KAHLER MOISEZON SPACES
WHICH ARE PROJECTIVE ALGEBRAIC

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Abstract. The Moisezon theorem is extended to complex spaces with isolated singularities.

A problem of fundamental concern is determining sufficient conditions under which a complex space \( X \) is algebraic. Ideally, these conditions involve simple intrinsic analytic features of \( X \). An \( n \)-dimensional manifold \( X \) is called Kähler if it admits a hermitian metric whose associated \((1, 1)\)-form \( \omega \) is closed, and it is called Moisezon if it admits \( n \) algebraically independent global meromorphic functions. A fundamental theorem of Moisezon states that if a manifold \( X \) is Kähler and Moisezon, then \( X \) is a projective algebraic variety (i.e., \( X \) is the common zero locus of a set of homogeneous polynomials in projective space). The Kähler and Moisezon conditions can be generalized to singular complex spaces; however, it is no longer the case that these two conditions will imply algebraicity (an example is provided in [7]). In this paper we investigate situations under which a complex space \( X \) which is Kähler and Moisezon must be projective algebraic. For example, we show that if a compact space of dimension at least four is a complete intersection with isolated singularities, then the Kähler and Moisezon conditions imply projective algebraicity.

In Section 1 we investigate conditions which allow us to extend holomorphic line bundles over singular points of a complex space. In Section 2 we use a singular version of the Kodaira embedding theorem to reduce the problem to one of extending line bundles over singular points.

1. EXTENDING LINE BUNDLES OVER SINGULAR POINTS

Let \( X \) be a complex space with isolated singularities and structure sheaf \( \mathcal{O}_X \). Recall that an invertible sheaf is a locally free \( \mathcal{O}_X \)-module of rank one and that the set of these is the Picard group \( H^1(X, \mathcal{O}_X^*) \). We seek sufficient conditions for a holomorphic line bundle \( L \) on the regular part of \( X \) to extend to \( X \) as an invertible sheaf. For a small neighborhood \( U \) of a singularity \( x \in X \), the exact sequence of sheaves \( 0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0 \) and the exact sequence

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for cohomology with supports induce the following commutative diagram with exact rows and exact columns:

$$
\begin{array}{ccc}
H^1(U, \mathcal{O}_X) & \xrightarrow{\alpha} & H^1(U-x, \mathcal{O}_X) \\
\downarrow \gamma & & \downarrow \delta \\
H^1(U, \mathcal{O}_X^*) & \xrightarrow{\beta} & H^1(U-x, \mathcal{O}_X^*) \\
\downarrow & & \downarrow \\
H^2(U, \mathcal{O}_X) & \xrightarrow{\gamma} & H^2(U-x, \mathcal{O}_X) \\
\end{array}
$$

If $\beta$ is surjective, then any line bundle $L$ over $U-x$ must extend to $U$. Among the several possibilities, the most useful way to establish the surjectivity of $\beta$ is to suppose that $H^2(U, \mathcal{O}_X) = H^2(U-x, \mathcal{Z}) = 0$. In this case, $\alpha$ and $\delta$ are surjective, and the representative of $L$ in $H^1(U-x, \mathcal{O}_X^*)$ lifts to $H^1(U, \mathcal{O}_X)$ and maps down to $H^1(U, \mathcal{O}_X)$ via $\gamma$. By the commutativity of the diagram, this gives a line bundle over $U$ whose representative in $H^1(U, \mathcal{O}_X)$ restricts to the representative of $L$ in $H^1(U-x, \mathcal{O}_X^*)$. We have established the following lemma:

**Lemma 1.1.** If $X$ is a complex space with isolated singularities for which there exists a sufficiently small neighborhood $U$ of each singularity $x$ with $H^2(U, \mathcal{O}_X) = H^2(U-x, \mathcal{Z}) = 0$, then any holomorphic line bundle $L$ over the regular part of $X$ is the restriction of an invertible sheaf on $X$.

The cohomological conditions of Lemma 1.1 are known to hold under more natural conditions on $X$. After recalling some pertinent definitions, we shall recall results of this type.

**Definition 1.2.** The homological codimension of a coherent analytic sheaf $\mathcal{F}$ (denoted $\text{codh}_X \mathcal{F}$) over an $n$-dimensional complex space $X$ at a point $x$ is $n-k$ where $k$ is defined as the smallest integer such that there exists a resolution of $\mathcal{F}$

$$
0 \rightarrow P_k \xrightarrow{d_k} P_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_0} P_0 \xrightarrow{d_0} \mathcal{F}_x \rightarrow 0
$$

where each $P_i$ is projective. The homological codimension of a sheaf is defined by $\text{codh} \mathcal{F} := \inf_{x \in X} \text{codh}_x \mathcal{F}$, and the homological codimension of $X$, $\text{codh} X$, is the homological codimension of the structure sheaf $\mathcal{O}_X$.

High homological codimension is useful in establishing vanishing of cohomology with supports. In particular, the following theorem shows that $\text{codh} X \geq 3$ implies the vanishing of $H^i_x(U, \mathcal{O}_X)$ as required in Lemma 1.1.

**Theorem 1.3** [9, Theorem 1.14]. Let $(X, \mathcal{O}_X)$ be a complex space and $x \in X$. Then for $U$ a small neighborhood of $x$, $H^i_x(U, \mathcal{O}_X) = 0$ for $i \leq q$ iff $\text{codh} X \geq q + 1$.

Results of Hamm provide natural conditions under which $H^2(U-x, \mathcal{Z})$ vanishes. Recall that $U$ is biholomorphic to the common zero locus of $s$ analytic equations in $\mathbb{C}^N$. Let $N(x)$, the embedding dimension at $x$, be the least possible $N$, and accordingly let $r(x) := N(x) - s(x)$.
Theorem 1.4 [3, 4]. Let \( X \) be an \( n \)-dimensional irreducible complex space with isolated singularities. For a small enough neighborhood \( U \) of \( x \), \( H^i(U - x, \mathbb{Z}) = 0 \) if either \( 0 < i \leq r(x) - 2 \) or \( 0 < i \leq 2n - N(x) - 1 \).

Summarizing we have

Theorem 1.5. Suppose \( X \) is an \( n \)-dimensional complex space with isolated singularities such that \( \text{codh} X \geq 3 \) and either \( r(x) \geq 4 \) or \( 2n - N(x) \geq 3 \) at all singular points \( x \). If \( L \) is a holomorphic line bundle on the regular part of \( X \), then \( L \) is the restriction of an invertible sheaf on \( X \).

Recall that a complex space is called Cohen-Macaulay if \( \text{codh}_x \mathcal{O}_X = \dim_x X \) at all points \( x \in X \).

Corollary 1.6. Suppose \( X \) is a complex space of dimension \( n \geq 4 \) with isolated singularities which is Cohen-Macaulay, and suppose that either \( r(x) \geq 4 \) or \( 2n - N(x) \geq 3 \) for all singular points \( x \). Then any holomorphic line bundle on the regular part of \( X \) is the restriction of an invertible sheaf on \( X \).

A complex space \( X \) of dimension \( n \) is called a local complete intersection if \( r(x) = n \) for all \( x \in X \).

Corollary 1.7 [2, Theorem 3.13.ii]. If a complex space \( X \) is a local complete intersection of dimension at least four, then any holomorphic line bundle \( L \) on the regular part of \( X \) is the restriction of an invertible sheaf on \( X \).

Proof. By [1, p. 200], local complete intersections are Cohen-Macaulay. \( \Box \)

2. Projective embeddings of spaces with isolated singularities

A function \( f \) on a singular space \( X \) is meromorphic (respectively plurisubharmonic, pluriharmonic) if at all points \( x \in X \) there exists a neighborhood \( U \) of \( x \) embedded in \( \mathbb{C}^N \) such that \( f \) is the restriction of a meromorphic (respectively plurisubharmonic, pluriharmonic) function on \( \mathbb{C}^N \). A compact irreducible complex space \( X \) is called Moishezon if the transcendence degree of the field of meromorphic functions on \( X \) equals the complex dimension of \( X \). A Kähler space is a complex space \( X \) such that there exists an open covering \( \mathcal{U} \) of \( X \) and strictly plurisubharmonic smooth functions \( \phi_U \) for each \( U \in \mathcal{U} \) such that \( \phi_U - \phi_V \) is pluriharmonic on \( U \cap V \) for all \( U, V \in \mathcal{U} \). These definitions generalize the standard definitions of Moishezon and Kähler manifolds; in particular, the forms \( \sqrt{-1} \partial \bar{\partial} \phi_U \) piece together to be a Kähler metric on the regular part of \( X \).

A previous result of the author provides a Kodaira-type embedding theorem for Kähler spaces with isolated singularities:

Theorem 2.1 [10]. Let \( X \) be a compact normal Kähler space with isolated singularities. Suppose \( X \) admits an invertible sheaf \( \mathcal{L} \) whose restriction to the regular part of \( X \) is a positive holomorphic line bundle \( L \). Then \( X \) is a projective algebraic variety.

Remark 2.2. Without the Kähler hypothesis, a theorem of Riemenschneider [8] shows that \( X \) is Moishezon.
The usefulness of this theorem is that no notion of positivity of a line bundle \( L \) is required near a singular point \( x \) of \( X \). All that is necessary is that \( L \) extends over \( x \) and defines rational maps \( \sigma_{L^p} : X \to \mathbb{P}^N \) given by global holomorphic sections of \( L^p \).

We now construct a positive line bundle on the regular part of a complex space which is both Kähler and Moishezon.

**Theorem 2.3** [6]. Every compact irreducible Moishezon space \( X \) has a projective algebraic desingularization \( \pi : \tilde{X} \to X \). In fact, there exists a finite sequence of monoidal transformations \( \phi : X_{i+1} \to X_i \), \( 0 \leq i < r \), such that

(i) \( X_0 = X \);

(ii) the center \( Z_i \) of \( \phi_i \) is a nonsingular projective algebraic variety which is nowhere dense in \( X_i \);

(iii) \( Z_i \subseteq \{ \text{the singular set of } X_i \} \) if \( X_i \) is singular;

(iv) \( X_r \) is a projective algebraic manifold.

**Theorem 2.4.** Suppose \( X \) is a normal irreducible complex space with isolated singularities such that any holomorphic line bundle on the regular part of \( X \) is the restriction of an invertible sheaf on \( X \). If \( X \) is Kähler and Moishezon, then \( X \) is projective algebraic.

*Proof.* By Theorem 2.3, there exists a desingularization \( \pi : \tilde{X} \to X \) with exceptional set \( E \) and \( \pi(E) \) supported on the singular set of \( X \). Since \( \tilde{X} \) is both Kähler and Moishezon, \( \pi \) is in fact a projective algebraic desingularization. Therefore, the positive hyperplane bundle \( O(1) \) over the Zariski open set \( \tilde{X} - E \) of the projective algebraic manifold \( \tilde{X} \) pushes forward to be a positive line bundle on the regular part of \( X \). The result follows from Theorem 2.1. □

We combine Theorem 2.4 with the results of Section 1 to assert our main results.

**Theorem 2.5.** Let \( X \) be a normal irreducible \( n \)-dimensional complex space with isolated singularities such that \( \text{codh } X \geq 3 \) and either \( r(x) \geq 4 \) or \( 2n - N(x) \geq 3 \) at all singular points \( x \in X \). If \( X \) is Kähler and Moishezon, then \( X \) is a projective variety.

*Proof.* The theorem follows from Theorem 1.5 and Theorem 2.4. □

**Remark 2.6.** When \( \text{codh } X \geq 4 \), the normality assumption found in Theorem 2.5 is automatically satisfied [5].

**Corollary 2.7.** Let \( X \) be an irreducible Cohen-Macaulay complex space with isolated singularities and of dimension at least four. Suppose that either \( r(x) \geq 4 \) or \( 2n - N(x) \geq 3 \) for all singular points \( x \in X \). If \( X \) is Kähler and Moishezon, then \( X \) is a projective algebraic variety.

**Corollary 2.8.** Suppose \( X \) is an irreducible local complete intersection of dimension at least four and with isolated singularities. If \( X \) is Kähler and Moishezon, then \( X \) is a projective algebraic variety.

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