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THE WEAK CONVERGENCE OF UNIT VECTORS TO ZERO IN HILBERT SPACE IS THE CONVERGENCE OF ONE-DIMENSIONAL SUBSPACES IN THE ORDER TOPOLOGY

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ABSTRACT. In this paper we deal with the (o)-convergence and the order topology in the hilbertian logic $\mathscr{L}(H)$ of closed subspaces of a separable Hilbert space H. We compare the order topology on $\mathscr{L}(H)$ with some other topologies. The main result is a theorem which asserts that the weak convergence of a sequence of unit vectors to zero in H is equivalent to the convergence of the sequence of one-dimensional subspaces generated by these vectors to the zero subspace in the order topology on $\mathscr{L}(H)$.

1. INTRODUCTION

The notion of (o)-*convergence* was introduced by G. Birkhoff (see [B1], [B2]) and, independently, by Kantorovich ([K]). Let $\mathscr{L}$ be a *quantum logic*, i.e., an *orthomodular lattice* (for definition see [V]). For any $a \in \mathscr{L}$ we denote by $a^\perp$ the *orthocomplement* of a. We say that the net a_α of elements of $\mathscr{L}$ (o)-converges to $a \in \mathscr{L}$ (written $a_\alpha \xrightarrow{(o)} a$), if there exist nets b_α, c_α such that $b_\alpha \leqq a_\alpha \leqq c_\alpha$ and $b_\alpha \nearrow a$, $c_\alpha \searrow a$. ($b_\alpha \nearrow a$, $c_\alpha \searrow a$ means that b_α is increasing, $\bigvee b_\alpha = a$, and c_α is decreasing, $\bigwedge c_\alpha = a$.) The *order topology* τ_o on $\mathscr{L}$ is the strongest topology such that (o)-convergence of a net implies the topological convergence. If $\mathscr{L}$ is separable (i.e. every set of pairwise orthogonal elements of $\mathscr{L}$ is at most countable), then in the definition of the order topology it suffices to use sequences instead of nets (see [S]). In this case, $a_n \xrightarrow{(o)} a$ is equivalent to the equality $\bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} a_k = a = \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} a_k$. The order topology was studied very intensively in recent years from various points of view, for example in [E], [EW]. The comparison with other topologies on $\mathscr{L}$ was studied in [PR1], [PR2], [PR3], [R1], [R2]. Relatively few results are known about the order topology on the *hilbertian logic* $\mathscr{L}(H)$ of closed subspaces of a separable infinite-dimensional Hilbert space H. If dim H is finite, then τ_0 is discrete on $\mathscr{L}(H)$. In most of the papers mentioned above there was used the assumption

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of (o)-continuity of L (it means that $x_\alpha \nearrow x$ implies $x_\alpha \wedge a \nearrow x \wedge a$), which is not satisfied on $\mathscr{L}(H)$ in the case of infinite dimension of H. This paper is a contribution to the study of order topology on $\mathscr{L}(H)$. In the following, the Hilbert space H is assumed to be infinite-dimensional, real or complex.

2. Comparison with other topologies on $\mathscr{L}(H)$

Let us denote $[H]$ the space of all bounded operators on H. Some known topologies are defined on $[H]$, for example the *uniform topology* and the *strong topology*. A sequence of operators A_n converges to operator A in the uniform topology, if $\|A_n - A\| \to 0$. $A_n \to A$ in the strong topology, if $\|A_n x - Ax\| \to 0$ for every $x \in H$. Identifying the closed subspace $M \subset H$ with the orthogonal projector P^M, projecting on M, we can consider $\mathscr{L}(H)$ as a subset of $[H]$. Denote by τ_{unif} the restriction of the uniform topology from $[H]$ to $\mathscr{L}(H)$ and by τ_{strong} the restriction of the strong topology from $[H]$ to $\mathscr{L}(H)$. Then, for M_n, $M \in \mathscr{L}(H)$,

$$M_n \to M \text{ in } \tau_{unif} \quad \text{iff} \quad \|P^{M_n} - P^M\| \to 0,$$

$$M_n \to M \text{ in } \tau_{strong} \quad \text{iff} \quad \|P^{M_n}x - P^M x\| \to 0 \text{ for every } x \in H.$$

In [P] was proved that τ_{strong} is metrizable and $M_n \to M$ in τ_{strong} iff $\mu(M_n) \to \mu(M)$ for every σ-additive measure μ on $\mathscr{L}(H)$. In the following if $M, N \in \mathscr{L}(H)$, $N \subset M$, then, for $M \wedge N^\perp$, we use the notation $M - N$. If $x \in H$ is nonzero, then $[x]$ denotes the one-dimensional subspace generated by x. For the zero vector and zero subspace we use the same symbol 0.

Lemma 2.1. *If* $\|x_n - x\| \to 0$, $x \neq 0$, *then* $[x_n] \to [x]$ *in* τ_{unif} (*and in* τ_{strong} *as well*).

Proof. It is a simple exercise (see also [P], Lemma 4.1).

Lemma 2.2. *If* $M_n, M \in \mathscr{L}(H)$, $M_n \xrightarrow{(o)} M$, *then* $M_n \to M$ *in* τ_{strong}.

Proof. Obviously, $M_n \nearrow M$ or $M_n \searrow M$ implies $M_n \to M$ in τ_{strong}. Let $M_n, M \in \mathscr{L}(H)$ be arbitrary, $M_n \xrightarrow{(o)} M$. There exist $A_n, B_n \in \mathscr{L}(H)$, $A_n \subset M_n \subset B_n$, $A_n \nearrow M$, $B_n \searrow M$. We obtain

$$\|P^{A_n}x - P^M x\| \to 0 \quad \text{and} \quad \|P^M x - P^{B_n}x\| \to 0 \tag{1}$$

for every $x \in H$. This implies $\|P^{B_n}x - P^{A_n}x\| \to 0$.

$$\begin{aligned} \|P^{B_n}x - P^{A_n}x\|^2 &= \|P^{B_n - A_n}x\|^2 \\ &= \|P^{B_n - M_n}x\|^2 + \|P^{M_n - A_n}x\|^2 \to 0. \end{aligned} \tag{2}$$

Then, according to (1) and (2), we obtain

$$\begin{aligned} \|P^{M_n}x - P^M x\| &\leqq \|P^{M_n}x - P^{A_n}x\| + \|P^{A_n}x - P^M x\| \\ &= \|P^{M_n - A_n}x\| + \|P^{A_n}x - P^M x\| \to 0, \end{aligned}$$

so $M_n \to M$ in τ_{strong}. The lemma is proved.

This lemma and the definition of the order topology imply immediately $\tau_{strong} \subset \tau_o$. The following two examples show that τ_o and τ_{unif} are not comparable.

Example 2.3. Let $\varphi_{k}{}_{k=1}^{\infty}$ be an orthonormal system in H and $M_n = \bigvee_{k=n}^{\infty}[\varphi_k]$, $n = 1, 2, \ldots$. $M_n \xrightarrow{(o)} 0$ and, hence, $M_n \to 0$ in τ_o, but $M_n \nrightarrow 0$ in τ_{unif}, because 0 is isolated point in τ_{unif}.

Example 2.4. Let φ_1, φ_2 be mutually orthogonal unit vectors and $\psi_n = \varphi_1 + \frac{1}{n}\varphi_2$, $n = 1, 2, \ldots$. Denote $B = \{[\psi_n];\ n = 1, 2, \ldots\}$. For any separable quantum logic $\mathscr{L}$, a subset $A \subset \mathscr{L}$ is closed in τ_o iff $a_n \in A$, $a_n \xrightarrow{(o)} a$ implies $a \in A$ ([B2]). We shall show that B is closed in τ_o. Let $M_n \in B$ be a sequence, which is not constant starting from any n. Then $\bigvee_{n=1}^{\infty}\bigwedge_{k=n}^{\infty} M_k$ is the zero subspace 0 and $\bigwedge_{n=1}^{\infty}\bigvee_{k=n}^{\infty} M_k$ is the two-dimensional subspace containing φ_1 and φ_2. So, M_n is not (o)-convergent. This implies that B is closed in τ_o and, hence, $[\psi_n] \nrightarrow [\varphi_1]$ in τ_o. However, $\|\psi_n - \varphi_1\| \to 0$ and, by Lemma 2.1, $[\psi_n] \to [\varphi_1]$ in τ_{unif} (and also in τ_{strong}).

Summarizing, we obtain

$$\tau_{strong} \subsetneqq \tau_{unif},$$
$$\tau_{strong} \subsetneqq \tau_o,$$
$$\tau_{strong} \subset \tau_{unif} \cap \tau_o.$$

Open problem. $\tau_{strong} = \tau_{unif} \cap \tau_o$?

3. The weak convergence of vectors as the convergence of corresponding subspaces in the order topology

A sequence φ_n converges to φ *weakly* in H, if the sequence of scalar products (φ_n, u) converges to (φ, u) for every $u \in H$.

Lemma 3.1. *If $\varphi_n \in H$, $\|\varphi_n\| = 1$, $n = 1, 2, \ldots$, then $\varphi_n \to 0$ weakly iff $[\varphi_n] \to 0$ in τ_{strong}.*

Proof. It follows from the equality $|(u, \varphi_n)| = \|P^{[\varphi_n]}u\|$, $u \in H$.

Lemma 3.2. *Let f_n be an orthonormal system in H and $g_n \in H$, $\|g_n\| = 1$, $\|f_n - g_n\| < 1/2^n$, $\bigvee_{i=1}^{n}[f_i] = \bigvee_{i=1}^{n}[g_i]$, $n = 1, 2 \ldots$, and $|(g_i, g_j)| < 1/2^{i+j}$, $i \neq j$. Then $[g_n] \xrightarrow{(o)} 0$.*

Proof. We have to show that $\bigwedge_{n=1}^{\infty}\bigvee_{i=n}^{\infty}[g_i] = 0$. We shall prove it by contradiction. Let us assume that there exists $u \in H$, $\|u\| = 1$, $u \in \bigvee_{i=n}^{\infty}[g_i]$ for every positive integer n. We shall show that this implies, for every integer $N > 0$ and real $\varepsilon > 0$, $\sum_{i=1}^{N}|(u, f_i)|^2 < \varepsilon^2$, which is a contradiction to the equality $1 = \|u\|^2 = \|\sum_{n=1}^{\infty}(u, f_n)f_n\|^2 = \sum_{n=1}^{\infty}|(u, f_n)|^2$. Let an arbitrary integer $N > 0$ and real ε,

$0 < \varepsilon < 1$, be given. Let us choose integer n_0 such that $n_0 > N$, $\frac{1}{2^{n_0-2}} < \frac{\varepsilon}{4}$ and

(1) $$\|u - \sum_{i=1}^{n_0}(u, f_i)f_i\| < \frac{\varepsilon}{4}.$$

We have $u \in \bigvee_{i=n_0}^{\infty}[g_i]$. Then there exist constants λ_i and integers $n_i \geqq n_0$, $i = 1, \dots, s$, such that

(2) $$\|u - \sum_{i=1}^{s}\lambda_i g_{n_i}\| < \frac{\varepsilon}{4}.$$

Since $\|u\| = 1$, we obtain from (2)

(3) $$(1 - \frac{\varepsilon}{4})^2 \leqq \|\sum_{i=1}^{s}\lambda_i g_{n_i}\|^2 \leqq (1 + \frac{\varepsilon}{4})^2.$$

Moreover,

(4) $$\|\sum_{i=1}^{s}\lambda_i g_{n_i}\|^2 = \sum_{\substack{i,j=1 \\ i\neq j}}^{s} \lambda_i\bar{\lambda}_j(g_{n_i}, g_{n_j}) + \sum_{i=1}^{s}|\lambda_i|^2.$$

Denote $d = \max\{|\lambda_i|^2;\ i = 1, \dots, s\}$.

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{s} \lambda_i\bar{\lambda}_j(g_{n_i}, g_{n_j}) \geqq -\sum_{\substack{i,j=1 \\ i\neq j}}^{s} |\lambda_i||\lambda_j||(g_{n_i}, g_{n_j}| \geqq -d\sum_{\substack{i,j=1 \\ i\neq j}}^{s} \frac{1}{2^{n_i+n_j}}$$
$$\geqq -d\sum_{i=n_0}^{\infty}\sum_{j=n_0}^{\infty}\frac{1}{2^i}\frac{1}{2^j} = -\frac{d}{2^{2n_0-2}}.$$

Hence, by (4),

$$\|\sum_{i=1}^{s}\lambda_i g_{n_i}\|^2 \geqq d(1 - \frac{1}{2^{2n_0-2}}) + (\sum_{i=1}^{s}|\lambda_i|^2 - d)$$

The immediate consequence is $\max|\lambda_i| = \sqrt{d} < 2$. In the opposite case,

$$\|\sum_{i=1}^{s}\lambda_i g_{n_i}\|^2 \geqq 4 - \frac{1}{2^{2n_0-4}} > (1 + \frac{\varepsilon}{4})^2,$$

which is a contradiction to (3). So, we have obtained

(5) $$\|\sum_{i=1}^{s}\lambda_i g_{n_i} - \sum_{i=1}^{s}\lambda_i f_{n_i}\| = \|\sum_{i=1}^{s}\lambda_i(g_{n_i} - f_{n_i})\| \leqq 2\sum_{i=n_0}^{\infty}\frac{1}{2^i} = \frac{1}{2^{n_0-2}} < \frac{\varepsilon}{4}.$$

From (1), (2), and (5) we obtain

$$\|\sum_{i=1}^{n_0}(u, f_i)f_i - \sum_{i=1}^{s}\lambda_i f_{n_i}\| < \varepsilon.$$

This implies

$$\|\sum_{i=1}^{n_0}(u, f_i)f_i - \sum_{i=1}^{s}\lambda_i f_{n_i}\|^2 = \sum_{i=1}^{n_0}|(u, f_i)|^2 + \sum_{i=1}^{s}|\lambda_i|^2 < \varepsilon^2.$$

Hence, $\sum_{i=1}^{N}|(u, f_i)|^2 \leqq \sum_{i=1}^{n_0}|(u, f_i)|^2 < \varepsilon^2$, which is the promised contradiction. Lemma 3.2 is proved.

This lemma enables us to claim the following theorem:

Theorem 3.3. *If $M_n \in \mathscr{L}(H)$ are one-dimensional, $n = 1, 2, \ldots$, then $M_n \to 0$ in τ_{strong} iff $M_n \to 0$ in τ_0.*

Proof. Since $\tau_{strong} \subset \tau_0$, the convergence in τ_0 implies the convergence in τ_{strong}. To show the opposite implication, it suffices to show that any sequence of one-dimensional subspaces converging to 0 in τ_{strong} contains a subsequence (o)-converging to 0. Let $\|\varphi_n\| = 1$, $n = 1, 2, \ldots$, $[\varphi_n] \to 0$ in τ_{strong}. We shall choose from $[\varphi_n]$ a subsequence (o)-converging to 0. By Lemma 3.1, $\varphi_n \to 0$ weakly. We shall prove that there exist a sequence g_n, which is a subsequence of φ_n with property $|(g_i, g_j)| < 1/2^{i+j}$, if $i \neq j$, and an orthonormal sequence f_n with the property

$$\bigvee_{i=1}^{n}[g_i] = \bigvee_{i=1}^{n}[f_i], \qquad n = 1, 2, \ldots, \|g_n - f_n\| < 1/2^n.$$

Sequences f_n, g_n will be constructed by mathematical induction. We use the Gramm-Schmidt orthogonalization process, modified slightly so that the sequence is orthogonalized before it is completely defined. Put $g_1 = f_1 = \varphi_1$. By assumption of the weak convergence of φ_n to 0, we have $(\varphi_n, \varphi_1) \to 0$. We can choose k_2 such that $|(\varphi_{k_2}, \varphi_1)| < 1/2^3$ and

$$\left\|\varphi_{k_2} - \frac{\varphi_{k_2} - (\varphi_{k_2}, \varphi_1)\varphi_1}{\|\varphi_{k_2} - (\varphi_{k_2}, \varphi_1)\varphi_1\|}\right\| < \frac{1}{2^2}.$$

We put $g_2 = \varphi_{k_2}$, $f_2 = (g_2 - (g_2, f_1)f_1)/\|g_2 - (g_2, f_1)f_1\|$. Then $\|g_2 - f_2\| < 1/2^2$, $|(g_1, g_2)| < 1/2^{1+2}$ and $(f_1, f_2) = 0$. If f_i, g_i with demanded properties are defined for $i = 1, 2, \ldots, m$, then from the weak convergence of φ_n to 0 there follows the existence of k_{m+1} such that

$$|(\varphi_{k_{m+1}}, \varphi_{k_i})| < \frac{1}{2^{m+1+i}}, \qquad i = 1, \ldots, m,$$

and

$$\left\|\varphi_{k_{m+1}} - \frac{\varphi_{k_{m+1}} - \sum_{i=1}^{m}(\varphi_{k_{m+1}}, f_i)f_i}{\|\varphi_{k_{m+1}} - \sum_{i=1}^{m}(\varphi_{k_{m+1}}, f_i)f_i\|}\right\| < \frac{1}{2^{m+1}}.$$

We put $g_{m+1} = \varphi_{k_{m+1}}$,

$$f_{m+1} = \frac{g_{m+1} - \sum_{i=1}^{m}(g_{m+1}, f_i)f_i}{\|g_{m+1} - \sum_{i=1}^{m}(g_{m+1}, f_i)f_i\|}.$$

Then $(f_{m+1}, f_i) = 0$, $i = 1, \ldots, m$. Obviously, $\bigvee_{i=1}^{n}[g_i] = \bigvee_{i=1}^{n}[f_i]$, $n = 1, 2, \ldots$, which follows from the properties of Gramm-Schmidt orthogonalization. This implies also $\bigvee_{i=1}^{\infty}[f_i] = \bigvee_{i=1}^{\infty}[g_i]$. Then, by Lemma 3.2, $[g_n] \xrightarrow{(o)} 0$. The theorem is proved.

Since a one-dimensional subspace can be represented with a unit vector, according to Lemma 3.1, we can formulate the result of the previous theorem as it is expressed in the title of this article.

Theorem 3.4. *A sequence φ_n of unit vectors converges to 0 weakly in H iff $[\varphi_n]$ converges to 0 in order topology in $\mathscr{L}(H)$.*

Let us remark that $[\varphi_n] \to 0$ in τ_0 iff it converges to 0 in $\tau_o \cap \tau_{unif}$. Thus, the result of Theorem 3.3 agrees with the conjecture $\tau_{strong} = \tau_o \cap \tau_{unif}$. The convergence in τ_0 and the (o)-convergence are not equivalent on an arbitrary quantum logic $\mathscr{L}$, in general. The same is true in the special case of $\mathscr{L}(H)$. We shall see it in the following example.

Example 3.4. Let φ_n be a complete orthonormal system in H. We shall show that the sequence $[(1/\sqrt{n})\varphi_1 + \varphi_{n+1}]$ does not (o)-converge to 0. For $n = 1, 2, \dots,$ denote by M_n the subspace generated by vectors $\varphi_2, \dots, \varphi_{n+1}$. For every $k = n+1, n+2, \dots,$ $\frac{1}{\sqrt{k}}\varphi_1 + \varphi_{k+1} \in M_n^\perp$. We show that $M_n^\perp = \bigvee_{k=n+1}^{\infty} [\frac{1}{\sqrt{k}}\varphi_1 + \varphi_{k+1}]$. Obviously, $M_n^\perp \supset \bigvee_{k=n+1}^{\infty} [\frac{1}{\sqrt{k}}\varphi_1 + \varphi_{k+1}]$. Let us assume that the opposite inclusion is not true. Then there exists $u \in M_n^\perp$, $u \neq 0$, $u \perp \frac{1}{\sqrt{k}}\varphi_1 + \varphi_{k+1}$, for $k \geqq n+1$. We have $u = (u, \varphi_1)\varphi_1 + \sum_{i=n+2}^{\infty} (u, \varphi_i)\varphi_i$. Then $0 = (u, \frac{1}{\sqrt{k}}\varphi_1 + \varphi_{k+1}) = \frac{1}{\sqrt{k}}(u, \varphi_1) + (u, \varphi_{k+1})$. Hence, $(u, \varphi_{k+1}) = -\frac{1}{\sqrt{k}}(u, \varphi_1)$. Since $u \neq 0$ implies $(u, \varphi_1) \neq 0$, we have obtained that the series of squares of $|(u, \varphi_{k+1})|$ is not convergent. This is a contradiction. Thus, $M_n^\perp = \bigvee_{k=n+1}^{\infty} [\frac{1}{\sqrt{k}}\varphi_1 + \varphi_{k+1}]$, and, hence,

$$0 \neq [\varphi_1] = \bigwedge_{n=1}^{\infty} M_n^\perp = \bigwedge_{n=1}^{\infty} \bigvee_{k=n+1}^{\infty} [\tfrac{1}{\sqrt{k}}\varphi_1 + \varphi_{k+1}].$$

So, $[\frac{1}{\sqrt{k}}\varphi_1 + \varphi_{k+1}]$ does not (o)-converge to 0. However, it converges to 0 in τ_0, because $\frac{1}{\sqrt{k}}\varphi_1 + \varphi_{n+1} \to 0$ weakly.

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