, $\inf_{n \ge 0} \int_X M^{*n} f \, d\nu > 0$.*

Proposition 5. *Suppose that $H \in \mathscr{B}$. Then there exists a measure μ on $\mathscr{A}_1$ equivalent to $m|_{\mathscr{A}_1}$ such that Λ is the classical conditional expectation from $L^\infty(\mathscr{A}_1; \mu)$ onto $L^\infty(\mathscr{B}_1; \mu)$.*

Proof. Define the mapping L on $L^1(\mathscr{A}_1; m)$ by

$$L(g \circ T) = \frac{(E^{\mathscr{B}}(hg)) \circ T}{h \circ T} .$$

Then

$$\begin{aligned} \int_X |L(g \circ T)| \, dm &\le \int_X \frac{(E^{\mathscr{B}}(h|g|)) \circ T}{h \circ T} \, dm = \int_X \chi_H h \frac{E^{\mathscr{B}}(h|g|)}{h} \, dm \\ &= \int_H E^{\mathscr{B}}(h|g|) \, dm = \int_H h|g| \, dm = \int_X |g \circ T| \, dm , \end{aligned}$$

showing that L is a positive contraction on $L^1(\mathscr{A}_1; m)$. Moreover, for $f \circ T$ in $L^\infty(\mathscr{A}_1)$ and $g \circ T$ in $L^1(\mathscr{B}_1)$,

$$\begin{aligned} \int_X f \circ T L(g \circ T) \, dm &= \int_X f \circ T \frac{(E^{\mathscr{B}}(hg)) \circ T}{h \circ T} \, dm = \int_H f E^{\mathscr{B}}(hg) \, dm \\ &= \int_H h(E^{\mathscr{B}} f) g \, dm = \int_X (E^{\mathscr{B}} f) \circ T (g \circ T) \, dm . \end{aligned}$$

This shows that $\Lambda = L^*$. We may now apply Proposition 4 with $C = X$. Let $f \circ T$ be a nonnegative element of $L^\infty(\mathscr{A}_1; m)$ which is not identically zero. Then $\chi_H f$ is not identically zero, and since H is in $\mathscr{B}$, $\chi_H E^{\mathscr{B}} f = E^{\mathscr{B}}(\chi_H f)$ is not identically zero. This shows that $\Lambda(f \circ T) = (E^{\mathscr{B}} f) \circ T$ is nonnegative and is positive on a set of positive measure. Since for all $n \geq 1$, $\Lambda^n = \Lambda = L^*$, we see that $\inf_{n\geq 0} \int_X L^{*n} f\, dm > 0$. Thus there is a strictly positive $L^1(\mathscr{A}_1)$ function $g \circ T$ such that $L(g \circ T) = g \circ T$. Let $d\mu = g\, dm$, and let F be the classical conditional expectation from $L^\infty(\mathscr{A}_1; \mu)$ to $L^\infty(\mathscr{B}_1; \mu)$. (Since the two measures appearing in this argument are equivalent, there is no difference in their L^∞ spaces. However, the conditional expectation operators are intimately related to their respective measures). Then for any $f \circ T$ in $L^\infty(\mathscr{A}_1; \mu)$ and $B \in \mathscr{B}$,

$$\begin{aligned}
\int_{T^{-1}B} F(f \circ T)\, d\mu &= \int_{T^{-1}B} f \circ T\, d\mu = \int_{T^{-1}B} f \circ T g \circ T\, dm \\
&= \int_{T^{-1}B} f \circ T L(g \circ T)\, dm = \int_X \chi_{T^{-1}B} f \circ T L(g \circ T)\, dm \\
&= \int_X \Lambda(\chi_{T^{-1}B} f \circ T)(g \circ T)\, dm \\
&= \int_X \chi_{T^{-1}B} \Lambda(f \circ T)(g \circ T)\, dm = \int_{T^{-1}B} \Lambda(f \circ T)\, d\mu .
\end{aligned}$$

Since the integrands at both ends of this chain of equalities are $\mathscr{B}_1$ measurable and $T^{-1}B$ is an arbitrary $\mathscr{B}_1$ set, $F(f \circ T) = \Lambda(f \circ T)$. □

Remark. This proof assures us of the existence of an invariant measure for L but does not describe it explicitly. In Proposition 6 we are able to construct invariant measures for L directly because H is $\mathscr{B}$-measurable. Although the proof we have given of Proposition 5 is nonconstructive, it should allow generalizations to other situations.

Proposition 6. *Suppose that* $H \in \mathscr{B}$.

(a) *There is a function* g *such that* $g \circ T$ *is strictly positive and in* L^1, *and* hg *is* $\mathscr{B}$*-measurable.*
(b) $L(g \circ T) = g \circ T$.
(c) *If* $m(H) < \infty$, *then* g *may be chosen to be* $1/h \circ T$.

Proof. Since $\mathscr{B}$ is σ-finite and $H \in \mathscr{B}$, we may write H as the union of a sequence of disjoint sets of finite measure from $\mathscr{B}$; say $H = \bigcup_{i=1}^\infty B_i$. Choose $\{\alpha_1, \alpha_2, \dots\}$ to be a sequence of positive numbers so that the $\mathscr{B}$-measurable function $\sum_{i=1}^\infty \alpha_i \chi_{B_i}$ is in L^1. Define $g = (1/h)\sum_{i=1}^\infty \alpha_i \chi_{B_i}$. Then hg is $\mathscr{B}$-measurable and

$$\int_X g \circ T\, dm = \int_X hg\, dm = \int_X \sum_{i=1}^\infty \alpha_i x_{B_i}\, dm < \infty .$$

In order to establish the validity of part (b), note that since hg is $\mathscr{B}$-measurable,

$$L(g \circ T) = \frac{(E^{\mathscr{B}}(hg)) \circ T}{h \circ T} = \frac{(hg) \circ T}{h \circ T} = g \circ T .$$

Finally, suppose that $m(H)$ is finite. Then $\int_X (1/h \circ T)\, dm = m(H)$, and $hg = \chi_H$. □

C^*-ALGEBRA CONDITIONAL EXPECTATION OPERATORS

Let $\mathfrak{A}$ be a unital C^*-algebra with identity element 1, and let $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{A}$. A *conditional expectation operator from* $\mathfrak{A}$ *onto* $\mathfrak{B}$ is a mapping $\Phi: \mathfrak{A} \to \mathfrak{B}$ satisfying

(i) $\Phi(b) = b$ for all $b \in \mathfrak{B}$,
(ii) $\Phi(ab) = \Phi(a)b$ and $\Phi(ba) = b\Phi(a)$ for all $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$,
(iii) $\Phi(a)$ is positive for all positive $a \in \mathfrak{A}$,

i.e., a conditional expectation operator from $\mathfrak{A}$ onto $\mathfrak{B}$ is a positive, $\mathfrak{B}$-linear projection from $\mathfrak{A}$ onto $\mathfrak{B}$. (Recall that if $x \in \mathfrak{A}$, then x is called *positive* if $x = x^*$ and the spectrum of x lies on the nonnegative real axis or, equivalently, if $x = y^*y$ for some $y \in \mathfrak{A}$.) Henceforth, the notation $\Phi: \mathfrak{A} \to \mathfrak{B}$ will always imply that $\Phi(\mathfrak{A}) = \mathfrak{B}$. When $\mathfrak{A} = L^\infty(\mathscr{A})$ and $\mathfrak{B} = L^\infty(\mathscr{B})$, the mapping $E^{\mathscr{B}}$ is a conditional expectation operator. (Of course, many examples of conditional expectation operators between nonabelian C^*-algebras exist and are of importance in the study of these algebras. A good general reference is Stratila [7].)

The following lemma is formulated and proved in [2]. We reproduce the proof here for the convenience of the reader.

Let $\mathfrak{A}_1$ be a second C^*-algebra, and let $\pi: \mathfrak{A} \to \mathfrak{A}_1$ be a $*$-homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_1$. Then $\mathfrak{B}_1 = \pi(\mathfrak{B})$ is a C^*-subalgebra of $\mathfrak{A}_1$. If $\Phi(\ker \pi) \subset \ker \pi$, then Φ induces a mapping $\Phi_1: \mathfrak{A}_1 \to \mathfrak{B}_1$ defined by $\Phi_1(\pi(x)) = \pi(\Phi(x))$.

Lemma 7. *Suppose $\mathfrak{A}_1$ is a C^*-algebra and $\pi: \mathfrak{A} \to \mathfrak{A}_1$ is a unital $*$-homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_1$ such that $\Phi(\ker \pi) \subset \ker \pi$. Let $\mathfrak{B}_1 = \pi(\mathfrak{B})$. Then Φ_1 is a conditional expectation operator from $\mathfrak{A}_1$ onto $\mathfrak{B}_1$.*

Proof. If $x \in \mathfrak{A}$, $\Phi_1(\Phi_1(\pi(x))) = \Phi_1(\pi(\Phi(x))) = \pi(\Phi(\Phi(x))) = \pi(\Phi(x)) = \Phi_1(\pi(x))$, so Φ_1 is a projection. Also, b, $c \in \mathfrak{B}$,

$$\begin{aligned}\Phi_1(\pi(b)\pi(x)\pi(c)) &= \Phi_1(\pi(bxc)) = \pi(\Phi(bxc)) \\ &= \pi(b\Phi(x)c) = \pi(b)\pi(\Phi(x))\pi(c) = \pi((b)\Phi_1(\pi(x))\pi(c).\end{aligned}$$

Thus, Φ_1 is $\mathfrak{B}_1$-linear. Finally, $\Phi_1(\pi(x)^*\pi(x)) = \Phi_1(\pi(x^*x)) = \pi(\Phi(x^*x))$. Since Φ is positive and π is a homomorphism, we see that Φ_1 is a positive map. Thus, Φ_1 is a conditional expectation operator. □

Thus, the mapping $\Lambda: L^\infty(\mathscr{A}_1) \to L^\infty(\mathscr{B}_1)$ given by $\Lambda(f \circ T) = (E^{\mathscr{B}} f) \circ T$ is a conditional expectation operator whenever it is well defined. Indeed, we can formulate the following generalization of Lemma 1 in the C^*-algebra setting. It can be proved in the same manner as Lemma 1.

Lemma 8. *Suppose $\mathfrak{A}_1$ is a C^*-algebra and $\pi: \mathfrak{A} \to \mathfrak{A}_1$ is a unital $*$-homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_1$ such that there exists a selfadjoint idempotent $p \in \mathfrak{A}$ with the property that $\pi(x) = 0$ if and only if $px = 0$. Let $\Phi: \mathfrak{A} \to \mathfrak{B}$ be a conditional expectation operator. Suppose that p commutes with each element of $\mathfrak{B}$. Then the induced conditional expectation operator $\Phi_1: \mathfrak{A}_1 \to \mathfrak{B}_1$ is well defined if and only if $p \in \mathfrak{B}$, thus if and only if p lies in the center of $\mathfrak{B}$.*

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