MENGER MANIFOLDS HOMEOMORPHIC TO THEIR n-HOMOTOPY KERNELS

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Abstract. We give a necessary and sufficient condition that an \( (n+1) \)-dimensional Menger manifold \( (\mu^{n+1}\text{-manifold}) \) is homeomorphic to its \( n \)-homotopy kernel. Such a \( \mu^{n+1} \)-manifold is called a \( \mu_\infty^{n+1} \)-manifold. We also prove the following results:

1. Each homeomorphism between two \( Z \)-sets in a \( \mu_\infty^{n+1} \)-manifold \( M \) extends to an ambient homeomorphism of \( M \) onto itself if it is \( n \)-homotopic to \( \text{id} \) in \( M \).

2. An \( n \)-homotopy equivalence between two \( \mu_\infty^{n+1} \)-manifolds is \( n \)-homotopic to a homeomorphism.

3. Each map from a \( \mu_\infty^{n+1} \)-manifold into a \( \mu^{n+1} \)-manifold is \( n \)-homotopic to an open embedding.

Introduction

All spaces considered in this paper are assumed to be locally compact separable metrizable and maps are continuous. In [Be], Bestvina introduced Menger manifolds and established the characterization theorem of such manifolds. An \( (n+1) \)-dimensional Menger manifold is a topological manifold modeled on the \( (n+1) \)-dimensional universal Menger compactum \( \mu^{n+1} \), which is also called a \( \mu^{n+1} \)-manifold. In [Ch3], Chigogidze introduced the notion of the \( n \)-homotopy kernel of a \( \mu^{n+1} \)-manifold and proved the following classification theorem for \( \mu^{n+1} \)-manifolds: Two \( \mu^{n+1} \)-manifolds have the same \( n \)-homotopy type if and only if their \( n \)-homotopy kernels are homeomorphic. There are close relations between Hilbert cube manifold (Q-manifold) theory, and Menger manifold theory, and the \( n \)-homotopy kernel of a \( \mu^{n+1} \)-manifold plays the role of the product \( X \times [0,1) \) of a \( Q \)-manifold \( X \) with \([0,1)\). It is said that \( X \) is \([0,1)\)-stable if it is homeomorphic to \( (\cong) X \times [0,1) \).

Wong [Wo] showed that a Q-manifold \( X \) is \([0,1)\)-stable if and only if \( X \) is properly contractible to \( \infty \); that is, for any compactum \( K \) in \( X \) there is a proper map \( j_K : X \to X \setminus K \) which is properly homotopic to \( \text{id}_X \). Replacing a proper homotopy with a proper \( n \)-homotopy, we have the notion of properly \( n \)-contractible to \( \infty \). Moreover we say that \( X \) is properly locally \( (n-)\)contractible
at $\infty$ if for any compactum $K \subset X$ there is a compactum $L \subset X$ with $K \subset L$ such that for each compactum $L' \subset X$ with $L \subset L'$ there exists a proper map $j_{L'} : X \setminus L \to X \setminus L'$ which is properly $(n)$-homotopic to $\text{id}_{X \setminus L}$ in $X \setminus K$.

In this paper we define $\mu_{\infty}^{n+1}$-manifolds as $\mu^{n+1}$-manifolds which are properly $n$-contractible to $\infty$ and properly locally $n$-contractible at $\infty$ and show the following characterization theorem for $\mu_{\infty}^{n+1}$-manifolds.

**Theorem I.** Let $M$ be a $\mu^{n+1}$-manifold. Then $M$ is a $\mu_{\infty}^{n+1}$-manifold if and only if $M$ is homeomorphic to its $n$-homotopy kernel $\text{Ker}(M)$.

We will show that two $n$-homotopic proper maps into a $\mu_{\infty}^{n+1}$-manifold are properly $n$-homotopic (see Lemma 3.1). Thus we can remove the requirement of an $n$-homotopy between $\mu_{\infty}^{n+1}$-manifold to be proper, whence we obtain the following $Z$-set unknotting theorem for $\mu_{\infty}^{n+1}$-manifolds.

**Theorem II.** Each homeomorphism between two $Z$-sets in a $\mu_{\infty}^{n+1}$-manifold extends to an ambient homeomorphism of $M$ onto itself if it is $n$-homotopic to $\text{id}$ in $M$.

From Theorem 2.2 in [Ch3], it follows that two $\mu_{\infty}^{n+1}$-manifolds of the same $n$-homotopy type are homeomorphic. Similarly to [C1, Theorem 5], we can clarify the relation between $n$-homotopy equivalences and homeomorphisms, that is:

**Theorem III.** An $n$-homotopy equivalence between two $\mu_{\infty}^{n+1}$-manifolds is $n$-homotopic to a homeomorphism.

Moreover, similarly to [0,1)-stable $Q$-manifolds [C1, Lemma 3.6], we can strengthen the open embedding theorem [Ch 2, Ch 3].

**Theorem IV.** Each map from a $\mu_{\infty}^{n+1}$-manifold into a $\mu^{n+1}$-manifold is $n$-homotopic to an open embedding.

1. Preliminaries

We say two (proper) maps $f, g : X \to Y$ are (properly) $n$-homotopic (notation: $f \simeq^n g$, $f \simeq^n_p g$, respectively) if, for any (proper) map $\alpha : Z \to X$ from a space $Z$ with dim $Z \leq n$ into $X$, the compositions $f\alpha$ and $g\alpha$ are (properly) homotopic in the usual sense. The notion of $n$-homotopy equivalence is defined in the obvious way.

**Proposition 1.1** [Hu]. Let $f : X \to Y$ be a map, where dim $X \leq n$ and $Y$ is LC$^n$. Then for any open cover $\mathcal{U}$ of $Y$, there are maps $\varphi : X \to P$ and $\psi : P \to Y$ such that $f$ and $\psi\varphi$ are $\mathcal{U}$-homotopic, where $P$ is a locally finite polyhedron with dim $P \leq n$. In particular, we can choose $\psi$ as a proper map.

Let us recall that a map $f : X \to Y$ is said to be $n$-invertible if for any space $Z$ with dim $Z \leq n$ and any map $\alpha : Z \to Y$ there exists a map $\beta : Z \to X$ such that $f\beta = \alpha$.

**Proposition 1.2** [Ch2]. Every $\mu^{n+1}$-manifold admits a proper $(n+1)$-invertible $UV^n$-surjection onto a $Q$-manifold.

**Proposition 1.3** [Ch3]. Two $\mu^{n+1}$-manifolds admitting proper $UV^n$-surjections onto the same LC$^n$-space are homeomorphic.
The following theorem is due to Bestvina [Be], where it is stated in terms of \( \mu \)-homotopy. However, as is known [Ch1], the notion of \( \mu \)-homotopy coincides with one of \( n \)-homotopy for maps between locally compact \( LC^n \)-spaces of dimension at most \( n + 1 \).

**Theorem 1.1 (Z-set unknotting theorem).** Let \( M \) be a \( \mu^{n+1} \)-manifold and \( f : A \to B \) be a homeomorphism between Z-sets in \( M \). If \( f \approx_{\mu} \text{id}_A \) in \( M \), then \( f \) extends to a homeomorphism \( h : M \to M \).

An \( n \)-homotopy kernel of a \( \mu^{n+1} \)-manifold \( M \) is defined to be the complement \( M \setminus f(M) \) of the image of an arbitrary Z-embedding \( f : M \to M \) with \( f \approx_{\mu} \text{id}_M \). Using the Z-set unknotting theorem, two \( n \)-homotopy kernels are homeomorphic by an ambient homeomorphism of \( M \) onto itself. By \( \text{Ker}(M) \), we denote a representative of \( n \)-homotopy kernels of \( M \). The following proposition is actually proved in [Ch3].

**Proposition 1.4.** For each \( \mu^{n+1} \)-manifold \( M \) there exists a proper \((n + 1)\)-invertible UV-surjection \( f_n : \mu^{n+1} \to Q \) satisfying the following condition:

\[(*) \quad f_n^{-1}(X) \text{ is a } \mu^{n+1} \text{-manifold for any locally compact } LC^n \text{-space } X \subset Q.\]

**Theorem 1.2 [Dr].** There exists an \((n + 1)\)-invertible UV-surjection \( f_n : \mu^{n+1} \to Q \) satisfying the following condition:

\[(*) \quad f_n^{-1}(X) \text{ is a } \mu^{n+1} \text{-manifold for any locally compact } LC^n \text{-space } X \subset Q.\]

**Theorem 1.3 [Ch4].** For each locally finite polyhedron \( K \), there exists a proper \((n + 1)\)-invertible UV-surjection \( f_K : M_K \to K \) from a \( \mu^{n+1} \)-manifold \( M_K \) onto \( K \) satisfying the following conditions:

(a) \( f_K^{-1}(L) \) is a \( \mu^{n+1} \)-manifold for any closed subpolyhedron \( L \) of \( K \);

(b) \( f_K^{-1}(Z) \) is a Z-set in \( f_K^{-1}(L) \) for any Z-set \( Z \) in a closed subpolyhedron \( L \) of \( K \).

Let \( f : X \to Y \) be a proper map. We say that \( f \) induces an epimorphism of \( j \)-th homotopy groups of ends if for every compactum \( C \subset Y \) there exists a compactum \( K \subset Y \) such that for each point \( x \in X \setminus f^{-1}(K) \) and every map \( \alpha : (S^j, *) \to (Y \setminus K, f(x)) \) there exists a map \( \hat{\alpha} : (S^j, *) \to (X \setminus f^{-1}(C), x) \) and a homotopy \( f\hat{\alpha} \approx \alpha \) rel. * in \( Y \setminus C \). Also we say that \( f \) induces a monomorphism of \( j \)-th homotopy groups of ends if for every compactum \( C \subset Y \) there exists a compactum \( K \subset Y \) such that for every map \( \alpha : S^j \to X \setminus f^{-1}(K) \) with \( f\hat{\alpha} \approx * \) in \( Y \setminus K \) it follows that \( \hat{\alpha} \approx * \) in \( X \setminus f^{-1}(C) \). It is said that \( f \) induces an isomorphism of \( j \)-th homotopy groups of ends if \( f \) induces both the epimorphism and monomorphism of \( j \)-th homotopy groups of ends.

**Theorem 1.4 [Be].** Let \( f : M \to N \) be a proper map between \( \mu^{n+1} \)-manifolds. If \( f \) induces an isomorphism of homotopy groups of \( \text{dim} \leq n \) and an isomorphism of homotopy groups of ends of \( \text{dim} \leq n \), then \( f \) is properly \( n \)-homotopic to a homeomorphism.

2. Characterization of \( \mu^{n+1} \)-manifolds

A space \( X \) is said to be properly \((n-)\)-contractible to \( \infty \) if for any compactum \( K \) in \( X \) there exists a proper map \( j_K : X \to X \setminus K \) which is properly...
(n-)homotopic to $\text{id}_X$. If for any compactum $K \subset X$ there exists a compactum $L \subset X$ with $K \subset L$ such that for each compactum $L' \subset X$ with $L \subset L'$ there exists a proper map $j_{L'} : X \setminus L \to X \setminus L'$ which is properly (n-)homotopic to $\text{id}_X \setminus L$ in $X \setminus K$, then a space $X$ is said to be properly locally (n-)contractible at $\infty$. It is easy to see that for any space $X$, $X \times [0, 1)$ is properly contractible to $\infty$ and properly locally contractible at $\infty$.

Lemma 2.1. Let $X$ be properly n-contractible to $\infty$ and properly locally n-contractible at $\infty$. Then for each compact cover $\{X_i\}_{i \in \omega}$ of $X$ with $X_i \subset \text{int} X_{i+1}$, there exists a subcover $\{X_{i_k} | k \in \omega, 0 = i_0 < i_1 < i_2 < \cdots \}$ and a collection of proper maps $\{f_k : X \to X \setminus X_{i_k}\}_{k \in \omega}$ such that $f_0 = \text{id}_X$ and $f_{k-1} \simeq_p f_k$ in $X \setminus X_{i_{k-2}}$ for $k \geq 1$, where $X_{i_{-1}} = \emptyset$.

Proof. For technical reasons we assume that $X_0 = \emptyset$. Let $L_{-2} = L_{-1} = L_0 = \emptyset$. We shall inductively choose integers $0 = i_0 = i_1 = i_0 < i_1 < i_2 < \cdots$ and construct compacta $L_{i_k-1} \subset X_{i_k} \subset L_{i_k}$ and proper maps $j_k : X \setminus L_{i_k-2} \to X \setminus X_{i_k}$, $k \in \omega$, satisfying the following conditions:

1. $j_0 = \text{id}_X$.
2. For each compactum $M \supset L_k$ there is a proper map $j_M : X \setminus L_k \to X \setminus M$ such that $j_M \simeq_p \text{id}_X \setminus L_k$ in $X \setminus X_{i_{k-2}}$.
3. $j_k \simeq_p \text{id}_X \setminus L_{k-2}$ in $X \setminus X_{i_{k-2}}$.

Let $i_1 = 1$. Since $X$ is properly n-contractible to $\infty$ and properly locally n-contractible at $\infty$, there exist a proper map $j_1 : X \to X \setminus X_{i_1}$ with $j_1 \simeq_p \text{id}$ and a compactum $L_1 \supset X_1$ satisfying (2). Since $X = \bigcup_{i \in \omega} X_i$ and $X_i \subset \text{int} X_{i+1}$, there exists $i_2 > i_1$ such that $X_{i_2} \supset L_1$. As in the above arguments there exist a proper map $j_2 : X \to X \setminus X_{i_2}$ with $j_2 \simeq_p \text{id}_X$ and a compactum $L_2 \supset X_{i_2}$ satisfying (2).

Assume that, for $k \geq 2$, $i_0 < i_1 < \cdots < i_k$, $L_k$, and $j_k : X \setminus L_{k-2} \to X \setminus X_{i_k}$ have been constructed. Choose $i_{k+1} > i_k$ so that $X_{i_{k+1}} \supset L_k$. Since $X_{i_{k+1}} \supset L_{k-1}$, by the property (2) of $L_{k-1}$, there exists a proper map $j_{X_{i_{k+1}}} : X \setminus L_{k-1} \to X \setminus X_{i_{k+1}}$ such that $j_{X_{i_{k+1}}} \simeq_p \text{id}_X \setminus L_{k-1}$ in $X \setminus X_{i_{k-1}}$. Then put $j_{k+1} = j_{X_{i_{k+1}}}$. Since $X$ is properly locally n-contractible at $\infty$, there exists a compactum $L_{k+1} \supset X_{i_{k+1}}$, satisfying (2).

Now define $f_k = j_k \cdots j_0 : X \to X \setminus X_{i_k}$ for $k \in \omega$ and observe that the collections of compacta $\{X_{i_k}\}_{k \in \omega}$ and maps $\{f_k\}_{k \in \omega}$ are as desired. \(\square\)

As is stated in the introduction, a $\mu_{n+1}^{\omega}$-manifold is a $\mu^{n+1}$-manifold which is properly n-contractible to $\infty$ and properly locally n-contractible at $\infty$. Theorem I is contained in the following.

Theorem 2.1 (Characterization). For a $\mu^{n+1}$-manifold $M$ the following conditions are equivalent:

1. $M$ is a $\mu_{n+1}^{\omega}$-manifold.
2. $M \cong \text{Ker}(M)$.
3. There is a proper $(n+1)$-invertible UV$n$-surjection $f : M \to X$ onto some $[0, 1)$-stable Q-manifold $X$.
4. There is a proper $(n+1)$-invertible UV$n$-surjection $g : M \to Y$ onto a space $Y$ which is properly n-contractible to $\infty$ and properly locally n-contractible at $\infty$.
Proof. We shall prove that (1) \(\Rightarrow\) (2). First we shall choose a compact cover \(\{M_i\}_{i \in \omega}\) of \(M\) with \(M_i \subset \text{int} M_{i+1}\), \(i \in \omega\), such that the topological frontier \(\text{Fr} M_i\) is a Z-set in \(M \setminus \text{int} M_i\). To this end, fix a proper \(UV^n\)-surjection \(g : M \to X\) onto a \(Q\)-manifold \(X\). Then choose a compact cover \(\{X_i\}_{i \in \omega}\) of \(X\) consisting of \(Q\)-manifold with \(X_i \subset \text{int} X_{i+1}\) such that \(\text{Fr} X_i\) is a Z-set in both \(X_i\) and \(X \setminus \text{int} X_i\), \(i \in \omega\) (see [C2, CS]). For each \(i \in \omega\), by the relative triangulation theorem for \(Q\)-manifolds [C3], we may assume that \(X = P \times Q\), \(X_i = P_i \times Q\), and \(X \setminus \text{int} X_i = P'_i \times Q\) for a locally finite polyhedron \(P\) and closed subpolyhedra \(P_i\), \(P'_i \subset P\). Let \(f_P : M_P \to P\) be a proper \(UV^n\)-surjection from a \(\mu^{n+1}\)-manifold \(M_P\) onto \(P\) satisfying condition (b) in Theorem 1.3. Since the composition \(\pi_P : M \to P\) is proper \(UV^n\) (where \(\pi_P : P \times Q \to P\) is the canonical projection), there is a homeomorphism \(k : M_P \to M\) by Proposition 1.3. Then by property (b) of \(f_P\), \(f_P^{-1}(P'_i \cap P'_j)\) is a Z-set in \(f_P^{-1}(P'_i)\) and so is the topological frontier of \(f_P^{-1}(P'_i)\) now let \(M_i = k f_P^{-1}(P'_i), i \in \omega\). Then the compact cover \(\{M_i\}_{i \in \omega}\) of \(M\) is the required one.

By Lemma 2.1, there is a collection of maps \(\{f_i : M \to M \setminus M_i\}_{i \in \omega}\) such that \(f_0 = \text{id}_M\), \(f_i \simeq f_{i+1}\) in \(M \setminus M_{i-1}\) for \(i \in \omega\). Using the Z-embedding approximation theorem for \(\mu^{n+1}\)-manifolds [Be, 2.3.8], we can choose \(f_i\) as a Z-embedding for each \(i \in \omega\). Put \(K_i = M \setminus f_i(M)\) for \(i \geq 1\). Then since \(f_i \simeq f_{i+1}\), by the definition of \(n\)-homotopy kernels, we have \(K_i \cong \text{Ker}(M)\).

By Theorem 1.1, since \(f_i \simeq f_{i+1}\) is in \(M \setminus M_{i-1}\) and \(\text{Fr} M_{i-1}\) is a Z-set in \(M \setminus M_{i-1}\), there exists a homeomorphism \(h_i : M \to M\) such that \(h_i f_i = f_{i+1}\) and \(h_i | M_{i-1} = \text{id}\). Note that \(h_i(K_i) = K_{i+1}\). Now we define \(h : K_1 \to M\) by \(h = \lim_{i \to \infty} h_i \cdots h_1\). Then \(h | h^{-1}(\text{int} M_i) = h_{i+2} \cdots h_1 | h^{-1}(\text{int} M_i)\). In fact, suppose that \(h(x) \neq h_{i+2} \cdots h_1(x)\) for some \(x \in h^{-1}(\text{int} M_i)\). Then there is an open subset \(U\) of \(\text{int} M_i\) such that \(h(x) \in U \subset U\) and \(h_{i+2} \cdots h_1(x) \notin U\). Since \(h | h^{-1}(\text{int} M_i) = \text{id}\) for \(j \geq i + 2\), \(h_j \cdots h_1(x) = h_{i+2} \cdots h_1(x) \notin U\) for all \(j \geq i + 2\). This contradicts the definition of \(h\).

One can easily see that \(h\) is injective. Moreover, since \(M = \cup_{i \in \omega} M_i\) and \(\text{int} h^{-1}(K_i) = K_i \supset M_i\), it follows that \(h\) is surjective. To finish the proof, it only remains to note that \(h\) is open. Thus \(h\) is a homeomorphism.

To prove (2) \(\Rightarrow\) (3), assume \(M \cong \text{Ker}(M)\). Then, by Proposition 1.4, there is a proper \((n+1)\)-invertible \(UV^n\)-surjection \(g : M \to M \times [0, 1]\). Let \(h : M \to Y\) be a proper \(UV^n\)-surjection onto a \(Q\)-manifold \(Y\) (Proposition 1.2). Since \(Y \times [0, 1]\) is a \([0, 1]\)-stable \(Q\)-manifold, the composition \((h \times \text{id}_{[0, 1]})g : M \to Y \times [0, 1]\) is the required one.

(3) \(\Rightarrow\) (4) is trivial.

Finally we shall show that (4) \(\Rightarrow\) (1). Let \(h : M \to X\) be a proper \((n+1)\)-invertible \(UV^n\)-surjection onto a space \(X\) properly \(n\)-contractible to \(\infty\) and properly locally \(n\)-contractible at \(\infty\). Let \(K\) be a compactum in \(M\). Then there exists a compactum \(L'\) in \(X\) with \(h(K) \subset L'\) such that for each compactum \(F'\) with \(L' \subset F'\) there exist proper maps \(i'_{h(K)} : X \to X \setminus h(K)\) and \(j'_F : X \setminus L' \to X \setminus F'\) such that \(i'_{h(K)} \simeq \text{id}_X\) in \(X\) and \(j'_F \simeq \text{id}_{X \setminus L'}\) in \(X \setminus h(K)\). Let \(L = h^{-1}(L')\) and \(F\) be a compactum containing \(L\). Since \(h\) is proper \((n+1)\)-invertible, there exist proper maps \(i_K : M \to M \setminus K\) and \(j_F : M \setminus L \to M \setminus F\) such that \(h i_K = i'_{h(K)} h\) and \(h j_F = j'_{h(F)} h\).
Consider a proper map \( \alpha : Z \to M \setminus L \ (\subset M \setminus h^{-1}h(K)) \), where \( \dim Z \leq n \). We shall now show that \( j_F \alpha \) is properly homotopic to \( \alpha \) in \( M \setminus K \). From Proposition 1.1, we may assume without loss of generality that \( Z \) is a locally finite polyhedron. Let \( H : (X \setminus L') \times [0, 1] \to X \setminus h(K) \) be a proper homotopy from \( \text{id}_{X \setminus L'} \) to \( j_{h(F)}' \). Then \( H(h \alpha \times \text{id}) : Z \times [0, 1] \to X \setminus h(K) \) is a proper homotopy from \( h \alpha \) to \( j_{h(F)}'h \alpha = h j_F \alpha \). Since \( h |_{M \setminus h^{-1}h(K)} : M \setminus h^{-1}h(K) \to X \setminus h(K) \) is proper \( UV^n \), by [La, §3, Lemma A], there exists a proper homotopy \( F : Z \times [0, 1] \to M \setminus h^{-1}h(K) \) from \( \alpha \) to \( j_F \alpha \). Thus \( j_F \simeq_p \text{id}_{M \setminus L} \) in \( M \setminus K \). Similarly, we can conclude \( i_F \simeq_p \text{id}_M \). \( \square \)

3. Proofs of Theorems II, III, and IV

Lemma 3.1. Let \( f : X \to Y \) be a map from a locally compact space \( X \) into a \( LC^n \)-space \( Y \) admitting a proper \( (n+1) \)-invertible \( UV^n \)-surjection onto a space \( Y \times [0, 1) \). Then \( f \) is \( n \)-homotopic to a proper map whenever \( \dim X \leq n+1 \). Moreover, if \( f \) is a proper map \( n \)-homotopic to a proper map \( g : X \to Y \), then \( f \simeq^n_p g \).

Proof. Fix a proper map \( p : X \to [0, 1) \), and let \( h : Y \to Y \times [0, 1) \) be a proper \( (n+1) \)-invertible \( UV^n \)-surjection. Let \( q : X \to Y \times [0, 1) \) be the map defined by \( q(x) = (h(x), p(x)) \), where \( h(x) = (h_1(x), h_2(x)) \), \( x \in X \). Then \( q \) is proper and homotopic to \( hf \). By the \( (n+1) \)-invertibility of \( h \), there is a map \( f' : X \to Y \) such that \( hf' = q \). Note that \( f' \) is proper and \( hf' \simeq hf \). Thus by the lifting property of \( h \) [La, §3, Lemma A], we conclude that \( f \simeq^n f' \).

Next suppose that \( f \) is a proper map \( n \)-homotopic to a proper map \( g : X \to Y \). Let \( \alpha : Z \to X \) be a proper map, where \( \dim Z \leq n \). We shall show that \( f \alpha \simeq_p g \alpha \). By Proposition 1.1, we may assume without loss of generality that \( Z \) is a locally finite polyhedron. Let \( \{Y_i\}_{i \in \omega} \) be a compact cover of \( Y \) with \( Y_0 = \emptyset \) and \( Y_i \subset \text{int } Y_{i+1} \), \( i \in \omega \). Then for each \( i \geq 1 \), let \( Z_i \) be a compact subpolyhedron of \( Z \) such that \( (h f \alpha)^{-1}(W_i) \cup (h g \alpha)^{-1}(W_i) \subset Z_i \subset \text{int } Z_{i+1} \), where \( Z_0 = \emptyset \) and \( W_i = Y_i \times [0, 1 - 2^{-i}] \). Since \( f \simeq^n g \), we can fix a homotopy \( G_0 : Z \times [0, 1) \to Y \) from \( f \alpha \) to \( g \alpha \). For \( k \geq 1 \), we shall inductively construct a homotopy \( G_k : (Z \setminus \text{int } Z_{2k-2}) \times [0, 1) \to Y \setminus h^{-1}(W_{2k-2}) \) from the restriction \( f \alpha |_{Z_{2k-2}} \) of \( f \alpha \) to the one \( g \alpha |_{Z_{2k-2}} \) of \( g \alpha \) satisfying the following conditions:

\[
(1)_k \quad G_k((Z \setminus \text{int } Z_{2k}) \times [0, 1]) \subset Y \setminus h^{-1}(W_{2k-2});
\]

\[
(2)_k \quad G_k = G_{k-1} \text{ on } \text{Fr } Z_{2k-2} \times [0, 1].
\]

Let \( F_i : [0, 1) \to [1 - 2^{-i}, 1) \) be the map defined by \( F_i(t) = 1 + (t - 1)2^{-i} \) for each \( i \geq 1 \). Suppose that a homotopy \( G_k : (Z \setminus \text{int } Z_{2k-2}) \times [0, 1) \to Y \setminus h^{-1}(W_{2k-2}) \) has been constructed for \( k \in \omega \). Then let \( A_{k+1} = (Z \setminus \text{int } Z_{2k}) \times [0, 1] \cup \text{Fr } Z_{2k} \times [0, 1) \) and \( B_{k+1} = (hG_k)^{-1}(W_{2k+1}) \cap (Z \setminus \text{int } Z_{2k+2}) \times [0, 1] \). Since \( A_{k+1} \) and \( B_{k+1} \) are disjoint closed, we can choose \( \beta : (Z \setminus \text{int } Z_{2k}) \times [0, 1) \to [0, 1] \) such that \( \beta(A_{k+1}) = 0 \) and \( \beta(B_{k+1}) = 1 \). Define \( G'_{k+1} : (Z \setminus \text{int } Z_{2k}) \times [0, 1) \to Y \setminus W_{2k-2} \) by

\[
G'_{k+1}(w) = (sk(w), (1 - \beta(w))tk(w) + \beta(w)F_{2k+2}tk(w)),
\]

where \( hG_k(w) = (sk(w), tk(w)) \), \( w \in (Z \setminus \text{int } Z_{2k}) \times [0, 1] \). By the lifting property [La], there is a homotopy \( G_{k+1} : (Z \setminus \text{int } Z_{2k}) \times [0, 1) \to Y \setminus W_{2k-3} \) from \( f \alpha |_{Z_{2k-2}} \) to \( g \alpha |_{Z_{2k-2}} \) with \( hG_{k+1} = G'_{k+1} \) and \( G_{k+1} = G_k \) on \( A_{k+1} \) (i.e., satisfying \( (2)_{k+1} \)) such that \( G_{k+1} \) satisfies \( (1)_{k+1} \).
We define $H : Z \times [0, 1] \to Y$ by $H = G_k$ on each $(Z_{2k} \setminus \text{int} Z_{2k-2}) \times [0, 1]$. Then $H$ is a well-defined homotopy from $f \sim a$ to $g \alpha$. Note that since $h$ is proper, $\{h^{-1}(W_i)\}_{i \in \omega}$ is a compact cover of $Y$ with $h^{-1}(W_i) \subset h^{-1}(W_{i+1})$. Thus it follows from our construction that $H$ is proper. The proof is finished. \hfill \Box

Proof of Theorem II. The theorem directly follows from Theorem 1.1 and Lemma 3.1. \hfill \Box

Lemma 3.2. If $f : M \to N$ is a proper $n$-homotopy equivalence between $\mu^n_{N^{n+1}}$-manifolds, then $f$ induces an isomorphism of homotopy groups of ends of dim $\leq n$.

Proof. By Theorem 2.1, we can fix proper $(n+1)$-invertible $UV^n$-surjections $g : M \to X \times [0, 1)$ and $h : N \to Y \times [0, 1)$, where $X$ and $Y$ are some $Q$-manifolds. Let $C$ be a compactum in $N$. Then there is a compactum $C'' \subset Y$ such that $C'' \times [0, t'] \supset h(C)$ for some $t' \in (0, 1)$. Since $h$ is proper, $C' = h^{-1}(C'' \times [0, t'])$ is a compactum with $C' \supset C$. Note that, since $f$ is proper, $g(f^{-1}(C'))$ is a compactum in $X \times [0, 1)$. Thus there exists $t_1 \in (0, 1)$ such that $L = \pi_X(g(f^{-1}(C'))) \times [0, t_1) \supset g(f^{-1}(C'))$, where $\pi_X : X \times [0, 1) \to X$ is the canonical projection. Similarly, since $g$ is proper, there exists $t_2 \in (0, 1)$ such that $K' = \pi_Y(h(f^{-1}(L))) \times [0, t_2) \supset h(f^{-1}(L))$, where $\pi_Y : Y \times [0, 1) \to Y$ is the canonical projection. Put $K = h^{-1}(K')$, and let $x_0 \in M \setminus f^{-1}(K)$, $j \leq n$, and $\alpha : (S^j, *) \to (N \setminus K, f(x_0))$. Since $f$ is an $n$-homotopy equivalence, there exists $\alpha_1 : (S^j, *) \to (M, x_0)$ such that $f \alpha_1 \simeq \alpha$ rel. $\ast$. Since $\alpha_1^{-1}(x_0)$ and $\alpha_1^{-1}(g_1^{-1}(L))$ are disjoint closed sets in $S^j$, we can choose a map $\beta : S^j \to [0, 1]$ such that $\beta(\alpha_1^{-1}(x_0)) = 0$ and $\beta(\alpha_1^{-1}(g^{-1}(L))) = 1$. Say $g \alpha_1(x) = (\pi_X g \alpha_1(x), t(x)) \in X \times [0, 1), x \in S^j$. Define $\alpha_2 : (S^j, *) \to (X \times [0, 1), g(x_0))$ by

$$\alpha_2(x) = (\pi_X g \alpha_1(x), (1 - t_1) \cdot t(x) + t_1) \cdot (1 - \beta(x) \cdot t(x)),$$

Clearly $\alpha_2 \simeq g \alpha_1$ rel. $\ast$ and $\alpha_2(S^j) \cap L = \emptyset$. Using the lifting property [La] of the proper $UV^n$-surjection $g$, there exists $\tilde{\alpha} : (S^j, *) \to (M, x_0)$ such that $\text{img} \tilde{\alpha} \cap L = \emptyset$ and $\tilde{\alpha} \simeq \alpha_1$ rel. $\ast$. Hence we have $f \tilde{\alpha} \simeq g \alpha_1$ rel. $\ast$. Since $f \tilde{\alpha} \simeq g \alpha_1$ rel. $\ast$, by the same technique we performed above, we can choose a homotopy so that $f \tilde{\alpha} \simeq \alpha$ rel. $\ast$ in $N \setminus C$.

Next let $\gamma : S^j \to M \setminus f^{-1}(K)$ be a map such that $f \gamma \simeq \ast$ in $N \setminus K$. Since $f$ is an $n$-homotopy equivalence, $g \gamma \simeq \ast$ in $X \times [0, 1)$. By sliding the $[0, 1)$-factor of the homotopy upward as the above, we have $g \gamma \simeq \ast$ in $X \times [0, 1) \setminus L$. By the lifting property of $g$ [La], it follows that $\gamma \simeq \ast$ in $X \setminus f^{-1}(C)$. Thus we conclude that $f$ induces an isomorphism of homotopy groups of ends of dim $\leq n$. \hfill \Box

Proof of Theorem III. Let $f : M \to N$ be an $n$-homotopy equivalence between $\mu^n_{N^{n+1}}$-manifolds. Then by Lemma 3.1 there is a proper map $h : M \to N$ such that $f \simeq h$; consequently, $h$ is a proper $n$-homotopy equivalence. By Lemma 3.2 and Theorem 1.4, $h$ is properly $n$-homotopic to a homeomorphism. Thus $f$ is $n$-homotopic to a homeomorphism. \hfill \Box
Proof of Theorem IV. Let $f : M \to N$ be a map from a $\mu_{n+1}^\infty$-manifold to a $\mu_{n+1}^\infty$-manifold. By replacing $N$ with $\text{Ker}(N)$, we may assume that $N$ is also a $\mu_{n+1}^\infty$-manifold. By the triangulation theorem for $\mu_{n+1}^\infty$-manifold [Dr], we can fix proper $(n+1)$-invertible UV$^n$-surjections $g : M \to K$ and $h : N \to L$, where $K$ and $L$ are locally finite polyhedra of dimension at most $n+1$. Then by the $(n+1)$-invertibility, $g$ has a section $p : K \to M$ (i.e., $gp = \text{id}_K$). Since $N$ is a $\mu_{n+1}^\infty$-manifold, by Lemma 3.1, $f$ is $n$-homotopic to a proper map $f' : M \to N$. Then $\phi = hfp : K \to L$ is a proper map. Let $M(\phi)$ be the mapping cylinder of $\phi$, that is, a space obtained from the disjoint union $K \times [0, 1] \oplus L$ by identifying $(x, 1)$ with $\phi(x)$, $x \in K$. Define $c_\phi : M(\phi) \to L$ by $c_\phi(x, t) = \phi(x)$, $x \in K$. Let $f_n : \mu_{n+1} \to Q$ be a proper $(n+1)$-invertible UV$^n$-surjection satisfying the condition $(\ast)$ in Theorem 1.2. Embed $M(\phi)$ into $Q$, whence $f_n^{-1}(M(\phi))$ is a $\mu_{n+1}^\infty$-manifold. We denote the restriction of $f_n$ to $f_n^{-1}(M(\phi))$ by $f_n|$. Observe that $f_n^{-1}(K \times \{0\}) \cong M$ and $f_n^{-1}(L) \cong N$ by Proposition 1.3. We identify $f_n^{-1}(K \times \{0\})$, $f_n^{-1}(L)$ with $M$, $N$ respectively. Abusing notation, by $g : M \to K \times \{0\}$, $h : N \to L$ we denote the restrictions of $f_n$ to $M$, $N$ respectively. Using the $(n+1)$-invertibility of $h$, we can fix a section $q : L \to N$ of $h$. Note that since $c_\phi f_n| : f_n^{-1}(M(\phi)) \to L$ and $h : N \to L$ are proper UV$^n$-surjections, $f_n^{-1}(M(\phi)) \cong N$ by Proposition 1.3. Observe that the map $q c_\phi f_n|$ is an $n$-homotopy equivalence between $\mu_{n+1}^\infty$-manifolds $f_n^{-1}(M(\phi))$ and $N$. Then by Theorem III, there is a homeomorphism $s : f_n^{-1}(M(\phi)) \to N$ such that $s \simeq n q c_\phi f_n|$. Note that $M' = f_n^{-1}(K \times \{0, 1\})$ is open in $f_n^{-1}(M(\phi))$ and is a $\mu_{n+1}^\infty$-manifold by Theorem 2.1. Since the inclusion $i : M = f_n^{-1}(K \times \{0\}) \hookrightarrow M'$ is an $n$-homotopy equivalence, by Theorem III, we can choose a homeomorphism $r : M \to M'$ with $r \simeq n i$. Then the map $sr : M \to N$ is an open embedding which is $n$-homotopic to $q c_\phi f_n|i = q \phi g = qh f'p g$. Since $pg \simeq_p \text{id}_M$ and $qh \simeq_p \text{id}_N$, we have $q h f' p g \simeq_p f' \simeq^n f$. The proof is finished. \hfill \square

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References


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