NUMERICAL MESHES AND COVERING MESHES OF APPROXIMATE INVERSE SYSTEMS OF COMPACTA

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Abstract. Mardesic and Rubin (1989) introduced approximate inverse systems of metric compacta by the conditions (A1)*-(A3)*. Mardesic and Watanabe (1988) introduced approximate inverse systems of topological spaces by the conditions (A1)-(A3). In this note we show that any approximate inverse system of metric compacta satisfies (A1)-(A3) if and only if it satisfies (A1)*-(A3)* for some matrices (see Theorem 1).

S. Mardesic and L. Rubin [1] introduced the notion of approximate inverse system \( \mathcal{A} = \{(X_a, d_a), e_a, p_{aa'}, A\} \) of metric compacta. Hence, \( (A, \leq) \) is a directed preordered infinite set. \( (X_a, d_a) \) is a compactum endowed with a metric \( d_a \), and \( p_{aa'} : X_{a'} \to X_a \) is a mapping defined whenever \( a \leq a' \) and is such that \( p_{aa} \) is the identity mapping. The real numbers \( e_a > 0, a \in A \), are called numerical meshes. We require the following conditions:

\[(A1)^* \quad (\forall a_2 > a_1 \geq a) d_a(p_{aa}, p_{a_1a_2}, p_{aa_2}) \leq e_a.\]
\[(A2)^* \quad (\forall a \in A)(\forall \eta > 0)(\exists a' \geq a)(\forall a_2 > a_1 \geq a') d_a(p_{aa}, p_{a_1a_2}, p_{aa_2}) \leq \eta.\]
\[(A3)^* \quad (\forall a \in A)(\forall \eta > 0)(\exists a' \geq a)(\forall a'' \geq a') (\forall x, x' \in X_{a''}) d_{a''}(x, x') \leq e_{a''} \implies d_a(p_{aa''}(x), p_{aa''}(x')) \leq \eta.\]

In [6] S. Mardesic and T. Watanabe introduced the notion of an approximate inverse system \( \mathcal{A} = \{X_a, \mathcal{U}_a, p_{aa'}, A\} \) of topological spaces. Here \( A \) is a directed preordered infinite set, \( \mathcal{U}_a \) is a normal open covering of \( X_a, a \in A \), also called a mesh, and \( p_{aa'} : X_{a'} \to X_a \) is a mapping defined whenever \( a \leq a' \) and is such that \( p_{aa} \) is the identity mapping. We require three conditions (A1)-(A3), which are natural analogues of conditions (A1)*-(A3)*. Before we state these conditions, let us denote that \( \text{Cov}(X) \) is the set of all normal open coverings of a space \( X \). For \( \mathcal{U}, \mathcal{V} \in \text{Cov}(X) \), \( \mathcal{V} \prec \mathcal{U} \) means that \( \mathcal{V} \) refines \( \mathcal{U} \). If \( \mathcal{V} \in \text{Cov}(X) \) and \( f, f' : X \to Y \) are mappings, \( (f, f') \prec \mathcal{V} \) means that \( f \) and \( f' \) are \( \mathcal{V} \)-near mappings, i.e., for each \( x \in X \) there is a \( V \in \mathcal{V} \) such that \( f(x), f'(x) \in V \).

\[(A1) \quad (\forall a_2 \geq a_1 \geq a)(p_{aa}, p_{a_1a_2}, p_{aa_2}) \prec \mathcal{U}_a.\]
\[(A2) \quad (\forall a \in A)(\forall \mathcal{U} \in \text{Cov}(X_a))(\exists a' \geq a)(\forall a_2 \geq a_1 \geq a')(p_{aa}, p_{a_1a_2}, p_{aa_2}) \prec \mathcal{U}.\]
These approximate inverse systems are noncommutative inverse systems. They have many applications in dimension theory and shape theory (see [2, 5–7, 9–11]).

In this note we investigate the relation between conditions (A1)–(A3) and (A1)*–(A3)*. Our purpose is the following Theorem 1.

**Theorem 1.** Let \((A, \leq)\) be a directed preordered cofinite infinite set; \(X_a, a \in A\), be a compact metric space; \(p_{aa'}: X_a \to X_a, a \leq a',\) be a mapping; and \(p_{aa}, a \in A,\) be the identity mapping. Then the following conditions are equivalent:

(A) There are coverings \(\mathcal{U}_a \in \text{Cov}(X_a), a \in A,\) such that \(\mathcal{U} = \{X_a, \mathcal{U}_a, p_{aa'}, A\}\) satisfies (A1)–(A3).

(B) There are metrics \(d_a\) on \(X_a, a \in A,\) which induce the topology of \(X_a,\) such that \(\mathcal{U} = \{(X_a, d_a), \epsilon_a, p_{aa'}, A\}, \epsilon_a = 1\) for \(a \in A,\) satisfies (A1)*–(A3)*.

(C) For any real numbers \(\epsilon_a > 0, a \in A,\) there are metrics \(d_a\) on \(X_a, a \in A,\) which induce the topology of \(X_a,\) such that \(\mathcal{U} = \{(X_a, d_a), \epsilon_a, p_{aa'}, A\}\) satisfies (A1)*–(A3)*.

(D) There are real numbers \(\epsilon_a > 0\) and metrics \(d_a\) on \(X_a, a \in A,\) which induce the topology of \(X_a,\) such that \(\mathcal{U} = \{(X_a, d_a), \epsilon_a, p_{aa'}, A\}\) satisfies (A1)*–(A3)*.

For our proof we need some lemmas. Let \(\mathcal{U}, \mathcal{V} \in \text{Cov}(X).\) For any subset \(K\) of \(X,\) we put \(st(K, \mathcal{U}) = \bigcup\{U \in \mathcal{U}: U \cap K \neq \emptyset\}.\) When \(K = \{x\}\) is a singleton set, \(st(x, \mathcal{U})\) denotes \(st(\{x\}, \mathcal{U}).\) Let \(st \mathcal{U}\) be a covering \(\{st(U, \mathcal{U}): U \in \mathcal{U}\}\) of \(X.\) Inductively, we define \(st^{n+1} \mathcal{U} = st(st^n \mathcal{U})\) and \(st^0 \mathcal{U} = \mathcal{U}\) for each integer \(n.\) We say \(st^n \mathcal{U}\) is the \(n\)th star covering of \(\mathcal{U}.\) When \(st^n \mathcal{U} < \mathcal{V},\) we say \(\mathcal{V}\) is an \(n\)-refinement of \(\mathcal{U}.\) Note that an open covering \(\mathcal{W}\) of \(X\) is normal provided there is a sequence of open coverings \(\mathcal{W}_i, i = 1, 2, \ldots,\) of \(X\) such that \(st \mathcal{W}_{i+1} < st \mathcal{W}_i\) and \(\mathcal{W}_1 = \mathcal{W}.\) We call such a sequence a normal sequence of \(\mathcal{W}.\) Let \(\mathcal{W}^\Delta\) be a normal covering \(\{st^0(x, \mathcal{U}): x \in X\}\) of \(X.\) Clearly \(\mathcal{W}^\Delta < st \mathcal{U}.\) Note that any open covering of a compact Hausdorff space is normal (see [8]). We can easily show the following:

**Lemma 2.** If \(\mathcal{U} = \{X_a, \mathcal{U}_a, p_{aa'}, A\}\) satisfies (A1)–(A3), then \(st^n \mathcal{U} = \{X_a, st^n \mathcal{U}_a, p_{aa'}, A\}\) satisfies (A1)–(A3) for each integer \(n.\)

Let \((X, d)\) be a compact metric space. For any \(e > 0,\) let \(S_d(x, e) = \{y \in X: d(x, y) < e\}\) for any \(x \in X\) and let \(\mathcal{S}_d(e) = \{S_d(x, e): x \in X\}.\)

**Lemma 3.** Let \((X, d)\) be a compact metric space. For any \(\mathcal{U} \in \text{Cov}(X),\) there exists a metric \(d^*\) on \(X\) satisfying

(i) \(d^*\) induces the topology of \(X,\)
(ii) \(\mathcal{U} < \mathcal{S}_d^*(2^{-5}) < st^8 \mathcal{U} < \mathcal{S}_d^*(2^{-1}) < st^8 \mathcal{U}.\)

**Proof.** Since \(d\) is a metric on \(X,\) \(\{\mathcal{S}_d(2^{-n}): n = 1, 2, \ldots,\}\) generates a uniformity \(\mu\) of \(X.\) Clearly, \(\text{Cov}(X)\) is also a uniformity of \(X.\) Since \(X\) is compact, we have the unique uniformity of \(X.\) Thus

(1) \(\mu = \text{Cov}(X)\).
Take any \( \mathcal{U} \in \text{Cov}(X) \). Since \( \mathcal{U} \) is normal, we have a sequence of open coverings \( \mathcal{U}_i, i = 1, 2, \ldots, \) of \( X \) such that
\[
(2) \quad \mathcal{U}_1 = \mathcal{U} \quad \text{and} \quad \mathcal{U}_i > \text{st} \mathcal{U}_{i+1} \quad \text{for each integer } i.
\]

By (1) and (2) inductively it is easy to make a sequence of open coverings \( \mathcal{V}_i, i = 1, 2, \ldots, \) of \( X \) such that
\[
(3) \quad \mathcal{V}_i > \mathcal{V}_i \quad \text{and} \quad \mathcal{V}_i \supset \mathcal{V}_{i+1} \quad \text{for each } i.
\]
\[
(4) \quad \mathcal{V}_i > \text{st} \mathcal{V}_{i+1} \quad \text{for each } i.
\]

By (3) we have that
\[
(5) \quad \{\text{st}(x, \mathcal{V}_i) : i = 1, 2, \ldots\} \text{ is a neighborhood base of } x \in X.
\]

Now, let \( \mathcal{W}_1 = \text{st} \mathcal{U}, \mathcal{W}_2 = \text{st}^2 \mathcal{U}, \ldots, \mathcal{W}_i = \text{st}^i \mathcal{U}, \mathcal{W}_8 = \mathcal{U}, \mathcal{W}_9 = \mathcal{V}, \mathcal{W}_10 = \mathcal{V}_3, \ldots, \mathcal{W}_j = \mathcal{V}_{j-7}, \ldots. \) Thus by (4) and (5) we have that
\[
(6) \quad \mathcal{W}_i > \text{st} \mathcal{W}_{i+1} \quad \text{for each } i.
\]
\[
(7) \quad \{\text{st}(x, \mathcal{W}_i) : i = 1, 2, \ldots\} \text{ is a neighborhood base of } x \in X.
\]

By (6), (7), and the proofs of 2-16 Theorem and 2-18 Corollary of [8, pp. 13-15], there is a metric \( d^* \) on \( X \) such that
\[
(8) \quad d^* \text{ induces the topology of } X,
\]
\[
(9) \quad \mathcal{W}_i^{1 + 3} < \mathcal{I}_{d^*} (2^{-i}) < \mathcal{W}_i^\Delta \quad \text{for each } i.
\]

Since \( \mathcal{W}_i < \mathcal{W}_i^\Delta \) and \( \mathcal{W}_i^\Delta < \text{st} \mathcal{W}_i \), by (9) for \( i = 1, 5 \) we have condition (ii).
(8) means condition (i). Hence we have Lemma 3.

**Proof of Theorem 1.** First, we show (A) \( \rightarrow \) (B). We assume condition (A) and take any \( a \in A \). By Lemma 3 there is a metric \( d^*_a \) on \( X_a \) such that
\[
(1) \quad d^*_a \text{ induces the topology of } X_a,
\]
\[
(2) \quad \mathcal{U}_a < \mathcal{I}_{d^*_a} (2^{-5}) < \text{st}^4 \mathcal{U}_a < \mathcal{I}_{d^*_a} (2^{-1}) < \text{st}^8 \mathcal{U}_a.
\]

Let \( d^*_a(x, x') = 2^4 d^*_a(x, x') \) for \( x, x' \in X_a \). Thus \( d^*_a \) is a metric and by (1) we have
\[
(3) \quad d^*_a \text{ induces the topology of } X_a.
\]

We show that \( \mathcal{X}^* = \{(X_a, d^*_a), e_a, p_{aa'}, A\} \), \( e_a = 1 \) for \( a \in A \), satisfies (A1)\*–(A3)\*. We consider (A1)\* for \( \mathcal{X}^* \). Take any \( a_2 \geq a_1 \geq a \). By (A1) for \( \mathcal{X} \), \( (p_{aa_1}, p_{aa_2}, p_{aa_3}) < \mathcal{U}_a \). By (2) \( d^*_a(p_{aa_1}, p_{aa_2}, p_{aa_3}) < 2 \cdot 2^{-5} = 2^{-4} \). Thus \( d^*_a(p_{aa_1}, p_{aa_2}, p_{aa_3}) < 1 = e_a \). This means condition (A1)\* for \( \mathcal{X}^* \).

We consider (A2)\* for \( \mathcal{X}^* \). Take any \( a \in A \) and any \( \eta > 0 \). We apply (A2) for \( \mathcal{X} \) to \( a \) and \( \mathcal{I}_{d^*_a} (\eta/2) \). Thus there exists an \( a' \geq a \) such that for each \( a_2 \geq a_1 \geq a' \), \( (p_{aa_1}, p_{aa_2}, p_{aa_3}) < \mathcal{I}_{d^*_a} (\eta/2) \). This means that \( d^*_a(p_{aa_1}, p_{aa_2}, p_{aa_3}) < \eta \). Thus we have condition (A2)\* for \( \mathcal{X}^* \).

We consider (A3)\* for \( \mathcal{X}^* \). Take any \( a \in A \) and any \( \eta > 0 \). By the assumption, \( \mathcal{X} \) satisfies (A3). Thus by Lemma 2, \( \text{st}^8 \mathcal{X} \) also satisfies (A3). By applying (A3) for \( \text{st}^8 \mathcal{X} \) there is an \( a' \geq a \) such that for any \( a'' \geq a' \)
\[
(4) \quad p_{aa''} (\mathcal{I}_{d^*_a} (\eta/2)) > \text{st}^8 \mathcal{U}_{aa''}.
\]
Take any \( a'' \geq a' \) and any points \( x, x' \in X_{a''} \) such that \( d^*_{a''}(x, x') \leq \varepsilon_a'' = 1 \). Since \( d^*_{a'}(x, x') \leq 2^{-\delta} \), \( x, x' \in S_{d^*_{a'}}(x, 2^{-\delta}) \). Thus by (2) \( x, x' \in S_{d^*_{a'}}(x, 2^{-\delta}) \subset U \) for some \( U \in \text{st}^8 \mathcal{Z}_{a''} \). By (4) \( U \subset p_{a''}(S_{d^*_{a'}}(z, \eta/2)) \) for some \( z \in X_a \). Then \( p_{a''}(x), p_{a''}(x') \in p_{a''}(U) \subset S_{d^*_{a''}}(z, \eta/2) \), and hence \( d^*_{a''}(p_{a''}(x), p_{a''}(x')) < \eta \). This means condition (A3)* for \( \mathcal{Z}^{**} \). Therefore we have (A) \( \rightarrow \) (B).

We show (B) \( \rightarrow \) (C). We may assume that \( \mathcal{Z} = \{(X_a, d_a), k_a, p_{a''}, A\} \), \( k_a = 1 \) for \( a \in A \), satisfies (A1)*-(A3)*. Take any real numbers \( \varepsilon_a > 0, a \in A \). We put \( d_a^*(x, x') = \varepsilon_a d_a(x, x') \) for \( x, x' \in X_a \) and \( a \in A \). Clearly \( d_a^* \) is a metric on \( X_a \), and it is not difficult to show that \( \mathcal{Z}^{**} = \{(X_a, d_a^*), \varepsilon_a, p_{a''}, A\} \) satisfies (A1)*-(A3)*. Therefore we have (B) \( \rightarrow \) (C).

Clearly, we have (C) \( \rightarrow \) (D) and (D) \( \rightarrow \) (A) is Theorem 1 of [3]. Hence, we complete the proof of Theorem 1.

Remark 4. We consider the following condition:

(E) For any metric \( d_a \) on \( X_a \) which induces the topology of \( X_a \), there are real numbers \( \varepsilon_a > 0, a \in A \), such that \( \mathcal{Z} = \{(X_a, d_a), \varepsilon_a, p_{a''}, A\} \) satisfies (A1)*-(A3)*.

Clearly (E) \( \rightarrow \) (D). However, in general, (D) \( \rightarrow \) (E) does not hold because Example 1 of [3] satisfies (A) but not (E).

REFERENCES

7. S. Mardešić and N. Uglešić, Approximate inverse systems which admit meshes, preprint.

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