STAR-SHAPED COMPLEXES AND EHRHART POLYNOMIALS

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Abstract. We study Ehrhart polynomials of star-shaped triangulations of balls by means of Cohen-Macaulay rings and canonical modules.

A polyhedral complex \( \Gamma \) in \( \mathbb{R}^N \) is a finite set of convex polytopes in \( \mathbb{R}^N \) such that

1. if \( P \in \Gamma \) and \( \mathcal{F} \) is a face of \( P \), then \( \mathcal{F} \in \Gamma \), and
2. if \( P, \mathcal{C} \in \Gamma \), then \( P \cap \mathcal{C} \) is a face of \( P \) and of \( \mathcal{C} \).

We are concerned with a polyhedral complex \( \Gamma \) in \( \mathbb{R}^N \) which satisfies the following conditions:

1. every vertex \( \alpha \) of \( P \in \Gamma \) has integer coordinates, i.e., \( \alpha \in \mathbb{Z}^N \), and
2. the underlying space \( X := \bigcup_{P \in \Gamma} P(\subset \mathbb{R}^N) \) of \( \Gamma \) is homeomorphic to the \( d \)-ball.

Let \( \partial X \) denote the boundary of \( X \); thus \( \partial X \) is homeomorphic to the \( (d-1) \)-sphere. Given an integer \( n > 0 \), write \( nX \) for \( \{ n \alpha; \alpha \in X \} \) and define \( i(X, n) \) to be \( \#(nX \cap \mathbb{Z}^N) \), the cardinality of \( nX \cap \mathbb{Z}^N \). In other words, \( i(X, n) \) is equal to the number of rational points \( (\alpha_1, \alpha_2, \ldots, \alpha_N) \in X \) with each \( \alpha_i \in \mathbb{Z} \). It is known that

1. \( i(X, n) \) is a polynomial in \( n \) of degree \( d \), called the Ehrhart polynomial of \( X \),
2. \( i(X, 0) = 1 \), and
3. \( (-1)^d i(X, -n) = \#(n(X - \partial X) \cap \mathbb{Z}^N) \) for every \( 1 \leq n \in \mathbb{Z} \).

Define the sequence \( \delta_0, \delta_1, \delta_2, \ldots \) of integers by the formula

\[
(1 - \lambda)^{d+1} \left[ 1 + \sum_{n=1}^{\infty} i(X, n) \lambda^n \right] = \sum_{i=0}^{\infty} \delta_i \lambda^i.
\]

Then

1. \( \delta_0 = 1 \) and \( \delta_1 = \#(X \cap \mathbb{Z}^N) - (d + 1) \),
2. \( \delta_i = 0 \) for each \( i > d \), and
3. \( \delta_d = \#(X - \partial X) \cap \mathbb{Z}^N \).

We say that \( \delta(X) = (\delta_0, \delta_1, \ldots, \delta_d) \) is the \( \delta \)-vector of \( X \). We refer the reader to, e.g., [6, Chapter IX], for geometric proofs of the above fundamental results.
due to Ehrhart. Note that, even though $X$ is not necessarily convex, the proofs in [6] are valid without modification since $X$ is homeomorphic to the $d$-ball.

Some algebraic technique\(^1\) is indispensable for the study of combinatorics on $\delta$-vectors. Fix a field $k$, and let $\xi_1, \xi_2, \ldots, \xi_N$, $t$ be (commutative) indeterminates over $k$. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in nX \cap \mathbb{Z}^N$, then we set $\xi^\alpha t^n = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_N^{\alpha_N} t^n$. We write $[A_k(\Gamma)]_n$ for the vector space spanned by all monomials $\xi^\alpha t^n$ with $\alpha \in nX \cap \mathbb{Z}^N$. Thus, in particular, $\dim_k [A_k(\Gamma)]_n = i(X, n)$. Let $A_k(\Gamma)$ denote $\bigoplus_{n \geq 0} [A_k(\Gamma)]_n$ with $[A_k(\Gamma)]_0 = k$, and define multiplication $(\xi^\alpha t^n)(\xi^\beta t^m)$ of monomials $\xi^\alpha t^n$ and $\xi^\beta t^m$ in $A_k(\Gamma)$ as follows: $(\xi^\alpha t^n)(\xi^\beta t^m) = \xi^{\alpha+\beta} t^{n+m}$ if there exists $\mathcal{P} \in \Gamma$ with $\alpha \in n\mathcal{P}$ and $\beta \in m\mathcal{P}$; $(\xi^\alpha t^n)(\xi^\beta t^m) = 0$ otherwise. Then $A_k(\Gamma)$ is a noetherian (i.e., finitely generated) graded algebra over $k$ and the Hilbert series $F(A_k(\Gamma), \lambda) := \sum_{n=0}^{\infty} \dim_k [A_k(\Gamma)]_n \lambda^n$ is $(\delta_0 + \delta_1 \lambda + \delta_2 \lambda^2 + \cdots + \delta_d \lambda^d)/(1-\lambda)^{d+1}$. Let $\Omega(A_k(\Gamma)) = \bigoplus_{n \geq 1} [\Omega(A_k(\Gamma))]_n$ be the graded ideal of $A_k(\Gamma)$ which is generated by those monomials $\xi^\alpha t^n$ such that $0 < n \in \mathbb{Z}$ and $\alpha \in n(X - \partial X) \cap \mathbb{Z}^N$. Since $X$ is homeomorphic to the $d$-ball, $A_k(\Gamma)$ is Cohen-Macaulay [10, Lemma 4.6]. Thus, a well-known technique of commutative algebra enables us to obtain $\delta(X) \geq 0$, i.e., each $\delta_i \geq 0$ (cf. Stanley [8]). On the other hand, the same technique as in the proof of [2, Theorem (5.6.1)] enables us to show that $\Omega(A_k(\Gamma))$ is the canonical module of $A_k(\Gamma)$.

We say that $X$ is "star-shaped" with respect to a point $a \in X - \partial X$ if $t\alpha + (1-t)\beta \in X - \partial X$ for every point $\beta \in X$ and for each real number $0 < t < 1$.

**Theorem.** We employ the same notation as used above. Suppose that the set $(X - \partial X) \cap \mathbb{Z}^N$ is nonempty and that the underlying space $X$ is star-shaped with respect to some $v_1 \in (X - \partial X) \cap \mathbb{Z}^N$. Then the $\delta$-vector $\delta(X) = (\delta_0, \delta_1, \ldots, \delta_d)$ of $X$ satisfies the linear inequalities as follows:

\begin{align*}
(5.1) \quad \delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_d + \delta_{d-1} + \cdots + \delta_{d-i}, & \quad 0 \leq i \leq \lfloor d/2 \rfloor; \\
(5.2) \quad \delta_1 \leq \delta_i, & \quad 2 \leq i \leq d.
\end{align*}

**Sketch of proof.** First, recall that a simplicial complex in $\mathbb{R}^N$ is a polyhedral complex $\Delta$ in $\mathbb{R}^N$ such that every convex polytope belonging to $\Delta$ is a simplex in $\mathbb{R}^N$. Fix an arbitrary simplicial complex $\Delta(0)$ in $\mathbb{R}^N$ with the vertex set $\partial X \cap \mathbb{Z}^N$ whose underlying space is the boundary $\partial X$ of $X$. Since $X$ is star-shaped with respect to $v_1 \in (X - \partial X) \cap \mathbb{Z}^N$, we can define the cone $\Delta(1)$ over $\Delta(0)$ with apex $v_1$, i.e., $\Delta(1)$ is the simplicial complex in $\mathbb{R}^N$ which consists of those simplices $\sigma$ such that either $\sigma \in \Delta(0)$ or $\sigma$ is the convex hull of $\tau \cup \{v_1\}$ in $\mathbb{R}^N$ for some $\tau \in \Delta(0)$. The vertex set of $\Delta(1)$ is $(\partial X \cap \mathbb{Z}^N) \cup \{v_1\}$ and the underlying space of $\Delta(1)$ is $X$. Let $(X - \partial X) \cap \mathbb{Z}^N = \{v_1, v_2, \ldots, v_l\}$ and, for each $2 \leq j \leq \ell$, construct a simplicial complex $\Delta(j)$ with the vertex set $(\partial X \cap \mathbb{Z}^N) \cup \{v_1, v_2, \ldots, v_j\}$ and with the underlying space $X$ by the same way as in [7]. We write $\Delta$ for $\Delta(\ell)$. Then the element $\theta = \xi^{v_1} t + \xi^{v_2} t + \cdots + \xi^{v_l} t$ of $[\Omega(A_k(\Delta))]_1$ is a nonzero divisor on $A_k(\Delta)$. Hence, it follows from a standard technique of commutative algebra [11] (see also [4]) that $\sum_{0 \leq i \leq d} \delta_i \leq \sum_{0 \leq i \leq d-j} \delta_{d-j}$ for every $0 \leq i \leq \lfloor d/2 \rfloor$. On the other hand, let $h(\Delta) = (h_0, h_1, \ldots, h_d, 0)$ be the $h$-vector (e.g., [9]) of the simplicial complex

\(^1\)We refer to, e.g., [6, Chapter IV] for "Commutative Algebra for Combinatorialists".
\( \Delta \). Then \( h_1 \leq h_i \) for each \( 2 \leq i < d \) (cf. [7]). Also, \( h_1 = \delta_1 \). Since \( h_i \leq \delta_i \), \( 0 \leq i \leq d \), by [1], we have \( \delta_1 \leq \delta_i \) for each \( 2 \leq i < d \) as desired. Q.E.D.

**Remark.** (a) In the above sketch of proof, let \( A_k(\Delta)^* \) denote the graded subalgebra of \( A_k(\Delta) \) generated by \([A_k(\Delta)]_1 \) over \( k \). Then \( A_k(\Delta)^* \) coincides with the Stanley-Reisner ring [9] of the simplicial complex \( \Delta \). Thus \( A_k(\Delta)^* \) is Cohen-Macaulay with the Hilbert series

\[
F(A_k(\Delta)^*, \lambda) = (h_0 + h_1 \lambda + h_2 \lambda^2 + \cdots + h_d \lambda^d)/(1 - \lambda)^{d+1}.
\]

Moreover, \( A_k(\Delta) \) is finitely generated as a module over \( A_k(\Delta)^* \).

(b) By the similar method as in [3, Theorem (1.3)], without the hypothesis that \((X - \partial X) \cap \mathbb{Z}^N \) is nonempty and \( X \) is star-shaped, we can prove that the \( \delta \)-vector \( \delta(X) = (\delta_0, \delta_1, \ldots, \delta_d) \) of \( X \) satisfies the linear inequality

\[
\delta_d + \delta_{d-1} + \cdots + \delta_{d-i} \leq \delta_0 + \delta_1 + \cdots + \delta_i + \delta_{i+1}
\]

for every \( 0 \leq i \leq \lfloor (d - 1)/2 \rfloor \).

**Example.** Let \( N = d = 3 \) and \( X = P \cup \mathcal{G} \), where \( P \subset \mathbb{R}^3 \) (resp. \( \mathcal{G} \subset \mathbb{R}^3 \)) is the tetrahedron with the vertices \((1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)\) (resp. \((1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0)\)). Then \((X - \partial X) \cap \mathbb{Z}^3 = \{(0, 0, 0)\}\) and \( X \) is not star-shaped with respect to \((0, 0, 0)\). However, \( X \) is star-shaped with respect to, e.g., \((1/3, 1/3, 1/3)\). We have \( \delta(X) = (1, 2, 1, 1) \) which fails to satisfy (5.1) for \( i = 1 \) and (5.2) for \( i = 2 \).

**Corollary** [3, 7, 11]. Let \( P \subset \mathbb{R}^N \) be an integral convex polytope of dimension \( d \), and suppose that \((P - \partial P) \cap \mathbb{Z}^N \) is nonempty. Then the \( \delta \)-vector \( \delta(P) = (\delta_0, \delta_1, \ldots, \delta_d) \) of \( P \) satisfies the following linear inequalities:

\[
\begin{align*}
(6.1) \quad & \delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_d + \delta_{d-1} + \cdots + \delta_{d-i}, & 0 \leq i \leq \lfloor d/2 \rfloor; \\
(6.2) \quad & \delta_d + \delta_{d-1} + \cdots + \delta_{d-i} \leq \delta_0 + \delta_1 + \cdots + \delta_i + \delta_{i+1}, & 0 \leq i \leq \lfloor (d - 1)/2 \rfloor; \\
(6.3) \quad & \delta_i \leq \delta_{i+1}, & 2 \leq i < d.
\end{align*}
\]

We conclude the paper with a remark about the question when \( A_k(\Gamma) \) is Gorenstein. For a while, we assume that \( N = d \) and the origin of \( \mathbb{R}^d \) is contained in the interior of \( X \). We say that \( \delta(X) = (\delta_0, \delta_1, \ldots, \delta_d) \) is symmetric if \( \delta_i = \delta_{d-i} \) for every \( 0 \leq i \leq d \). It follows from, e.g., [5] that \( X \) is star-shaped with respect to the origin if \( \delta(X) \) is symmetric. On the other hand, \( \delta(X) \) is symmetric if and only if there exists a polyhedral complex \( \Gamma \) in \( \mathbb{R}^d \) with the underlying space \( X \) such that \( A_k(\Gamma) \) is Gorenstein, i.e., the canonical module \( \Omega(A_k(\Gamma)) \) of \( A_k(\Gamma) \) is generated by a single element of \( A_k(\Gamma) \).

**References**


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