D-SETS AND BG-FUNCTORS IN KAZHDAN-LUSZTIG THEORY

YI MING ZOU

(Communicated by Roe Goodman)

Abstract. By using Deodhar's combinatorial setting and Bernstein-Gelfand projective functors, this paper provides some necessary and sufficient conditions for a highest weight category to have a Kazhdan-Lusztig theory. A consequence of these conditions is that in the semisimple Lie algebra case, the Kazhdan-Lusztig conjecture on the multiplicities of a Verma module implies the nonnegativity conjecture on the coefficients of Kazhdan-Lusztig polynomials.

One of the central topics in representation theory in recent years is the so-called Kazhdan-Lusztig theory. The Kazhdan-Lusztig polynomials play a key role in this theory. These polynomials can be defined by using a distinguished basis of the Hecke algebra associated to a Coxeter group. In [KL1], there are two conjectures about these polynomials: (a) For any Coxeter group, the coefficients of these polynomials are nonnegative integers; (b) If the Coxeter group is the Weyl group of a complex semisimple Lie algebra, then the multiplicities of the composition series of a Verma module are given by the values of these polynomials at 1. Conjecture (b) is usually referred to as the Kazhdan-Lusztig conjecture and was proven in [BB] and [BK] shortly thereafter. Conjecture (a) is now known to be true for all crystallographic Coxeter groups (for a more up-to-date reference on recent developments of Kazhdan-Lusztig theory, we refer to [DS]). It was shown in [D] that if the coefficients of the Kazhdan-Lusztig polynomials of a Coxeter group are nonnegative, then these polynomials can be defined by using certain sets derived from the elements of the Coxeter group. In fact, these sets give a closed formula for the Kazhdan-Lusztig polynomials under the nonnegativity assumption (see [D]). Since the Kazhdan-Lusztig polynomials are not easy to get at in general, the results in [D] give strong evidence for the importance of the nonnegativitiness. In an attempt to understand the results of [D], we observed that in the semisimple Lie algebra case, conjecture (b) implies conjecture (a). The connection is provided by some tensor functors called projective functors defined in [BG]. In this paper, we will give some necessary and sufficient conditions for the validity of the Kazhdan-Lusztig conjecture in certain special cases of the highest weight categories defined by CPS (see [CPS1]).

Received by the editors June 2, 1993; the contents of this paper have been presented to the Nineteenth Holiday Symposium held in December 1992 at New Mexico State University.

1991 Mathematics Subject Classification. Primary 22E47, 17B10; Secondary 22E46, 17B35.
Key words and phrases. D-sets, BG-functors.
and [CPS2]). In particular, our results will show that in the semisimple Lie algebra case, conjecture (b) implies conjecture (a).

This paper is arranged as follows: In section 1, we recall the definition of the highest weight categories, D-sets and BG-functors. In section 2, we discuss the relationship between D-sets and BG-functors. Some necessary and sufficient conditions for some special highest weight categories to have a Kazhdan-Lusztig theory (in the sense of [DS]) were given in section 3.

1. Definitions and notation

1.1. Highest weight categories. We recall the definition of highest weight categories given by CPS.

Let $\mathcal{C}$ be an abelian category over a field $F$. Then $\mathcal{C}$ is called locally artinian if it admits arbitrary union of its subobjects of finite length. In addition, we assume that $\mathcal{C}$ satisfies the Grothendieck condition: $B \cap (\cup A_a) = \cup (B \cap A_a)$ for a subobject $B$ and a family of subobjects $\{A_a\}$ of an object $X$. A composition factor $S$ (also called a subquotient) of an object $A$ in $\mathcal{C}$ is a composition factor of a subobject of finite length. The multiplicity of $S$ in $A$, denoted $[A : S]$, is defined to be the supremum of the multiplicity of $S$ in all subobjects of $A$ of finite length. A poset $\Lambda$ is said to be interval-finite provided that, for every $\mu \leq \lambda$ in $\Lambda$, the “interval” $[\mu, \lambda] = \{\tau \in \Lambda : \mu \leq \tau \leq \lambda\}$ is finite.

**Definition** (cf. [CPS1]). A locally artinian category $\mathcal{C}$ over $F$ is called a highest weight category if there exists an interval-finite poset $\Lambda$ (the “weights” of $\mathcal{C}$) satisfying the following conditions:

(a) There is a complete collection $\{L(\lambda)\}_{\lambda \in \Lambda}$ of nonisomorphic simple (irreducible) objects of $\mathcal{C}$ indexed by the set $\Lambda$.

(b) There is a collection $\{M(\lambda)\}_{\lambda \in \Lambda}$ of objects of $\mathcal{C}$ and, for each $\lambda$, a subobject $M'(\lambda) \subseteq M(\lambda)$ such that $M(\lambda)/M'(\lambda) \cong L(\lambda)$ and all composition factors $L(\mu)$ of $M'(\lambda)$ satisfy $\mu < \lambda$. For $\lambda, \mu \in \Lambda$, we have that $\dim F \text{Hom}_F(M(\lambda), M(\mu))$ and $[M(\lambda) : L(\mu)]$ are finite.

(c) Each simple object $L(\lambda)$ has a projective cover $P(\lambda)$ in $\mathcal{C}$. Also, each $P(\lambda)$ has a filtration

$$P(\lambda) = P^0 \supset P^1 \supset P^2 \supset \cdots \supset P^n \supset P^{n+1} = (0)$$

such that:

(i) $P^0/P^1 \cong M(\lambda)$,

(ii) for $k \geq 1$, $P^k/P^{k+1} \cong M(\mu_k)$ for some $\mu_k > \lambda$.

We call the objects $M(\lambda)$ ($\lambda \in \Lambda$) described in (b) Verma modules. For $\mu \in \Lambda$, we will use the notation $P(\lambda) : M(\mu)$ to denote the number of $i$'s in $[0, n]$ such that $P^i/P^{i+1} \cong M(\mu)$.

**Remark.** Our highest weight categories are actually the “duals” of some special highest weight categories defined in [CPS1]; the highest weight category of CPS is more general. For examples of highest weight categories, we refer to [CPS1] and [CPS2].

1.2. Coxeter groups and Hecke algebras. We further assume that there is a Coxeter group $(W, S)$ ($W$ is the group, $S$ is the generating set, see [H] for the definition of a Coxeter group) associated to our highest weight category $\mathcal{C}$ such that $W$ acts on $\Lambda$ and satisfies the following conditions:
(i) for each $x \in \Lambda$, there is a unique subgroup $W_x$ of $W$ such that for any $x \in W_x$, $[M(x \cdot \lambda), L(\mu)] \neq 0$ only if $\mu = y \cdot \lambda$ for some $y \in W_x$.
(ii) if $\lambda_0 \in W_x \cdot \lambda$ is a maximal element, then $x \cdot \lambda_0 \geq y \cdot \lambda_0$ or $x \leq y$ in the Chevalley order (known as Bruhat order before) of $W$.

The Coxeter groups which we will be interested in are just Weyl groups.

Let $\mathbf{H}$ be the Hecke algebra associated to $(W, S)$; then $\mathbf{H}$ is a free $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ module with basis $\{T_x : x \in W\}$ and multiplication defined by:

$$T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w), \\ qT_{ws} + (q-1)T_w & \text{if } \ell(ws) < \ell(w), \end{cases}$$

where $w \in W$, $s \in S$, and $\ell$ is the length function of $W$. Recall that the following elements of $\mathbf{H}$

$$C_y = q^{-\ell(y)/2} \sum_{x \leq y} P_{x,y} T_x, \quad y \in W,$$

form a distinguished basis of $\mathbf{H}$, where $P_{x,y}$ are the Kazhdan-Lusztig polynomials.

1.3. **Duality condition.** Let $\mathbf{C}$ be a highest weight category. Then $\mathbf{C}$ is said to satisfy the duality condition provided that for $\lambda, \mu \in \Lambda$,

$$(1.3.1) \quad [M(\lambda) : L(\mu)] = (P(\mu) : M(\lambda)).$$

1.4. **D-sets.** We recall some notions introduced in [D].

Let $(W, S)$ be a Coxeter group. Let $e$ be the identity of $W$. For any $y \in W$ and a fixed reduced expression $y = s_1 s_2 \cdots s_k$, $s_i \in S$, $i = 1, \ldots, k$, we define $\mathcal{S}_y$ to be the set which consists all $k$-tuples $x = (x_1, x_2, \ldots, x_k)$ such that $x_i = s_i$ or $e$, $i = 1, 2, \ldots, k$. Thus, the cardinality of $\mathcal{S}_y$ is $2^k$. For our convenience, we set $\mathcal{S}_e = \{e\}$. We define the length of $x = (x_1, x_2, \ldots, x_k)$ to be the length of $x_1 x_2 \cdots x_k$, and we also use $\ell$ to denote the length of $x$. Hence $\ell(x) = \ell(x_1 x_2 \cdots x_k)$. For $x = (x_1, x_2, \ldots, x_k)$, let $\pi(x) = x_1 x_2 \cdots x_k$. Define a map $d: \mathcal{S}_y \to \mathbb{Z}^+$ by

$$d(x) = |\{j \in [2, k] : x_1 \cdots x_{j-1} s_j < x_1 \cdots x_{j-1}\}|.$$

A Coxeter group $(W, S)$ is said to satisfy the $D$-condition, if for any $y \in W$ and a fixed reduced expression $y = s_1 \cdots s_k$, one can define (inductively on $k$) a subset $\delta(y)$ of $\mathcal{S}_y$ such that:

1. $\delta(e) = e$.
2. If for all $x \in \mathcal{S}_y$,

$$(1.4.1) \quad d(x) \leq (\ell(y) - \ell(x) - 1)/2,$$

then $\delta(y) = \mathcal{S}_y$. Otherwise, there are subsets $B_i$ of $W$, $1 \leq i \leq t$, consisting of elements $x \in W$ with $\ell(x) < \ell(y)$, such that one can find a reduced expression for each $x \in B_i$, $1 \leq i \leq t$, and the corresponding sets

$$B_i = \bigcup_{x \in B_i} \delta(x).$$
and embeddings \( t_i : \mathcal{B}_i \to \mathcal{J}_y \) such that:

(i) \( t_i(\delta(x)) \cap t_i(\delta(x')) = \emptyset \) if \( x \neq x' \), and \( t_i(\mathcal{B}_i) \cap t_j(\mathcal{B}_j) = \emptyset \) if \( i \neq j \);

(ii) if \( x \in \mathcal{J}_y \) and \( d(x) > (\ell(y) - \ell(x) - 1)/2 \), then \( x \in t_i(\mathcal{B}_i) \) for some \( 1 \leq i \leq t \);

(iii) \( \delta(y) = \mathcal{J}_y - \bigcup_{i=1}^t t_i(\mathcal{B}_i) \).

(3) The following formula holds:

\[
P_{x,y} = \sum_{x \in \delta(y) \setminus \delta(z)} q^{d(x)}.
\]

**Definition.** If \( \langle W, S \rangle \) satisfies the \( D \)-condition, then for each \( y \in W \) and a fixed reduced expression \( y = s_1 \cdots s_k \), the set \( \delta(y) \) defined above is called a \( D \)-set.

**Remarks.**

1. Note that for each \( y \in W \), one can have several \( D \)-sets (see [D, Section 4]).

2. It follows from the results in [D] that \( \langle W, S \rangle \) satisfies the \( D \)-condition if and only if the corresponding Kazhdan-Lusztig polynomials have nonnegative coefficients. It is also known that all crystallographic Coxeter groups satisfy the \( D \)-condition.

Suppose that \( \langle W, S \rangle \) satisfies the \( D \)-condition. Let \( y \in W \). Fix a reduced expression of \( y \) and a \( D \)-set \( \delta(y) \) corresponding to this reduced expression. Denote by \( m_x(\delta(y)) \) the number of appearances of an element \( x \) in the \( B_i \)'s, and let

\[
(1.4.2) \quad B = \bigcup_{i=1}^t B_i.
\]

1.5. **Bernstein-Gelfand projective functors** (cf. [BG]). Let \( \mathcal{C} \) be a highest weight category. Suppose that there is a Coxeter group \( \langle W, S \rangle \) acting on the poset \( \Lambda \) and satisfying the assumptions in 1.2. Suppose further that the duality condition (1.3.1) is also satisfied.

**Definition.** An indecomposable functor \( F : \mathcal{C} \to \mathcal{C} \) is called an indecomposable \( BG \)-functor, if:

(i) \( F \) is exact.

(ii) There is a \( \lambda \in \Lambda \) such that if \( \lambda' \) is the maximal element of \( \mathcal{W}_\lambda(\lambda) \), then \( F(M(\lambda')) = P(\lambda'') \) for some \( \lambda'' \in \mathcal{W}_\lambda(\lambda) \).

(iii) For any \( \mu \notin \mathcal{W}_\lambda(\lambda) \), \( F(M(\mu)) = 0 \).

(iv) If \( F^K \) is the operator induced by \( F \) on the Grothendieck group of \( \mathcal{C} \), then \( F^K \) commutes with the \( W \)-action.

A functor \( F : \mathcal{C} \to \mathcal{C} \) is called a \( BG \)-functor if \( F \) is the direct sum of indecomposable \( BG \)-functors.

We assume that the composition \( F_1 \circ F_2 \) of two \( BG \)-functors is again a \( BG \)-functor.

We say that \( \mathcal{C} \) (with the action of a Coxeter group \( \langle W, S \rangle \)) has enough \( BG \)-functors provided that for any \( P(\lambda), \lambda \in \Lambda \), there is an indecomposable \( BG \)-functor \( F_\lambda \) and an object \( M(\mu) \) (see Definition 1.1(b)) of \( \mathcal{C} \) such that \( F_\lambda(M(\mu)) = P(\lambda) \).
It is known from the results of [BG] (see Theorems 3.3 and 3.4 in [BG]) that for the highest weight category of a complex semisimple Lie algebra there are enough BG-functors. In [BG], these functors were defined by taking tensor products in the category with finite-dimensional objects in the same category.

2. \(D\)-SETS AND PROJECTIVE OBJECTS

In this section, we assume that we have a highest weight category \(\mathcal{O}\) with a Coxeter group \((W, S)\) acting on the poset \(\Lambda\) such that:

1. There exists at least one \(\lambda \in \Lambda\) such that for any \(x \neq y \in W\), we have \(x \cdot \lambda \neq y \cdot \lambda\).
2. For any \(x \in W\), \([M(x \cdot \lambda) : L(\mu)] \neq 0\) iff \(\mu = y \cdot \lambda\) for some \(y \geq x\).
3. The duality condition (1.3.1) holds.

Proposition 1. Assume that \(\mathcal{O}\) has enough BG-functors. Then for each \(y \in W\) and a reduced expression \(y = s_1 \cdots s_k\), there is a BG-functor \(G_y\) such that:

1. \(G_y(M(\lambda)) = D_y\) is a projective object of \(\mathcal{O}\).
2. \([D_y : M(p)] \neq 0\) iff \(\mu = x \cdot \lambda\) for some \(x \in W\) such that \(e \leq x \leq y\).
3. Let \(\mathcal{I}_y\) be the set corresponding to \(y = s_1 \cdots s_k\) defined in 1.4. For any \(x\) such that \(e \leq x \leq y\), let \(D_y(x)\) be the number of elements \(x = (x_1, \ldots, x_k) \in \mathcal{I}_y\) such that \(x_1 \cdots x_k = x\). Then \([D_y : M(p)] = D_y(x)\).

Remark. From this proposition, we see that \(D_y\) corresponds to \(\mathcal{I}_y\). Also note that since \(\mathcal{I}_y\) depends on the chosen reduced expression of \(y\), \(D_y\) depends on the chosen reduced expression of \(y\).

Proof. Under the assumption of the proposition, there is an indecomposable BG-functor \(F_x\) for each \(x \in W\) such that

\[
F_x(M(\lambda)) = P(x \cdot \lambda),
\]

where \(P(x \cdot \lambda)\) is the projective cover of \(L(x \cdot \lambda)\) (hence \(P(x \cdot \lambda)\) is indecomposable). Also, under the assumption, there is a one-to-one correspondence between \(W\) and the set \(W\) of elements of the Grothendieck group \(G\) of \(\mathcal{O}\) given by \(\text{ch} M(x \cdot \lambda)\), \(x \in W\), so we can identify them. Since \(W\) acts on \(\Lambda\), hence on \(G\), by assumption, we see that \(W\) fixes the subgroup \(\mathbb{Z}[W]\) of \(G\). Since each \(F_x^K\) \((x \in W)\) commutes with the \(W\) action, \(F_x^K\) can be viewed as an operator of right multiplication on \(\mathbb{Z}[W]\) by

\[
f_x = F_x^K(\text{ch} M(\lambda)) = F_x^K(e) \in \mathbb{Z}[W].
\]

With this convention, if

\[
f_x = \sum_{v \in W} m_v v, \quad m_v \in \mathbb{Z}_+,
\]

then by the duality condition \(\langle P(x \cdot \lambda) : M(v \cdot \lambda) \rangle = m_v\). The elements of \(\mathbb{Z}[W]\) can be viewed as functions on \(W\) as follows: if

\[
g = \sum_{x \in W} a_x x,
\]

then for \(y \in W\),

\[
g(y) = \sum_{x \in W} a_x \delta_{x,y},
\]

where \(\delta_{x,y} = 1\) if \(x = y\), and \(\delta_{x,y} = 0\) otherwise.
Therefore since the duality condition holds, for any \( x \in W \), we have

\[
f_x(y) = (P(y \cdot \lambda) : M(x \cdot \lambda)) = [M(x \cdot \lambda) : L(y \cdot \lambda)].
\]

In particular, we have \( f_{s_j} = s_j + e \) for any \( s_j \in S \). Let

\[
G_y = F_{s_1} \circ F_{s_2} \circ \cdots \circ F_{s_k}.
\]

Then \( G_y \) is a BG-functor of \( \mathcal{C} \) and satisfies (i)-(iii). In fact,

\[
G_y^K(e) = (s_1 + e) \cdots (s_k + e),
\]

and any element that appears on the right-hand side is of the form \( x_1 x_2 \cdots x_k \), with \( x_j = s_j \) or \( e \). Comparing this fact with the definition of \( \mathcal{S}_y \), we see that (iii) holds. (ii) follows from the fact that any \( x \in W \) such that \( e \leq x \leq y = s_1 \cdots s_k \) is of the form \( s_{j_1} \cdots s_{j_k} \), where \((j_1, \ldots, j_k)\) is a subsequence of \((1, \ldots, k)\) (see [H]). Q.E.D.

Under the assumption of Proposition 1, each \( y \in W \) corresponds to a unique element \( f_y \) of \( Z[W] \), these elements can be viewed both as operators of \( Z[W] \) and as functions on \( Z[W] \). They have the following properties (see [BG 4.5]):

(i) \( f_y(x) \in \mathbb{Z}_+ \) and \( f_y(x) \geq f_y(x') \) for \( x' \geq x \) and \( f_y(x) > 0 \) iff \( y \geq x \) and \( f_y(y) = 1 \).

(ii) If \( s \in S \) is a reflection, then \( f_s(s) = f_s(e) = 1 \), and \( f_s(x) = 0 \) if \( x \neq s \), \( e \).

(iii) If \( y > x \) and \( \ell(y) = \ell(x) + 1 \), then \( f_y(x) = 1 \).

(iv) Let \( s \in S \), \( y \in W \) be such that \( sy < y \). Then \( f_y(sx) = f_y(x) \) for any \( x \in W \). Similarly, if \( ys < y \), then \( f_y(xs) = f_y(x) \).

(v) \( f_y(x) = f_{y^{-1}}(x^{-1}) \) for any \( y, x \in W \).

From the proof of Proposition 1, we see that for a fixed reduced expression of \( y \), there is a one-to-one correspondence between the terms of \( G_y^K(e) \) and \( \mathcal{S}_y \), so we can identify them. Also note that since the composition of finite number of BG-functors is again a BG-functor, \( G_y \) has decomposition

\[
G_y = F_y \oplus_{x \in E} m_{y,x} F_x,
\]

for a subset \( E \) of \( W \) consisting of some \( x \in W \) such that \( x < y \) (\( E \) may be empty) and some \( m_{y,x} \in \mathbb{Z}_+ \). Therefore,

\[
G_y^K(e) = f_y + \sum_{x \in E} m_{y,x} f_x.
\]

**Proposition 2.** Notation is as above. Assume further that the Kazhdan-Lusztig polynomials associated to \( W \) have nonnegative coefficients. Suppose that for any \( y \in W \) and a chosen reduced expression \( y = s_1 \cdots s_k \), the following condition holds: \( x = (x_1, \ldots, x_k) \in \mathcal{S}_y \) does not appear in \( f_y \) (i.e., \( \pi(x) \) does not contribute to a term for \( f_y \)) \( \iff \) there is \( z \in \mathcal{S}_y \) with \( d(z) > (\ell(y) - \ell(x) - 1)/2 \) such that (i) \( z = \pi(x) \in E \), and (ii) \( \pi(x) \) appears in \( f_z \). Then the Kazhdan-Lusztig conjecture is true.

**Proof.** Since there is a one-to-one correspondence between \( \mathcal{S}_y \) and the terms in

\[
G_y^K(e) = f_y + \sum_{x \in E} m_{y,x} f_x,
\]
we can define \( \delta(y) \) to be the subset of \( S_y \) corresponding to \( f_y \). It is easy to see that these sets \( \delta(y), y \in W \), satisfy (1) and (2) of (1.4). On the other hand, since the Kazhdan-Lusztig polynomials have nonnegative coefficients, Algorithm 4.11 in [D] applies. Thus one can find a minimal subset \( \mathcal{E}_{\text{min}} \) of \( S_y \) which defines \( P_{x,y} \) [D, Theorem 4.12]. By comparing the algorithm in [D] and our definition of \( \delta(y) \), we see that we can choose \( \mathcal{E}_{\text{min}} = \delta(y) \). Hence the Kazhdan-Lusztig conjecture is true. Q.E.D.

3. Some equivalent conditions to the Kazhdan-Lusztig conjecture

Let \( \mathcal{C} \) be a highest weight category with a Coxeter group \( (W, S) \) acting on the poset \( \Lambda \) satisfying conditions (1)–(3) in section 2.

Let \( \mathcal{H} \) be the Hecke algebra corresponding to \( W \). Recall that the Kazhdan-Lusztig conjecture says

\[
[M(x \cdot \lambda) : L(y \cdot \lambda)] = P_{x,y}(1).
\]

**Theorem.** Under the assumptions of Proposition 2.1, the following are equivalent:

(i) The Kazhdan-Lusztig conjecture is true.

(ii) There is a homomorphism \( \psi : \mathcal{H} \to \mathbb{Z}[W] \) such that \( \psi(q) = 1 \) and \( \psi(C'_x) = f_x \), for any \( x \in W \).

(iii) The nonzero coefficients of all Kazhdan-Lusztig polynomials given by \( \mathcal{H} \) are positive integers, and for any \( y \in W \) and a fixed reduced expression \( y = s_1s_2\cdots s_k \), the subset \( E \) of \( W \) defined by (2.1.1) equals the subset of \( B \) of \( W \) defined by (1.4.2) and \( m_x(\delta(y)) = m_{y,x} \).

**Proof.** (i) \( \Rightarrow \) (ii). Assume that the Kazhdan-Lusztig conjecture is true. Then by the duality condition, we have

\[
(P(y \cdot \lambda) : M(x \cdot \lambda)) = [M(x \cdot \lambda) : L(y \cdot \lambda)] = P_{x,y}(1).
\]

Let \( \phi \) be the isomorphism \( \mathcal{H}/(q - 1) \cong \mathbb{Z}[W] \), and let \( p : \mathcal{H} \to \mathcal{H}/(q - 1) \) be the canonical homomorphism. Then \( \psi = \phi \circ p \) is the homomorphism we are looking for, since

\[
\psi(C'_y) = \sum_{x \leq y} P_{x,y}(1)x = f_y.
\]

(ii) \( \Rightarrow \) (iii). Assume that \( \psi : \mathcal{H} \to \mathbb{Z}[W] \) is a homomorphism such that \( \psi(q) = 1 \), \( \psi(C'_x) = f_x \) for all \( x \in W \). Then since \( C'_x, x \in W \) form a basis of \( \mathcal{H} \), \( f_x, x \in W \) form a basis of \( \mathbb{Z}[W] \) over \( \mathbb{Z} \). If \( s \in S \), then we have

\[
\psi(C'_x \cdot C'_s) = f_x \cdot f_s,
\]

where \( C'_s = q^{-1/2}(T_s + T_e) \). Now if \( \ell(xs) > \ell(x) \), then setting \( xs = y \), we have

\[
f_x \cdot f_s = f_y + \sum_{z<y} a_z f_z \quad \text{for some } a_z \in \mathbb{Z}_+.
\]

On the other hand we have (see [KL, section 2])

\[
C'_x \cdot C'_s = C'_y + \sum_{z<y} \mu(z, y)C'_z,
\]
Thus by applying $\psi$ to (3.1), we have

$$f_x \cdot f_y = f_y + \sum_{z < y} \mu(z, y) f_z.$$  

(We adopt the convention that $\mu(z, y)$ may be zero.) Hence $a_z = \mu(z, y)$. This implies in particular that the $\mu(z, y)$'s are all nonnegative integers. Therefore by [D, Corollary 3.8], the polynomials $L_z$ introduced by [D, Proposition 3.7] have nonnegative integral coefficients for all $z \in W(y) = \{z \in W : z \leq y\}$. Hence by [D, Theorem 4.12], the $D$-condition is satisfied and we can find a $D$-set $\delta(y)$ of $\mathcal{S}_y$ such that

$$P_{x, y} = \sum_{x \in \delta(y)} q^{d(x)}.$$

This implies in particular that all $P_{x, y}$ have nonnegative integer coefficients. Also, since (3.2) holds, from

$$\pi(\mathcal{S}_y) = \pi(\delta(y)) + \pi \left( \bigcup_{i=1}^{t} \mathcal{B}_i \right),$$

we have

$$G_y^K(e) = f_y + \sum_{x \in B} m_x(\delta(y)) f_x.$$  

Since $\{f_x, x \in W\}$ is a basis of $\mathbb{Z}[W]$, by comparing the above identity with (2.1.2), we see that $E = B$ and $m_x(\delta(y)) = m_y, x$ as desired.

(iii) $\Rightarrow$ (i). If all coefficients of $P_{x, y}$ are nonnegative, then by [D, Theorem 4.12], one can find $\delta(y)$ such that (3.2) holds. On the other hand, $E = B$ and $m_x(\delta(y)) = m_y, x$ imply that there is a one-to-one correspondence between the elements of $\delta(y)$ and the terms of $f_y$. So for $x \leq y$,

$$P_{x, y}(1) = \{x \in \delta(y) : \pi(x) = x\} = f_y(x) = (P(y \cdot \lambda) : M(x \cdot \lambda)) = [M(y \cdot \lambda) : L(x \cdot \lambda)].$$

That is, the Kazhdan-Lusztig conjecture is true. Q.E.D.

Acknowledgment

The author wishes to express his indebtedness to V. V. Deodhar. The author learned these materials under his guidance when the author was a Ph.D. student at Indiana University.

References


Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201

E-mail address: ymzou@convex.csd.uwm.edu